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**Based on the latest  
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## O-LEVEL CLASSIFIED ADDITIONAL MATHEMATICS

New Edition 2008

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GCE O Level Past Years' Examination Papers and Syllabus Support Materials:*

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ISBN-13 978-981-274-281-0

ISBN-10 981-274-281-6

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## O LEVEL ADDITIONAL MATHEMATICS EXAMINATION

NOVEMBER 2007	(N2007)1 - 9
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ANSWERS ( <i>separate booklet</i> )	A1 - A19
<i>Specimen Papers</i>	
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# O LEVEL ADDITIONAL MATHEMATICS (4038)

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## SCHEME OF ASSESSMENT

Paper	Duration	Description	Marks	Weighting
Paper 1	2 h	There will be 11 – 13 questions of varying marks and lengths testing more on the fundamental skills and concepts. Candidates are required to answer <b>all</b> questions.	80	44%
Paper 2	$2\frac{1}{2}$ h	There will be 9 – 11 questions of varying marks and lengths. Candidates are required to answer <b>all</b> questions.	100	56%

# CONTENT OUTLINE

Knowledge of the content of the O Level Mathematics syllabus is assumed in the syllabus below and will not be tested directly, but it may be required indirectly in response to questions on other topics.

	Topic/Sub-topics	Content
<b>1</b>	<b>Algebra</b>	
1.1	Quadratic equations and inequalities	Include: <ul style="list-style-type: none"> <li>• conditions for a quadratic equation to have:               <ol style="list-style-type: none"> <li>(i) two real roots</li> <li>(ii) two equal roots</li> <li>(iii) no real roots</li> </ol>               and related conditions for a given line to:               <ol style="list-style-type: none"> <li>(i) intersect a given curve</li> <li>(ii) be a tangent to a given curve</li> <li>(iii) not intersect a given curve</li> </ol> </li> <li>• solution of quadratic inequalities, and the representation of the solution set on the number line</li> <li>• conditions for <math>ax^2+bx+c</math> to be always positive (or always negative)</li> <li>• relationships between the roots and coefficients of the quadratic equation <math>ax^2+bx+c=0</math></li> </ul>
1.2	Indices and surds	Include: <ul style="list-style-type: none"> <li>• four operations on indices and surds</li> <li>• rationalising the denominator</li> <li>• solving equations involving indices and surds</li> </ul>
1.3	Polynomials	Include: <ul style="list-style-type: none"> <li>• multiplication and division of polynomials</li> <li>• use of remainder and factor theorems</li> <li>• factorisation of polynomials</li> <li>• solving cubic equations</li> </ul>
1.4	Simultaneous equations in two unknowns	Include: <ul style="list-style-type: none"> <li>• solving simultaneous equations with at least one linear equation, by substitution</li> <li>• expressing a pair of linear equations in matrix form and solving the equations by inverse matrix method</li> </ul>
1.5	Partial fractions	Include cases where the denominator is no more complicated than: <ul style="list-style-type: none"> <li>• <math>(ax+b)(cx+d)</math></li> <li>• <math>(ax+b)(cx+d)^2</math></li> <li>• <math>(ax+b)(x^2+c^2)</math></li> </ul>

	Topic/Sub-topics	Content
1.6	Binomial expansions	Include: <ul style="list-style-type: none"> <li>• use of the Binomial Theorem for positive integer <math>n</math></li> <li>• use of the notations <math>n!</math> and <math>\binom{n}{r}</math></li> <li>• use of the general term <math>\binom{n}{r} a^{n-r} b^r</math>, <math>0 &lt; r \leq n</math></li> </ul> Exclude: <ul style="list-style-type: none"> <li>• proof of the theorem</li> <li>• knowledge of the greatest term and properties of the coefficients</li> </ul>
1.7	Exponential, logarithmic and modulus functions	Include: <ul style="list-style-type: none"> <li>• functions <math>a^x</math>, <math>e^x</math>, <math>\log_a x</math>, <math>\ln x</math> and their graphs</li> <li>• laws of logarithms</li> <li>• equivalence of <math>y = a^x</math> and <math>x = \log_a y</math></li> <li>• change of base of logarithms</li> <li>• function <math> x </math> and graph of <math> f(x) </math>, where <math>f(x)</math> is linear, quadratic or trigonometric</li> <li>• solving simple equations involving exponential, logarithmic and modulus functions</li> </ul>
2	Geometry and Trigonometry	
2.1	Trigonometric functions, identities and equations	Include: <ul style="list-style-type: none"> <li>• six trigonometric functions for angles of any magnitude (in degrees or radians)</li> <li>• principal values of <math>\sin^{-1} x</math>, <math>\cos^{-1} x</math>, <math>\tan^{-1} x</math></li> <li>• exact values of the trigonometric functions for special angles (<math>30^\circ</math>, <math>45^\circ</math>, <math>60^\circ</math>) or <math>(\frac{\pi}{6}</math>, <math>\frac{\pi}{4}</math>, <math>\frac{\pi}{3})</math> amplitude, periodicity and symmetries related to the sine and cosine functions</li> <li>• graphs of <math>y = a \sin(bx) + c</math>, <math>y = a \sin(\frac{x}{b}) + c</math>,  <math>y = a \cos(bx) + c</math>, <math>y = a \cos(\frac{x}{b}) + c</math> and <math>y = a \tan(bx)</math>, where <math>a</math> and <math>b</math> are positive integers and <math>c</math> is an integer</li> <li>• use of the following               <ul style="list-style-type: none"> <li>• <math>\frac{\sin A}{\cos A} = \tan A</math>, <math>\frac{\cos A}{\sin A} = \cot A</math>, <math>\sin^2 A + \cos^2 A = 1</math>, <math>\sec^2 A = 1 + \tan^2 A</math>, <math>\operatorname{cosec}^2 A = 1 + \cot^2 A</math></li> <li>• the expansions of <math>\sin(A \pm B)</math>, <math>\cos(A \pm B)</math> and <math>\tan(A \pm B)</math></li> <li>• the formulae for <math>\sin 2A</math>, <math>\cos 2A</math> and <math>\tan 2A</math></li> <li>• the formulae for <math>\sin A \pm \sin B</math> and <math>\cos A \pm \cos B</math></li> <li>• the expression for <math>a \cos \theta + b \sin \theta</math> in the form <math>R \cos(\theta \pm \alpha)</math> or <math>R \sin(\theta \pm \alpha)</math></li> </ul> </li> </ul>

	Topic/Sub-topics	Content
		<ul style="list-style-type: none"> <li>simplification of trigonometric expressions</li> <li>solution of simple trigonometric equations in a given interval</li> <li>proofs of simple trigonometric identities</li> </ul> <p>Exclude general solution of trigonometric equations</p>
2.2	Coordinate geometry in two dimensions	<p>Include:</p> <ul style="list-style-type: none"> <li>condition for two lines to be parallel or perpendicular</li> <li>mid-point of line segment</li> <li>finding the area of rectilinear figure given its vertices</li> <li>graphs of equations               <ul style="list-style-type: none"> <li><math>y = ax^n</math>, where <math>n</math> is a simple rational number</li> <li><math>y^2 = kx</math></li> </ul> </li> <li>coordinate geometry of the circle with the equation <math>(x - a)^2 + (y - b)^2 = r^2</math> and <math>x^2 + y^2 + 2gx + 2fy + c = 0</math></li> <li>transformation of given relationships, including <math>y = ax^n</math> and <math>y = kb^x</math>, to linear form to determine the unknown constants from the straight line graph</li> </ul> <p>Exclude:</p> <ul style="list-style-type: none"> <li>finding the equation of the circle passing through three given points</li> <li>intersection of two circles</li> </ul>
2.3	Proofs in plane geometry	<p>Include:</p> <ul style="list-style-type: none"> <li>□ symmetry and angle properties of triangles, special quadrilaterals and circles*</li> <li>□ mid-point theorem and intercept theorem for triangles</li> <li>□ tangent-chord theorem (alternate segment theorem), intersecting chords theorem and tangent-secant theorem for circles</li> <li>□ use of above properties and theorems</li> </ul>
3	Calculus	
3.1	Differentiation and integration	<p>Include:</p> <ul style="list-style-type: none"> <li>derivative of <math>f(x)</math> as the gradient of the tangent to the graph of <math>y = f(x)</math> at a point</li> <li>derivative as rate of change</li> <li>use of standard notations <math>f'(x)</math>, <math>f''(x)</math>, <math>\frac{dy}{dx}</math>, <math>\frac{d^2y}{dx^2}</math>, <math>[= \frac{d}{dx}(\frac{dy}{dx})]</math></li> <li>derivatives of <math>x^n</math>, for any rational <math>n</math>, <math>\sin x</math>, <math>\cos x</math>, <math>\tan x</math>, <math>e^x</math> and <math>\ln x</math>, together with constant multiples, sums and differences</li> <li>derivatives of composite functions</li> <li>derivatives of products and quotients of functions</li> <li>increasing and decreasing functions</li> <li>stationary points (maximum and minimum turning points and stationary points of inflexion)</li> </ul>

\* These are properties learnt in O Level Mathematics.



	Topic/Sub-topics	Content
		<ul style="list-style-type: none"> <li>• use of second derivative test to discriminate between maxima and minima</li> <li>• applying differentiation to gradients, tangents and normals, connected rates of change and maxima and minima problems</li> <li>• integration as the reverse of differentiation</li> <li>• integration of <math>x^n</math> for any rational <math>n</math>, <math>\sin x</math>, <math>\cos x</math>, <math>\sec^2 x</math> and <math>e^x</math>, together with constant multiples, sums and differences</li> <li>• integration of <math>(ax + b)^n</math> for any rational <math>n</math>, <math>\sin(ax + b)</math>, <math>\cos(ax + b)</math> and <math>e^{(ax + b)}</math></li> <li>• definite integral as area under a curve</li> <li>• evaluation of definite integrals</li> <li>• finding the area of a region bounded by a curve and lines parallel to the coordinate axes</li> <li>• finding areas of regions below the <math>x</math>-axis</li> <li>• application of differentiation and integration to problems involving displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration</li> </ul> <p>Exclude:</p> <ul style="list-style-type: none"> <li>• differentiation of functions defined implicitly and parametrically</li> <li>• finding the area of a region between a curve and an oblique line, or between two curves</li> <li>• use of formulae for motion with constant acceleration</li> </ul>

# MATHEMATICAL NOTATION

The list which follows summarizes the notation used in the Syndicate's Mathematics examinations. Although primarily directed towards A level, the list also applies, where relevant, to examinations at all other levels, i.e. O level, AO level and N level.

## 1. Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \dots\}$	the set with elements $x_1, x_2, \dots$
$\{x: \dots\}$	the set of all $x$ such that
$n(A)$	the number of elements in set $A$
$\emptyset$	the empty set
$\mathcal{U}$	universal set
$A'$	the complement of the set $A$
$\mathbb{N}$	the set of positive integers, $\{1, 2, 3, \dots\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \dots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \dots\}$
$\mathbb{Z}_n$	the set of integers modulo $n$ , $\{0, 1, 2, \dots, n-1\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q}: x > 0\}$
$\mathbb{Q}_0^+$	the set of positive rational numbers and zero, $\{x \in \mathbb{Q}: x \geq 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \in \mathbb{R}: x > 0\}$
$\mathbb{R}_0^+$	the set of positive real numbers and zero, $\{x \in \mathbb{R}: x \geq 0\}$
$\mathbb{R}^n$	the real $n$ tuples
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\not\subseteq$	is not a subset of
$\not\subset$	is not a proper subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leq x \leq b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leq x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leq b\}$
$(a, b)$	the open interval $\{x \in \mathbb{R}: a < x < b\}$
$yRx$	$y$ is related to $x$ by the relation $R$

## 2. Miscellaneous Symbols

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\cong$	is isomorphic to
$\propto$	is proportional to
$<$ ; $\ll$	is less than; is much less than
$\leq$ ; $\leqslant$	is less than or equal to; is not greater than
$>$ ; $\gg$	is greater than; is much greater than
$\geq$ ; $\geqslant$	is greater than or equal to; is not less than
$\infty$	infinity

## 3. Operations

$a + b$	$a$ plus $b$
$a - b$	$a$ minus $b$
$a \times b, ab, a \cdot b$	$a$ multiplied by $b$
$a \div b, \frac{a}{b}, a/b$	$a$ divided by $b$
$a : b$	the ratio of $a$ to $b$
$\sum_{i=1}^n a_i$	$a_1 + a_2 + \dots + a_n$
$\sqrt{a}$	the positive square root of the real number $a$
$ a $	the modulus of the real number $a$
$n!$	$n$ factorial for $n \in \mathbb{Z}^+ \cup \{0\}$ ( $0! = 1$ )
$\binom{n}{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$ , for $n, r \in \mathbb{Z}^+ \cup \{0\}$ , $0 \leq r \leq n$ $\frac{n(n-1)\dots(n-r+1)}{r!}$ , for $n \in \mathbb{Q}$ , $r \in \mathbb{Z}^+ \cup \{0\}$

#### 4. Functions

$f$	function $f$
$f(x)$	the value of the function $f$ at $x$
$f: A \rightarrow B$	$f$ is a function under which each element of set $A$ has an image in set $B$
$f: x \mapsto y$	the function $f$ maps the element $x$ to the element $y$
$f^{-1}$	the inverse of the function $f$
$g \circ f, gf$	the composite function of $f$ and $g$ which is defined by $(g \circ f)(x)$ or $gf(x) = g(f(x))$
$\lim_{x \rightarrow a} f(x)$	the limit of $f(x)$ as $x$ tends to $a$
$\Delta x; \delta x$	an increment of $x$
$\frac{dy}{dx}$	the derivative of $y$ with respect to $x$
$\frac{d^n y}{dx^n}$	the $n$ th derivative of $y$ with respect to $x$
$f'(x), f''(x), \dots, f^{(n)}(x)$	the first, second, ..., $n$ th derivatives of $f(x)$ with respect to $x$
$\int y dx$	indefinite integral of $y$ with respect to $x$
$\int_a^b y dx$	the definite integral of $y$ with respect to $x$ for values of $x$ between $a$ and $b$
$\frac{\partial y}{\partial x}$	the partial derivative of $y$ with respect to $x$
$\dot{x}, \ddot{x}, \dots$	the first, second, ... derivatives of $x$ with respect to time

#### 5. Exponential and Logarithmic Functions

$e$	base of natural logarithms
$e^x, \exp x$	exponential function of $x$
$\log_a x$	logarithm to the base $a$ of $x$
$\ln x$	natural logarithm of $x$
$\lg x$	logarithm of $x$ to base 10

#### 6. Circular Functions and Relations

$\sin, \cos, \tan,$ $\operatorname{cosec}, \sec, \cot$	} the circular functions
$\sin^{-1}, \cos^{-1}, \tan^{-1}$ $\operatorname{cosec}^{-1}, \sec^{-1}, \cot^{-1}$	} the inverse circular functions

## 7. Complex Numbers

<b>i</b>	square root of $-1$
<b>z</b>	a complex number, $z = x + iy$ $= r(\cos \theta + i \sin \theta)$ , $r \in \mathbb{R}_0^+$ $= re^{i\theta}$ , $r \in \mathbb{R}_0^+$
<b>Re z</b>	the real part of $z$ , $\text{Re}(x+iy) = x$
<b>Im z</b>	the imaginary part of $z$ , $\text{Im}(x+iy) = y$
<b> z </b>	the modulus of $z$ , $ x+iy  = \sqrt{(x^2+y^2)}$ , $ r(\cos \theta + i \sin \theta)  = r$
<b>arg z</b>	the argument of $z$ , $\arg(r(\cos \theta + i \sin \theta)) = \theta$ , $-\pi < \theta \leq \pi$
<b>z*</b>	the complex conjugate of $z$ , $(x+iy)^* = x-iy$

## 8. Matrices

<b>M</b>	a matrix <b>M</b>
<b>M<sup>-1</sup></b>	the inverse of the square matrix <b>M</b>
<b>M<sup>T</sup></b>	the transpose of the matrix <b>M</b>
<b>det M</b>	the determinant of the square matrix <b>M</b>

## 9. Vectors

<b>a</b>	the vector <b>a</b>
$\vec{AB}$	the vector represented in magnitude and direction by the directed line segment <b>AB</b>
$\hat{a}$	a unit vector in the direction of the vector <b>a</b>
<b>i, j, k</b>	unit vectors in the directions of the cartesian coordinate axes
<b> a </b>	the magnitude of <b>a</b>
$ \vec{AB} $	the magnitude of $\vec{AB}$
<b>a.b</b>	the scalar product of <b>a</b> and <b>b</b>
<b>a<b>x</b>b</b>	the vector product of <b>a</b> and <b>b</b>

## 10. Probability and Statistics

<b>A, B, C, etc.</b>	events
<b>A <math>\cup</math> B</b>	union of events <b>A</b> and <b>B</b>
<b>A <math>\cap</math> B</b>	intersection of the events <b>A</b> and <b>B</b>
<b>P(A)</b>	probability of the event <b>A</b>
<b>A'</b>	complement of the event <b>A</b> , the event 'not <b>A</b> '
<b>P(A B)</b>	probability of the event <b>A</b> given the event <b>B</b>
<b>X, Y, R, etc.</b>	random variables
<b>x, y, r, etc.</b>	value of the random variables <b>X, Y, R, etc.</b>
<b>x<sub>1</sub>, x<sub>2</sub>, ...</b>	observations
<b>f<sub>1</sub>, f<sub>2</sub>, ...</b>	frequencies with which the observations, <b>x<sub>1</sub>, x<sub>2</sub> ...</b> occur

$p(x)$	the value of the probability function $P(X = x)$ of the discrete random variable $X$
$p_1, p_2 \dots$	probabilities of the values $x_1, x_2, \dots$ of the discrete random variable $X$
$f(x), g(x) \dots$	the value of the probability density function of the continuous random variable $X$
$F(x), G(x) \dots$	the value of the (cumulative) distribution function $P(X \leq x)$ of the random variable $X$
$E(X)$	expectation of the random variable $X$
$E[g(X)]$	expectation of $g(X)$
$\text{Var}(X)$	variance of the random variable $X$
$B(n, p)$	binominal distribution, parameters $n$ and $p$
$N(\mu, \sigma^2)$	normal distribution, mean $\mu$ and variance $\sigma^2$
$\mu$	population mean
$\sigma^2$	population variance
$\sigma$	population standard deviation
$\bar{x}$	sample mean
$s^2$	unbiased estimate of population variance from a sample,
	$s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$
$\phi$	probability density function of the standardised normal variable with distribution $N(0, 1)$
$\Phi$	corresponding cumulative distribution function
$\rho$	linear product-moment correlation coefficient for a population
$r$	linear product-moment correlation coefficient for a sample
$\text{Cov}(X, Y)$	covariance of $X$ and $Y$

## Mathematical Formulae

### 1. ALGEBRA

#### Quadratic Equation

For the equation  $ax^2 + bx + c = 0$ ,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

#### Binomial expansion

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n,$$

where  $n$  is a positive integer and  $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

### 2. TRIGONOMETRY

#### Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$$

#### Formulae for $\Delta ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

MINISTRY OF EDUCATION, SINGAPORE  
in collaboration with  
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE  
General Certificate of Education Ordinary Level

**ADDITIONAL MATHEMATICS**

**4038/02**

Paper 2

For Examination from 2008

**SPECIMEN PAPER**

**2 hours 30 minutes**

Additional Materials: Answer Paper  
Graph paper

**READ THESE INSTRUCTIONS FIRST**

Write your Centre number, index number and name on all the work you hand in.

Write in dark blue or black pen.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Write your answers on the separate Answer Paper provided.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The use of a scientific calculator is expected, where appropriate.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 100.

This document consists of 6 printed pages.



Singapore Examinations and Assessment Board

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UNIVERSITY of CAMBRIDGE  
International Examinations

[Turn over



- 1 An engineering firm buys a new piece of equipment at a cost of \$72000. The value of this equipment decreases with time so that its value,  $V$ , after  $t$  months use is given by

$$V = 72000 e^{-kt},$$

where  $k$  is a positive constant. The value of the equipment is expected to be \$40000 after 30 months use.

- (i) Calculate the value, to the nearest \$100, of the equipment after 20 months use. [4]

The equipment is replaced when its value reaches  $\frac{1}{3}$  of its original value.

- (ii) Calculate the length of time, to the nearest month, the equipment has been in use when it is replaced. [2]

- 2 The function  $f$  is defined, for all values of  $x$ , by  $f(x) = 2\cos\left(\frac{x}{2}\right) - 1$ .

- (i) State the amplitude and period of  $f$ . [2]

The function  $g$  is defined, for  $0^\circ \leq x \leq 360^\circ$ , by  $g(x) = 2\cos\left(\frac{x}{2}\right) - 1$ .

- (ii) Find the  $x$ -coordinate of the point where the graph of  $y = g(x)$  crosses the  $x$ -axis. [2]

- (iii) Sketch the graph of  $y = g(x)$ . [2]

- (iv) Sketch the graph of  $y = |g(x)|$ . [1]

- 3 The equation of a curve is  $y = e^{x^3}$ .

- (i) Find expressions for  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ . [4]

- (ii) Find the  $x$ -coordinate of the stationary point and determine the nature of the stationary point. [3]

- 4 (i) Express  $\frac{x+21}{x^2-9}$  in partial fractions. [4]

- (ii) Hence evaluate  $\int_2^5 \frac{x+21}{x^2-9} dx$ . [4]

- 5 The roots of the quadratic equation  $2x^2 - 4x + 5 = 0$  are  $\alpha$  and  $\beta$ .

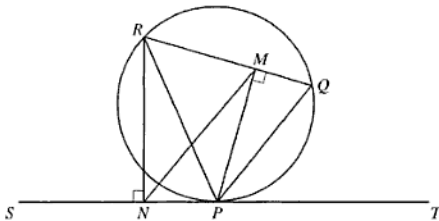
- (i) State the value of  $\alpha + \beta$  and of  $\alpha\beta$ . [2]

- (ii) Find the quadratic equation in  $x$  whose roots are  $\frac{1}{\alpha^2}$  and  $\frac{1}{\beta^2}$ . [6]

6 Without using a calculator

- (i) evaluate  $6^x$ , given that  $3^{x+1} \times 2^{2x+1} = 2^{x+2}$ , [4]
- (ii) find, in the form  $(a + b\sqrt{2})$  cm, the height of the right circular cylinder of volume  $\pi(7 + 4\sqrt{2})$  cm<sup>3</sup> whose radius is  $(1 + \sqrt{2})$  cm. [4]

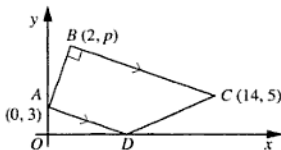
7



In the diagram above  $SPT$  is a tangent to a circle at the point  $P$ . The points  $Q$  and  $R$  lie on the circle. The line  $PM$  is perpendicular to the chord  $QR$  and the line  $RN$  is perpendicular to the tangent  $SPT$ .

- (i) By considering  $QP$  as a chord of the circle, find, with explanation, an angle equal to angle  $QPT$ . [2]
- (ii) Explain why a circle with  $PR$  as diameter passes through  $M$  and  $N$ . [2]
- (iii) Prove that the lines  $MN$  and  $QP$  are parallel. [5]

8

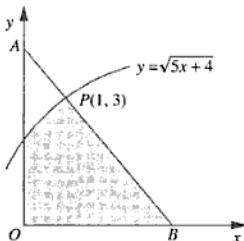


The diagram shows a trapezium  $ABCD$  in which  $A$  is  $(0, 3)$ ,  $C$  is  $(14, 5)$  and angle  $ABC$  is  $90^\circ$ . The point  $D$  lies on the  $x$ -axis and the point  $B$  has coordinates  $(2, p)$ , where  $p$  is a positive constant.

- (i) Express the gradient of  $AB$  and the gradient of  $BC$  in terms of  $p$ . [2]
- (ii) Hence find the value of  $p$ . [3]
- (iii) Find the coordinates of  $D$ . [3]
- (iv) Calculate the perimeter of  $ABCD$ , correct to 1 decimal place. [2]

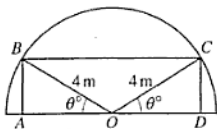
- 9 A particle moves in a straight line so that,  $t$  seconds after leaving a fixed point  $O$ , its velocity,  $v \text{ m s}^{-1}$ , is given by  $v = 16 + 6t - t^2$ . Find
- (i) the velocity of the particle when its acceleration is zero, [3]
  - (ii) the value of  $t$  when the particle is instantaneously at rest, [2]
  - (iii) the distance from  $O$  at which the particle is instantaneously at rest, [3]
  - (iv) the total distance travelled by the particle in the interval  $t = 0$  to  $t = 12$ . [3]

10



In the diagram above 1 unit represents 1 kilometre along each axis. The triangle  $OAB$  represents a park. The development of a new road will result in the park being reduced to the shaded region shown. One of the boundaries of this shaded region is represented by the curve  $y = \sqrt{5x + 4}$ . The side  $AB$  is normal to the curve at the point  $P(1, 3)$ .

- (i) Find the equation of the line  $AB$ . [5]
- (ii) Show that the length of  $OB$  is 3.5 units and find the length of  $OA$ . [2]
- (iii) Show that the development of the new road will reduce the park to approximately 85.5% of its original size. [6]



The diagram shows a rectangle  $ABCD$  inside a semicircle, centre  $O$  and radius  $4\text{ m}$ , such that angle  $BOA = \text{angle } COD = \theta^\circ$ . The perimeter of the rectangle is  $P\text{ m}$ .

- (i) Show that  $P = 16\cos\theta^\circ + 8\sin\theta^\circ$ . [3]
- (ii) Express  $P$  in the form  $R \cos(\theta^\circ - \alpha^\circ)$ . [6]
- (iii) Find the maximum value of  $P$  and the corresponding value of  $\theta$ . [2]
- (iv) Find the value of  $\theta$  for which  $P = 15$ . [2]

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# TOPIC 1 SETS

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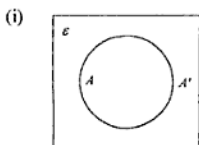
## FORMULAE AND IMPORTANT NOTES

1. A set is any well-defined list, collection, or class of objects. These objects are called elements or members of the set. A finite set has finite number of members. An infinite set has infinite number of members.
2. Sets will usually be denoted by capital letters  $A, B, C$ , etc. There are many ways to define sets. Here are some examples:  $A = \{x : 2 < x < 100\}$ ,  $B = \{(x, y) : 2x + 3y = 5\}$ ,  $C = \{x : x \text{ is a perfect square}\}$ ,  $D = \{a, b, c, d, e\}$ .
3. Quick reference

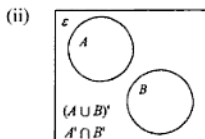
NOTATIONS	WHAT IT MEANS
$A \cup B$	Union of $A$ and $B$
$A \cap B$	Intersection of $A$ and $B$
$n(A)$	Number of elements in set $A$
$\in$	"... is an element of ..."
$\notin$	"... is not an element of ..."
$A'$	Complement of set $A$
$\emptyset$	The empty set or null set
$\mathcal{U}$	Universal set
$A \subseteq B$	$A$ is a subset of $B$
$A \subset B$	$A$ is a proper subset of $B$
$A \not\subseteq B$	$A$ is not a subset of $B$
$A \not\subset B$	$A$ is not a proper subset of $B$
$\{x_1, x_2, \dots\}$	The set with elements $x_1, x_2, \dots$
$\{x : \dots\}$	The set of all $x$ such that ...

4. (i) The null (empty) set is a set with no number. Example:  $\{a \text{ real number whose square is negative}\} = \emptyset$ . A set with "zero" as the only member is not a null set.  
(ii) A null set is a subset of all sets.  
(iii) If  $A \subseteq B$  then  $n(A) \leq n(B)$ . The converse needs not be true.  
(iv) If  $A \subset B$  then  $n(A) < n(B)$ . The converse needs not be true.  
(v) If  $A \subseteq B$  and  $B \subseteq A$  then  $A$  and  $B$  have the same members. We write  $A = B$ . The members of each of the two sets do not have to be arranged in the same order.

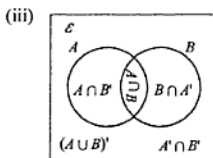
5. Study the following diagrams carefully. Pay particular attention to the notations that are used to define each region.



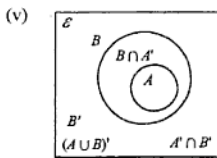
In diagram (i),  $A \cup A' = \mathcal{E}$



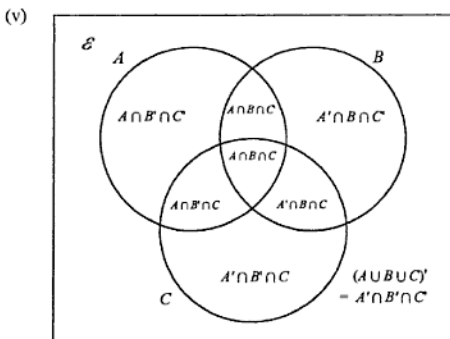
In diagram (ii),  
 $B \subset A'$   
 $A \subset B'$   
 $A \cap B = \emptyset$   
 $A \cup B' = B'$   
 $A' \cup B = A'$   
 $A \cap B' = A$   
 $A' \cap B = B$   
 $A$  and  $B$  are mutually exclusive.  
 They are disjoint sets.



In diagram (iii),  $A \cap B \neq \emptyset$

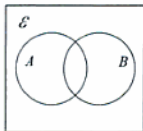


In diagram (iv),  
 $B \subset A'$   
 $B' \subset A'$   
 $A \cup B = B$   
 $A \subset B$   
 $A \cap B = A$   
 $B$  contains or includes  
 $A$ .  $A$  is a proper subset  
 of  $B$ .



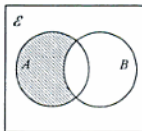
6. Study the name given to each shaded region of the following diagrams.

(i)



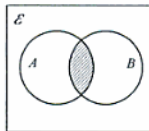
$$\emptyset = e'$$

(ii)



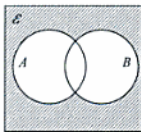
$$A \cap B' = (A' \cup B)'$$

(iii)



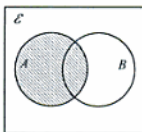
$$A \cap B = (A' \cup B')$$

(iv)



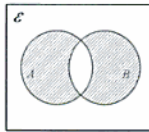
$$(A \cup B)' = A' \cap B'$$

(v)



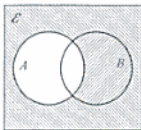
$$A$$

(vi)



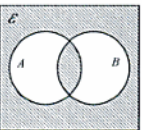
$$(A \cap B') \cup (B \cap A')$$

(vii)



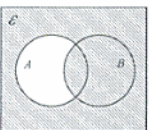
$$A'$$

(viii)



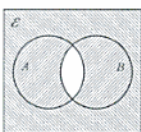
$$\begin{aligned} (A \cup B)' \cup (A \cap B) \\ = (A' \cap B') \cup (A \cap B) \\ = (A' \cup B) \cap (B' \cup A) \end{aligned}$$

(ix)



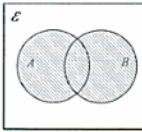
$$A' \cup B = (A \cap B)'$$

(x)



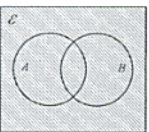
$$(A \cap B)' = A' \cup B'$$

(xi)



$$A \cup B$$

(xii)



$$e$$

7. (i) For any two sets:  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

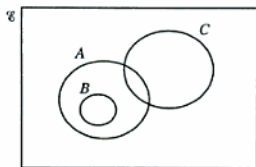
(ii) For any three sets:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

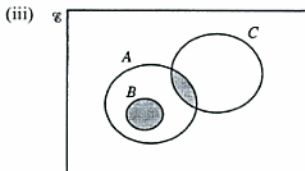
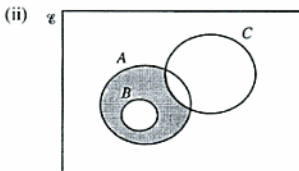
## PAST EXAMINATION QUESTIONS

1. The diagram shows a universal set  $\mathcal{E}$  and the three sets  $A$ ,  $B$  and  $C$ .

(i) Copy the diagram and shade the region representing  $(A \cup C) \cap B'$ .



For each of the diagrams below, express, in set notation, the set represented by the shaded area in terms of  $A$ ,  $B$  and  $C$ .

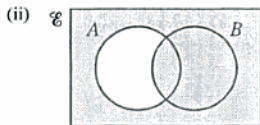
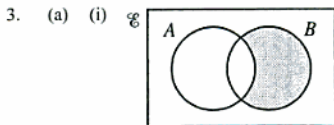


(N2003/P2/4)

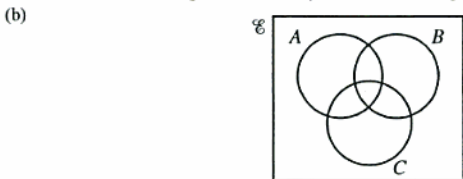
2. A youth club has facilities for members to play pool, darts and table-tennis. Every member plays at least one of the three games.  $P$ ,  $D$  and  $T$  represent the sets of members who play pool, darts and table-tennis respectively. Express each of the following in set language and illustrate each by means of a Venn diagram.

(i) The set of members who only play pool.

(ii) The set of members who play exactly 2 games, neither of which is darts. (N2004/P1/2)



For each of the Venn diagrams above, express the shaded region in set notation.



- (i) Copy the Venn diagram above and shade the region that represents  $A \cap B \cap C'$ .  
 (ii) Copy the Venn diagram above and shade the region that represents  $A' \cap (B \cup C)$ .

(N2005/P1/2)



4. Express each of the following statements in appropriate set notation.
- (i)  $x$  is not an element of set  $A$ .
  - (ii) The number of elements not in set  $B$  is 16.
  - (iii) Sets  $C$  and  $D$  have no common element.

(N2006/P1/1)

# TOPIC 2 IRRATIONAL ROOTS (SURDS)

## FORMULAE AND IMPORTANT NOTES

1. A surd is an irrational root of a number.

$$(i) \sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$(ii) \sqrt{a} \times \sqrt{a} = \sqrt{a^2} = (\sqrt{a})^2 = a$$

$$(iii) \sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{a \div b} = \sqrt{\frac{a}{b}}$$

$$(iv) \sqrt{a^2 b} = a\sqrt{b}$$

$$(v) a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$$

$$(vi) \frac{a\sqrt{b}}{c} = \frac{a}{c}\sqrt{b}$$

$$(vii) \frac{a}{\sqrt{a}} = \sqrt{a}$$

$$(viii) \frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$$

$$(ix) \frac{1}{a}\sqrt{b} = \frac{\sqrt{b}}{a} = \sqrt{\frac{b}{a^2}}$$

$$(x) \frac{a\sqrt{b}}{\sqrt{c}} = a\sqrt{\frac{b}{c}}$$

$$(xi) \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}$$

$$(xii) \sqrt{a^{2b}} = a^b$$

2. (i)  $\sqrt{a^{2b} \times c} = a^b \times \sqrt{c}$

(ii)  $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$  unless one or both of  $a$  and  $b$  are zero

(iii)  $\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$  unless  $b$  is zero

$$(iv) a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b} \qquad a\sqrt{b} + a\sqrt{c} = a(\sqrt{b} + \sqrt{c})$$

$$(v) a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b} \qquad a\sqrt{b} - a\sqrt{c} = a(\sqrt{b} - \sqrt{c})$$

$$(vi) (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

$$(vii) (a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

$$(viii) (\sqrt{a} + b)(\sqrt{a} - b) = a - b^2$$

$$(ix) (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b}) = a + b + 2\sqrt{ab}$$

$$(x) (a + \sqrt{b})^2 = (a + \sqrt{b})(a + \sqrt{b}) = a^2 + b + 2a\sqrt{b}$$

$$(xi) (\sqrt{a} + b)^2 = (\sqrt{a} + b)(\sqrt{a} + b) = a + b^2 + 2b\sqrt{a}$$

$$(xii) (\sqrt{a} - \sqrt{b})^2 = (\sqrt{a} - \sqrt{b})(\sqrt{a} - \sqrt{b}) = a + b - 2\sqrt{ab}$$

$$(xiii) (a - \sqrt{b})^2 = (a - \sqrt{b})(a - \sqrt{b}) = a^2 + b - 2a\sqrt{b}$$

$$(xiv) (\sqrt{a} - b)^2 = (\sqrt{a} - b)(\sqrt{a} - b) = a + b^2 - 2b\sqrt{a}$$

3. Rationalising the denominator:

$$(i) \frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

$$(ii) \frac{a}{b\sqrt{c}} = \frac{a}{b\sqrt{c}} \times \frac{\sqrt{c}}{\sqrt{c}} = \frac{a\sqrt{c}}{bc}$$

$$(iii) \frac{a}{\sqrt{b} + \sqrt{c}} = \frac{a}{\sqrt{b} + \sqrt{c}} \times \frac{\sqrt{b} - \sqrt{c}}{\sqrt{b} - \sqrt{c}} = \frac{a(\sqrt{b} - \sqrt{c})}{b - c}$$

$$(iv) \frac{a}{\sqrt{b} - \sqrt{c}} = \frac{a}{\sqrt{b} - \sqrt{c}} \times \frac{\sqrt{b} + \sqrt{c}}{\sqrt{b} + \sqrt{c}} = \frac{a(\sqrt{b} + \sqrt{c})}{b - c}$$

4. If  $a + \sqrt{b} = c + \sqrt{d}$ , where  $a$  and  $c$  are both rational and  $\sqrt{b}$  and  $\sqrt{d}$  are both irrational roots, then  $a = c$  and  $b = d$ .

## WORKED EXAMPLES

1. Simplify:  $\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\frac{48}{25}}$

$$\begin{aligned} \sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\frac{48}{25}} &= \sqrt{3 \times 9} - \sqrt{3 \times 4} + 2\sqrt{3 \times 25} - \sqrt{\frac{3 \times 16}{25}} \\ &= 3\sqrt{3} - 2\sqrt{3} + 10\sqrt{3} - \frac{4}{5}\sqrt{3} \\ &= (3 - 2 + 10 - \frac{4}{5})\sqrt{3} \\ &= 10\frac{1}{5}\sqrt{3} \end{aligned}$$

2. Express  $\frac{3\sqrt{2} - 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}}$  in the form  $a - b\sqrt{c}$ .

$$\begin{aligned} \frac{3\sqrt{2} - 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}} &= \frac{(3\sqrt{2} - 2\sqrt{3})^2}{(3\sqrt{2} + 2\sqrt{3})(3\sqrt{2} - 2\sqrt{3})} \\ &= \frac{(3\sqrt{2})^2 - 2(3\sqrt{2})(2\sqrt{3}) + (2\sqrt{3})^2}{(3\sqrt{2})^2 - (2\sqrt{3})^2} \\ &= \frac{18 - 12\sqrt{6} + 12}{18 - 12} \\ &= \frac{30 - 12\sqrt{6}}{6} \\ &= \frac{6(5 - 2\sqrt{6})}{6} \\ &= 5 - 2\sqrt{6} \end{aligned}$$

3. Without using tables or calculators evaluate  $\frac{1}{5-\sqrt{7}} - \frac{1}{5+\sqrt{7}}$ .

$$\begin{aligned} \frac{1}{5-\sqrt{7}} - \frac{1}{5+\sqrt{7}} &= \frac{5+\sqrt{7} - (5-\sqrt{7})}{(5-\sqrt{7})(5+\sqrt{7})} \\ &= \frac{5+\sqrt{7} - 5 + \sqrt{7}}{5 - (\sqrt{7})^2} \\ &= \frac{2\sqrt{7}}{25-7} \\ &= \frac{2\sqrt{7}}{18} \\ &= \frac{\sqrt{7}}{9} \end{aligned}$$

## PAST EXAMINATION QUESTIONS

1. The variable  $\theta$  and  $t$  are related by the equation  $\theta = \theta_0 e^{-kt}$ , where  $\theta_0$  and  $k$  are constants.  
When  $t = 30$ ,  $\theta = \frac{1}{2}\theta_0$ .
- Show that the value of  $k$ , correct to 4 decimal places, is 0.0231. When  $t = 40$ ,  $\theta = 28$ .
  - Calculate the value of  $\theta_0$ .
- When  $t = 50$ , calculate
- $\theta$ ,
  - $\frac{d\theta}{dt}$ .
- Find the average rate of change of  $\theta$  with respect to  $t$  over the interval  $0 \leq t \leq 50$ .  
(N2000/P2/3b)
2. Given that  $(a + \sqrt{5})(3 + b\sqrt{5}) = 26 + 11\sqrt{5}$ , find the possible values of  $a$  and of  $b$ .  
(N01/P2/8a)
3. Given that  $k = \frac{1}{\sqrt{3}}$  and that  $p = \frac{1+k}{1-k}$ , express in its simplest surd form
- $p$ ,
  - $p - \frac{1}{p}$ .
- (N2002/P2/3)
4. A rectangular block has a square base. The length of each side of the base is  $(\sqrt{3} - \sqrt{2})$  m and the volume of the block is  $(4\sqrt{2} - 3\sqrt{3})$  m<sup>3</sup>. Find, without using a calculator, the height of the block in the form  $(a\sqrt{2} + b\sqrt{3})$  m, where  $a$  and  $b$  are integers.  
(N2003/P1/4)
5. Given that  $\sqrt{a + b\sqrt{3}} = \frac{13}{4 + \sqrt{5}}$ , where  $a$  and  $b$  are integers, find, without using a calculator, the value of  $a$  and of  $b$ .  
(N2004/P2/2)
6. A cuboid has a square base of side  $(2 - \sqrt{3})$  m and a volume of  $(2\sqrt{3} - 3)$  m<sup>3</sup>. Find the height of the cuboid in the form  $(a + b\sqrt{3})$  m, where  $a$  and  $b$  are integers.  
(N2005/P1/4)
7. Express  $(2 - \sqrt{5})^2 - \frac{8}{3 - \sqrt{5}}$  in the form  $p + q\sqrt{5}$ , where  $p$  and  $q$  are integers.  
(N2006/P2/9a)
8. Without using a calculator find, in the form  $(a + b\sqrt{2})$  cm, the height of the right circular cylinder of volume  $\pi(7 + 4\sqrt{2})$  cm<sup>3</sup> whose radius is  $(1 + \sqrt{2})$  cm.  
(SP08/P2/6ii)

# TOPIC 3 INDICES

## FORMULAE AND IMPORTANT NOTES

1. The three laws of indices:

$$x^a \cdot x^b = x^{a+b}, x^a \div x^b = \frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$$

$$(x^a)^b = x^{ab}$$

2.  $x^a \cdot y^a = (xy)^a$ ,  $x^a \div y^a = \frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a$

3. Let  $a$  and  $b$  be positive integers

(i)  $x^a = x \cdot x \cdot x \dots$  to  $a$  factor

(ii)  $x^0 = 1$

(iii)  $x^{-1} = \frac{1}{x}$ ,  $\left(\frac{1}{x}\right)^{-1} = x$ ,  $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$

$$x^{-a} = \frac{1}{x^a}, \left(\frac{1}{x}\right)^{-a} = x^a, \left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a = \frac{y^a}{x^a}$$

$$\frac{x^a}{y^b} = \frac{x^a y^{-b}}{x^a y^b} = x^a y^{-b} = \frac{1}{x^{-a} y^b}$$

(iv)  $x^{\frac{1}{2}} = \sqrt{x}$ ,  $x^{\frac{2}{3}} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$

$$x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}}$$

4. If  $y^b = x^a$  then  $y = x^{\frac{a}{b}}$

5.  $10^{-a} = \frac{1}{10^a}$  = the digit 1 that is  $a$  places after the decimal point

$$10^0 = 1$$

$10^a$  = the digit 1 followed by  $a$  zero(s)

## WORKED EXAMPLES

1. Find the value of  $\left(\frac{27}{64}\right)^{-\frac{2}{3}} \times \left(\frac{9}{64}\right)^{\frac{1}{2}} \div \left(\frac{3}{8}\right)^{-1}$ .

$$\begin{aligned} \left(\frac{27}{64}\right)^{-\frac{2}{3}} \times \left(\frac{9}{64}\right)^{\frac{1}{2}} \div \left(\frac{3}{8}\right)^{-1} &= \left[\left(\frac{3}{4}\right)^3\right]^{-\frac{2}{3}} \times \sqrt{\frac{9}{64}} \div \frac{8}{3} \\ &= \left(\frac{3}{4}\right)^{-2} \times \frac{3}{8} \times \frac{3}{8} \\ &= \frac{16}{9} \times \frac{9}{64} \\ &= \frac{1}{4} \end{aligned}$$

2. Solve the equation  $3^{2x} \cdot 4^{x+2} = 576$ .

$$3^{2x} \cdot 4^{x+2} = 576$$

$$(3^2)^x \cdot 4^x = \frac{576}{16}$$

$$9^x \cdot 4^x = 36$$

$$(9 \times 4)^x = 36$$

$$36^x = 36$$

$$x = 1$$

3. Simplify  $\frac{3 \cdot 2^n - 4 \cdot 2^{n-2}}{2^n - 2^{n-1}}$ .

$$\frac{3 \cdot 2^n - 4 \cdot 2^{n-2}}{2^n - 2^{n-1}} = \frac{3 \cdot 2^n - 2^2 \cdot 2^{n-2}}{2^n - 2^{n-1}}$$

$$= \frac{2^n(3-2)}{2^{n-1}(2-1)}$$

$$= 2^{n-(n-1)} \times 2$$

$$= 2 \times 2$$

$$= 4$$

## PAST EXAMINATION QUESTIONS

- Solve the equation  $2^{x+2} = 2^x + 10$ . (N2000/P2/3a)
- (a) Find, in its simplest form, the product of  $a^{\frac{1}{3}} + b^{\frac{2}{3}}$  and  $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ .  
(b) Given that  $2^{2x+2} \times 5^{x-1} = 8^x \times 5^{2x}$ , evaluate  $10^x$ . (N2002/P2/9)
- Without using a calculator, solve the equation  $\frac{2^{x-3}}{8^{-x}} = \frac{32}{4^{\frac{1}{2}x}}$ . (N2003/P2/2)
- Find the value of each of the integers  $p$  and  $q$  for which  $\left(\frac{25}{16}\right)^{-\frac{3}{2}} = 2^p \times 5^q$ . (N2005/P2/8a)
- Given that  $\frac{a^x}{b^{3-x}} \times \frac{b^x}{(a^{x+1})^2} = ab^6$ , find the value of  $x$  and of  $y$ . (N2006/P2/9b)
- Without using a calculator, evaluate  $6^x$ , given that  $3^{x+1} \times 2^{2x+1} = 2^{x+2}$ . (SP08/P2/6i)

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# TOPIC 4 LOGARITHMS

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## FORMULAE AND IMPORTANT NOTES

1. If  $y > 0$ ,  $a > 0$  and  $a \neq 1$ ,  $x$  is a real number, and  $y = a^x$  then  $x$  is the logarithm of  $y$  to the base  $a$ . We write  $x = \log_a y$ .
2.  $y = a^{\log_a y}$ .
3. If  $y = 10^x$  then  $x = \log_{10} y = \lg y$ , the common logarithm of  $y$ .
4. If  $y = e^x$  where  $e = 2.718281828\dots$  then  $x = \ln y$ , the natural logarithm of  $y$ .
5. ...  
 $\lg 0.01 = -2$   
 $\lg 0.1 = -1$   
 $\lg 1 = 0$   
 $\lg 10 = 1$   
 $\lg 100 = 2$   
....
6.  $\log_a a = \lg 10 = \ln e = 1$   
 $\log_a 1 = 0$   
 $\log_a a^b = b$
7.  $\log_a y + \log_a z \neq \log_a (y + z)$  and  $\log_a yz \neq \log_a y \times \log_a z$   
 $\log_a y + \log_a z = \log_a yz$
8.  $\log_a y - \log_a z \neq \log_a (y - z)$  and  $\log_a \frac{y}{z} \neq \frac{\log_a y}{\log_a z}$   
 $\log_a y - \log_a z = \log_a \frac{y}{z}$
9.  $\log_a y^b \neq (\log_a y)^b$  but  $\log_a y^b = b \times \log_a y$
10.  $\log_a y \times \log_y z = \log_a z$
11.  $\log_a y \times \log_y a = \log_a a = 1$
12.  $\log_a y = \frac{\log_a z}{\log_a z}$
13.  $\log_y z = \frac{\log_a z}{\log_a y} = \frac{\lg z}{\lg y} = \frac{\ln z}{\ln y}$
14. If  $y = a^x$  then  $\lg y = \lg a^x$   
 $\lg y = x \cdot \lg a$   
 $x = \frac{\lg y}{\lg a} = \frac{\ln y}{\ln a}$

15.

Base of logarithm of $y$	G.C.E. notation	Keys in calculator
$e$	$\ln y$	$\ln y$
10	$\lg y$	$\log y$
$a$	$\log_a y$	$\frac{\lg y}{\lg a}$ or $\frac{\ln y}{\ln a}$

**PAST EXAMINATION QUESTIONS**

- Solve the equation  $\lg(20 + 5x) - \lg(10 - x) = 1$ . (N97/P2/4b)
- Solve the equation  $2^{x+1} + 2^x = 9$ . (N98/2/5a)
- Solve the equation  $\lg(x - 8) + \lg\left(\frac{x}{2}\right) = 1 + \lg\left(\frac{x}{4}\right)$ . (N99/P2/3a)
- Solve the equation  $2^{x+2} = 2^x + 10$ . (N2000/P2/3a)
- (a) Given that  $41\lg(x\sqrt{y}) = 1.5 + \lg x - \lg y$ , where  $x$  and  $y$  are both positive, express, in its simplest form,  $y$  in terms of  $x$ .  
(b) Given that  $p = \log_a q$ , express, in terms of  $p$ , (i)  $\log_a\left(\frac{1}{q}\right)$ , (ii)  $\log_2 8q$ . (N2000/P2/5c, d)
- Solve the equations (i)  $\lg(2x + 5) = 1 + \lg x$ , (ii)  $\log_4 y + \log_2 y = 12$ . (N01/P2/7a)
- (a) At the beginning of 1960, the number of animals of a certain species was estimated at 20 000. This number decreased so that, after a period of  $n$  years, the population was  $20\,000e^{-0.055n}$ .  
Estimate  
(i) the population at the beginning of 1970,  
(ii) the year in which the population would be expected to have first decreased to 2000.  
(b) Solve the equation  $3^{x+1} - 2 = 8 \times 3^{x-1}$ . (N2004/P1/12EITHER)
- (a) Solve  $\log_7(17y + 15) = 2 + \log_7(2y - 3)$ .  
(b) Evaluate  $\log_9 8 \times \log_{16} p$ . (N2005/P1/7)
- Variables  $V$  and  $t$  are related by the equation  $V = 100e^{-kt}$ , where  $k$  is a constant. Given that  $V = 500$  when  $t = 21$ , find  
(i) the value of  $k$ ,  
(ii) the value of  $V$  when  $t = 30$ . (N2005/P2/1)
- (a) Solve the equation  $\lg(x + 12) = 1 + \lg(2 - x)$ .  
(b) Given that  $\log_2 p = a$ ,  $\log_3 q = b$  and  $\frac{p}{q} = 2^c$ , express  $c$  in terms of  $a$  and  $b$ . (N2006/P1/8)
- Solve the equation  $\log_3(2x - 1) - \log_3(x^2 - 2) = \frac{1}{\log_2 3}$ . (SP08/P1/7)



12. An engineering firm buys a new piece of equipment at a cost of \$72 000. The value of this equipment decreases with time so that its value, \$ $V$ , after  $t$  months use is given by  $V = 72\,000 e^{-kt}$ , where  $k$  is a positive constant. The value of the equipment is expected to be \$40 000 after 30 months use.

(i) Calculate the value, to the nearest \$100, of the equipment after 20 months use.

The equipment is replaced when its value reaches  $\frac{1}{3}$  of its original value.

(ii) Calculate the length of time, to the nearest month, the equipment has been in use when it is replaced. (SP08/P2/1)

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# TOPIC 5 SIMULTANEOUS LINEAR EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. To solve a pair of simultaneous linear equations with two variables is to find the co-ordinates of their point of intersection, if any.
2. There is no solution if the lines are parallel and an infinite number of solutions if the lines coincide.
3. There are generally four nongraphical methods of solutions:

(i) Equating two expressions

Solve:  $y = 3x - 4$  and  $2y = 7 + 4x$

Solution:  $2(3x - 4) = 7 + 4x$  etc.

(ii) Substitution

Solve:  $3y + 8x = 13$  and  $2x = 9 - 11y$

Solution:  $3y + 4(2x) = 13$

$3y + 4(9 - 11y) = 13$  etc.

(iii) Addition or subtraction

Solve:  $5x - 3y = 2$  and  $8x + 5y = 9$

Solution:  $5x - 3y = 2$  ... (i)

$8x + 5y = 9$  ... (ii)

(i)  $\times 5$ :  $25x - 15y = 10$  ... (iii)

(ii)  $\times 3$ :  $24x + 15y = 27$  ... (iv)

(iii) + (iv):  $49x = 37$  etc

(iv) Two-by-two determinant

Solve:  $ax + by = e$  and  $cx + dy = f$

$$\text{Solution: } x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - fb}{ad - cb}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - cb}$$

Note: If  $ad - cb = 0$  then the graphs of the pair of equations are either parallel or identical.

## WORKED EXAMPLES

1. Solve the equations  $5x - 4y = 8\frac{1}{6}$ ,  $2x + 3y = 14$ .

$$x = \frac{\begin{vmatrix} 8\frac{1}{6} & -4 \\ 14 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{8\frac{1}{6}(3) - 14(-4)}{5(3) - 2(-4)} = 3\frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 5 & 8\frac{1}{6} \\ 2 & 14 \end{vmatrix}}{23} = \frac{5(14) - 2(8\frac{1}{6})}{23} = 2\frac{1}{3}$$

2. Solve the equations  $2^{x+1} - 5^y = 131$ ,  $2^{x-4} + 5^{y-2} = 13$ .

$$2^{x+1} - 5^y = 131 \dots (i)$$

$$2^{x-4} + 5^{y-2} = 13 \dots (ii)$$

$$\text{From (i), } 2(2^x) - 5^y = 131 \dots (iii)$$

$$\text{From (ii), } \frac{1}{16}(2^x) + \frac{1}{25}(5^y) = 13 \dots (iv)$$

$$2^x = \frac{\begin{vmatrix} 131 & -1 \\ 13 & \frac{1}{25} \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ \frac{1}{16} & \frac{1}{25} \end{vmatrix}} = \frac{\frac{131}{25} - 13(-1)}{\frac{1}{25} - \frac{1}{16}(-1)} = \frac{18\frac{6}{25}}{\frac{37}{400}} = 128 = 2^7$$

$$5^y = \frac{\begin{vmatrix} 2 & 131 \\ \frac{1}{16} & 13 \end{vmatrix}}{\frac{52}{400}} = \frac{2(13) - \frac{1}{16}(131)}{\frac{52}{400}} = \frac{17\frac{11}{16}}{\frac{52}{400}} = 125 = 5^3$$

$$x = 7 \text{ and } y = 3$$

3. Given that  $\log_a(xy^3) = 10$  and  $\log_a(x^3y^2) = 16$ , find the value of  $\log_a\sqrt{xy}$ .

$$\log_a(xy^3) = 10 \rightarrow \log_a x + 3 \log_a y = 10$$

$$\log_a(x^3y^2) = 16 \rightarrow 3 \log_a x + 2 \log_a y = 16$$

$$\log_a x = \frac{\begin{vmatrix} 10 & 3 \\ 16 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}} = \frac{20 - 48}{2 - 9} = \frac{-28}{-7} = 4$$

$$\log_a y = \frac{\begin{vmatrix} 1 & 10 \\ 3 & 16 \end{vmatrix}}{-7} = \frac{16 - 30}{-7} = 2$$

$$\begin{aligned} \log_a \sqrt{xy} &= \frac{1}{2} \log_a xy \\ &= \frac{1}{2} (\log_a x + \log_a y) \\ &= \frac{1}{2} (4 + 2) \\ &= 3 \end{aligned}$$

## PAST EXAMINATION QUESTIONS

1. A particle starts from rest at a fixed point  $O$  and moves in a straight line towards a point  $A$ . The velocity,  $v \text{ ms}^{-1}$ , of the particle,  $t$  seconds after leaving  $O$ , is given by  $v = 6 - 6e^{-3t}$ . Given that the particle reaches  $A$  when  $t = \ln 2$ , find
- (i) the acceleration of the particle at  $A$ ,
  - (ii) the distance  $OA$ .

(N2005/P1/6)

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# TOPIC 6 SOLUTION OF QUADRATIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. If  $(ax - b)(cx - d) = 0$  then  $x = \frac{b}{a}$  or  $\frac{d}{c}$ .
2. If  $ax^2 + bx + c = 0$  then  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## WORKED EXAMPLES

1. Find the value of  $q$  such that  $2^{2q+3} - 2^{q+2} = 5 \cdot 2^q - 1$ .

$$2^{2q+3} - 2^{q+2} = 5 \cdot 2^q - 1$$

$$2^3 \cdot 2^{2q} - 2^2 \cdot 2^q - 5 \cdot 2^q + 1 = 0$$

$$8(2^q)^2 - 9(2^q) + 1 = 0$$

$$(8 \cdot 2^q - 1)(2^q - 1) = 0$$

$$\text{If } 8 \cdot 2^q - 1 = 0 \qquad \text{If } 2^q - 1 = 0$$

$$2^q = \frac{1}{8} = 2^{-3} \qquad 2^q = 1 = 2^0$$

$$q = -3 \qquad q = 0$$

$$\text{Ans: } q = -3 \text{ or } 0$$

2. Solve for  $x$  if  $9(\lg x)^2 = 17 \lg x + 2$ .

$$9(\lg x)^2 = 17 \lg x + 2$$

$$9(\lg x)^2 - 17 \lg x - 2 = 0$$

$$(9 \lg x + 1)(\lg x - 2) = 0$$

$$\text{If } 9 \lg x + 1 = 0 \qquad \text{If } \lg x - 2 = 0$$

$$\lg x = -\frac{1}{9} \qquad \lg x = 2$$

$$x = 0.774 \qquad x = 100$$

$$\therefore x = 0.774 \text{ or } 100$$

3. Solve the equation  $\sqrt{x-5} + \sqrt{x+4} = \frac{45}{\sqrt{x+4}}$ .

$$\sqrt{x-5} + \sqrt{x+4} = \frac{45}{\sqrt{x+4}}$$

$$\sqrt{(x-5)(x+4)} + (x+4) = 45$$

$$\sqrt{x^2 - x - 20} = 41 - x$$

$$x^2 - x - 20 = 1681 - 82x + x^2$$

$$81x = 1701$$

$$x = 21$$

## PAST EXAMINATION QUESTIONS

- Given that  $x = \lg a$  is a solution of the equation  $10^{2x+1} - 7(10^x) = 26$ , find the value of  $a$ .  
(N97/P2/4c)
- Solve the equation  $9^{2x} + 2(9^{x+1}) = 40$ .  
(N99/P2/3b)
- Sketch the graph of  $y = \ln x$ .
  - Determine the equation of the straight line which would need to be drawn on the graph of  $y = \ln x$  in order to obtain a graphical solution of the equation  $x^2 e^{x-2} = 1$ .  
(N2002/P2/8)
- Solve the equation  $\log_2 x - \log_4(x-4) = 2$ .  
(N2003/P2/3)
- Solve the equation  $\log_{16}(3x-1) = \log_4(3x) + \log_4(0.5)$ .  
(N2004/P2/5)
- Express the equation  $4^x - 2^{x+1} = 3$  as a quadratic equation in  $2^x$ .
  - Hence find the value of  $x$ , correct to 2 decimal places.  
(N2005/P2/8b)

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# TOPIC 7 SIMULTANEOUS LINEAR AND NON-LINEAR EQUATIONS

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## WORKED EXAMPLES

1. Solve the equations  $x^2 + 2xy + 4y^2 = 28$  and  $x + 2y = 6$ .

$$x + 2y = 6 \rightarrow 2y = 6 - x \dots(i)$$

$$x^2 + 2xy + 4y^2 = 28$$

$$\rightarrow x^2 + x(2y) + (2y)^2 = 28 \dots(ii)$$

$$\text{From (i) and (ii), } x^2 + x(6 - x) + (6 - x)^2 = 28$$

$$x^2 + 6x - x^2 + 36 - 12x + x^2 - 28 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$\text{If } x = 2, \text{ from (i) } y = \frac{1}{2}(6 - x) = \frac{1}{2}(6 - 2) = 2$$

$$\text{If } x = 4, \quad ; y = \frac{1}{2}(6 - 4) = 1$$

$$\text{Ans: } x = 2, y = 2 ; x = 4, y = 1$$

2. Solve the equations  $\frac{1}{x^2} + \frac{1}{y^2} = 13$  and  $\frac{1}{x} + \frac{1}{y} = 5$ .

$$\frac{1}{x^2} + \frac{1}{y^2} = 13 \dots(i)$$

$$\frac{1}{x} + \frac{1}{y} = 5 \dots(ii)$$

$$\text{From (ii) } \left(\frac{1}{x} + \frac{1}{y}\right)^2 = 5^2$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{2}{xy} = 25 \dots(iii)$$

$$(iii) - (i) \quad \frac{2}{xy} = 12$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)^2 = \frac{1}{x^2} + \frac{1}{y^2} - \frac{2}{xy}$$

$$= 13 - 12$$

$$= 1$$

$$\therefore \frac{1}{x} - \frac{1}{y} = 1 \dots(iv)$$

$$\text{or } \frac{1}{x} - \frac{1}{y} = -1 \dots(v)$$

$$(ii) + (iv) \quad \frac{2}{x} = 6$$

$$\frac{1}{x} = 3 \dots(vi)$$

$$x = \frac{1}{3}$$

$$\text{From (ii) and (vi) : } 3 + \frac{1}{y} = 5$$

$$\frac{1}{y} = 2$$

$$y = \frac{1}{2}$$

$$\text{(ii) + (v) } \frac{2}{x} = 4$$

$$\frac{1}{x} = 2 \dots \text{(vii)}$$

$$x = \frac{1}{2}$$

$$\text{From (ii) and (vii) } 2 + \frac{1}{y} = 5$$

$$\frac{1}{y} = 3$$

$$y = \frac{1}{3}$$

$$\therefore x = \frac{1}{3}, y = \frac{1}{2}; x = \frac{1}{2}, y = \frac{1}{3}$$

3. Solve the equations  $7x - 4y = 23$  and  $49x^2 - 16y^2 = 1081$ .

$$49x^2 - 16y^2 = (7x - 4y)(7x + 4y) = 1081$$

$$23(7x + 4y) = 1081$$

$$7x + 4y = 47 \dots \text{(i)}$$

$$7x - 4y = 23 \dots \text{(ii)}$$

$$\text{(i) + (ii) } 14x = 70$$

$$x = 5 \dots \text{(iii)}$$

$$\text{From (i) and (iii) } 7(5) + 4y = 47$$

$$4y = 12$$

$$y = 3$$

$$\text{Ans: } x = 5, y = 3.$$

## PAST EXAMINATION QUESTIONS

- Find the coordinates of the points of intersection of the line  $x + 2y = 10$  and the curve  $2y^2 - 7y + x = 0$ . (N97/P1/1)
- Solve the simultaneous equations  $3^x = 27(3^y)$ ,  $\lg(x + 2y) = \lg 5 + \lg 3$ . (N98/P2/5b)
- Solve the simultaneous equations  $3x - 2y = 1$ ,  $9x^2 + y = 7$ . (N2000/P1/1)
- Find the coordinates of the points of intersection of the line  $y = 2x + 3$  and the curve  $2x^2 + y^2 - 4x = 39$ . (N01/P1/1)
- The line  $4y = x + 11$  intersects the curve  $y^2 = 2x + 7$  at the point  $A$  and  $B$ . Find the coordinates of the mid-point of the line  $AB$ . (N2003/P2/1)
- Without using a calculator, solve, for  $x$  and  $y$ , the simultaneous equations
$$8^x \div 2^y = 64,$$
$$3^{4x} \times \left(\frac{1}{9}\right)^{y-1} = 81.$$
 (N2004/P1/3)



# TOPIC 8 LINEAR GRAPHS

## FORMULAE AND IMPORTANT NOTES

1. Let  $A(x_1, y_1)$  and  $B(x_2, y_2)$  be the end-points of a line segment.

(i) Distance from  $A$  to  $B = AB$   
$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

(ii) The midpoint of  $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- (iii) The angle of inclination ( $\theta$ ) of a straight line is the angle which the line makes with the positive direction of the axis of  $x$ .

The constant  $m$  ( $\tan \theta$ ) is called the gradient (slope) of the line, and  $m$  will be positive or negative according as to whether  $\theta$  is acute or obtuse.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient of a horizontal line is zero.

The gradient of a vertical line is infinite.

All parallel lines have the same gradient.

The gradients of two perpendicular lines are negative reciprocal of each other.

- (iv) The equation of the line through  $A$  and  $B$ :  $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ .

- (v) The equation of the perpendicular bisector of  $AB$ :

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

$$\text{or, } \frac{y - y_1}{x - x_1} \times \frac{y_2 - y_1}{x_2 - x_1} = -1$$

2. Every pair of values that satisfies the equation of a graph is the co-ordinates of a point on the graph and conversely.

Every straight line (linear graph) is represented by an equation of the first degree and conversely.

- (i) Special form:  $x = a$  This is a perpendicular line which crosses the  $x$ -axis at  $(a, 0)$ .

$y = b$  The equation of the  $y$ -axis is  $x = 0$ .

$y = b$  This is a horizontal line which crosses the  $y$ -axis at  $(0, b)$ .

$y = mx$  The equation of the  $x$ -axis is  $y = 0$ .

$y = mx$  This is a line through the origin with gradient  $m$ .

$x + y = a$  This is a line which intersects both axes at points  $a$  unit from the origin.

- (ii) Point-slope form:  $y - y_1 = m(x - x_1)$  where  $(x_1, y_1)$  is a given point and  $m$  is the given gradient of the line.

- (iii) Two-point form:  $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$  where  $(x_1, y_1)$  and  $(x_2, y_2)$  are two given points.

- (iv) Slope-intercept form:  $y = mx + b$  where the gradient is  $m$  and the line crosses the  $y$ -axis at  $(0, b)$ .

(v) Double intercept form:  $\frac{x}{a} + \frac{y}{b} = 1$  where the graph crosses the  $x$ -axis at  $(a, 0)$  and the  $y$ -axis at  $(0, b)$ .  $a$  and  $b$  are the  $x$ -intercept and  $y$ -intercept respectively.

(vi) General form:  $Ax + By + C = 0$  where  $A, B,$  and  $C$  are constants and  $A$  and  $B$  cannot be both zero.

3. The set of straight lines  $y = mx + b$ , where only  $m$  is a constant, is a set of parallel lines which have different  $y$ -intercepts for different values of  $b$ .  
The set of straight lines  $y = mx + b$ , where only  $b$  is a constant, is a set of lines through  $(0, b)$  with different gradients for different values of  $m$ .
4. The lines  $Ax + By = C$  and  $Ax + By = D$  are parallel. If  $C = D$ , they coincide. Therefore, the equation of the line that passes through  $(x_1, y_1)$  and is parallel to  $Ax + By = C$  is  $Ax + By = Ax_1 + By_1$ .
5. The lines  $Ax + By = C$  and  $Bx - Ay = D$  (or  $-Bx + Ay = D$ ) are perpendicular to each other. Therefore, the equation of the line that passes through  $(x_1, y_1)$  and is perpendicular to  $Ax + By = C$  is  $Bx - Ay = Bx_1 - Ay_1$  (or  $-Bx + Ay = -Bx_1 + Ay_1$ ).
6. The general form  $Ax + By + C = 0$  can be transformed into:
- (i) Slope-intercept form:  $y = -\frac{A}{B}x - \frac{C}{B}$ .
- (ii) Double intercept form:  $\frac{x}{-C/A} + \frac{y}{-C/B} = 1$ .
7. If the equations of the four sides or the co-ordinates of the four vertices of a quadrilateral are known, then the quadrilateral is a parallelogram if any one of the following can be proved:
- (i) All the opposite sides are parallel, i.e. they have equal gradients.
- (ii) All the opposite sides are equal.
- (iii) One pair of opposite sides are parallel and equal.
- (iv) All the opposite angles are equal.
- (v) The diagonals bisect each other.

If the four vertices of the quadrilateral are  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  and  $(x_4, y_4)$  successively this amounts to proving that  $(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}) = (\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2})$ .

8. The two intersecting diagonals divide a parallelogram into four triangles with equal areas.
9. A rectangle is an equiangular parallelogram. All the four vertices are right-angled. A rhombus is an equilateral parallelogram. All the sides are equal. A square is both a rectangle and a rhombus.

10. The area of a triangle  $ABC$ , where  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$  and  $C = (x_3, y_3)$

$$\text{is } \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3| \text{ units}^2.$$

11. The area of a polygon with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ ... $(x_{n-1}, y_{n-1})$  and  $(x_n, y_n)$  is

$$\frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n & x_1 \\ y_1 & y_2 & y_3 & \dots & y_{n-1} & y_n & y_1 \end{vmatrix} \text{ units}^2.$$

## PAST EXAMINATION QUESTIONS

1. A line drawn through the point  $A(4, 6)$ , parallel to the line  $2y = x - 2$ , meets the  $y$ -axis at the point  $B$ .

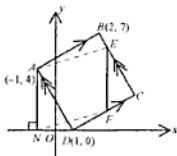
(i) Calculate the coordinates of  $B$ .

A line drawn through  $A$ , perpendicular to  $AB$ , meets the line  $2y = x - 2$  at the point  $C$ .

(ii) Calculate the coordinates of  $C$ .

(N97/P1/2)

2. Solutions to this question by accurate drawing will not be accepted. The points  $A(-1, 4)$ ,  $B(2, 7)$ ,  $C$  and  $D(1, 0)$  are the four vertices of a parallelogram. The point  $E$  lies on  $BC$  such that  $BE = \frac{1}{3}BC$ . Lines are drawn, parallel to the  $y$ -axis, from  $A$  to meet the  $x$ -axis at  $N$  and from  $E$  to meet  $CD$  at  $F$ .



(i) Calculate the coordinates of  $C$  and of  $E$ .

(ii) Find the equation of  $DC$  and calculate the coordinates of  $F$ .

(iii) Explain why  $AEFN$  is a parallelogram and calculate its area.

(N97/P1/15)

3. The line  $2y + x = 5$  intersects the curve  $y^2 + xy = 6$  at the points  $A$  and  $B$ . Find

(i) the coordinates of  $A$  and  $B$ ,

(ii) the distance  $AB$ .

(N98/P1/1)

4. Solutions to this question by accurate drawing will not be accepted.

The diagram shows a parallelogram  $ABCD$  in which  $A$  is  $(8, 2)$  and  $B$  is  $(2, 6)$ . The equation of  $BC$  is  $2y = x + 10$  and  $X$  is the point on  $BC$  such that  $AX$  is perpendicular to  $BC$ . Find

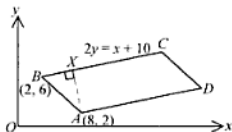
(i) the equation of  $AX$ ,

(ii) the coordinates of  $X$ .

Given also that  $BC = 5BX$ , find

(iii) the coordinates of  $C$  and of  $D$ ,

(iv) the area of the parallelogram  $ABCD$ .



(N98/P1/11)

5. Find the equation of the perpendicular bisector of the line joining the point  $(-5, 4)$  to the point  $(9, -3)$ .

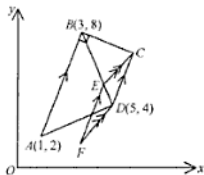
(N99/P1/1)

6. The point  $P(x, y)$  lies on the line  $7y = x + 23$  and is 5 units from the point  $(2, 0)$ . Calculate the coordinates of the two possible positions of  $P$ .

(N99/P1/9)

7. **Solutions to this question by accurate drawing will not be accepted.**

The diagram shows the trapezium  $ABCD$  in which  $A$  is the point  $(1, 2)$ ,  $B$  is  $(3, 8)$ ,  $D$  is  $(5, 4)$ , angle  $ABC = 90^\circ$  and  $AB$  is parallel to  $DC$ .



(i) Find the coordinates of  $C$ .

The point  $E$  lies on  $BD$  and is such that the area of  $\triangle CDE$  is  $\frac{1}{4}$  of the area of  $\triangle CDB$ .

(ii) Find the coordinates of  $E$ .

The point  $F$  is such that  $CDFE$  is a parallelogram.

(iii) Find the coordinates of  $F$  and the area of the parallelogram  $CDFE$ . (N99/P1/15)

8. The points  $A$  and  $B$  have coordinates  $(2, 2)$  and  $(10, 8)$  respectively. Find the equation of the perpendicular bisector of  $AB$ . (N2000/P1/2)

9. Points  $A$ ,  $B$  and  $C$  have coordinates  $(1, 2)$ ,  $(-2, 6)$  and  $(9, 8)$  respectively.

(i) Show that triangle  $ABC$  is right-angled.

(ii) Calculate the area of triangle  $ABC$ . (N01/P1/2)

10. The line joining  $A(5, 11)$  and  $B(0, 1)$  meets the  $x$ -axis at  $C$ .

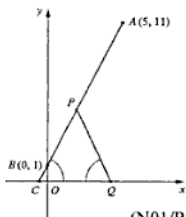
The point  $P$  lies on  $AC$  such that  $AP : PB = 3 : 2$ .

(i) Find the coordinates of  $P$ .

A line through  $P$  meets the  $x$ -axis at  $Q$  and angle  $PCQ = \text{angle } PQC$ . Find

(ii) the equation of  $PQ$ ,

(ii) the coordinates of  $Q$ .



(N01/P1/3)

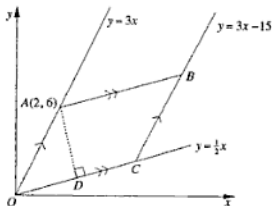
11. The line  $2y = 3x - 6$  intersects the curve  $xy = 12$  at the points  $P$  and  $Q$ . Find the equation of the perpendicular bisector of  $PQ$ . (N2002/P1/9)

12. **Solutions to this question by accurate drawing will not be accepted.**

The diagram, which is not drawn to scale, shows a parallelogram  $OABC$  where  $O$  is the origin and  $A$  is the point  $(2, 6)$ . The equations of  $OA$ ,  $OC$  and  $CB$  are  $y = 3x$ ,  $y = \frac{1}{2}x$  and  $y = 3x - 15$  respectively. The perpendicular from  $A$  to  $OC$  meets  $OC$  at the point  $D$ . Find

(i) the coordinates of  $C$ ,  $B$  and  $D$ ,

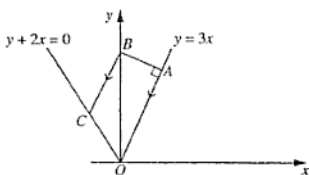
(ii) the perimeter of the parallelogram  $OABC$ , correct to 1 decimal place. (N2003/P1/11)



13. The line  $4y = 3x + 1$  intersects the curve  $xy = 28x - 27y$  at the point  $P(1, 1)$  and at the point  $Q$ . The perpendicular bisector of  $PQ$  intersects the line  $y = 4x$  at the point  $R$ . Calculate the area of triangle  $PQR$ .  
(N2004/P1/11)

14. The diagram shows a trapezium  $OABC$ , where  $O$  is the origin. The equation of  $OA$  is  $y = 3x$  and the equation of  $OC$  is  $y + 2x = 0$ . The line through  $A$  perpendicular to  $OA$  meets the  $y$ -axis at  $B$  and  $BC$  is parallel to  $AO$ . Given that the length of  $OA$  is  $\sqrt{250}$  units, calculate the coordinates of  $A$ , of  $B$  and of  $C$ .

(N2004/P2/11)



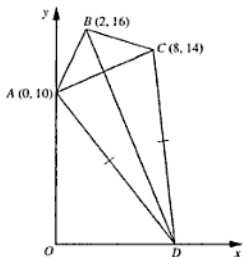
15. Solutions to this question by accurate drawing will not be accepted.

The diagram, which is not drawn to scale, shows a quadrilateral  $ABCD$  in which  $A$  is  $(0, 10)$ ,  $B$  is  $(2, 16)$  and  $C$  is  $(8, 14)$ .

- (i) Show that triangle  $ABC$  is isosceles.

The point  $D$  lies on the  $x$ -axis and is such that  $AD = CD$ . Find

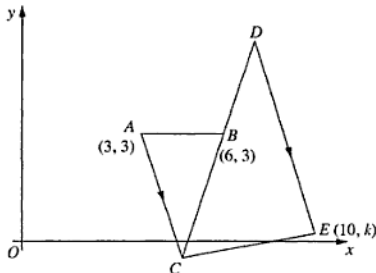
- (ii) the coordinates of  $D$ ,  
(iii) the ratio of the area of triangle  $ABC$  to the area of triangle  $ACD$ .  
(N2005/P1/10)



16. The line  $x + y = 10$  meets the curve  $y^2 = 2x + 4$  at the points  $A$  and  $B$ . Find the coordinates of the mid-point of  $AB$ .  
(N2005/P2/2)

17. The straight line  $2x + y = 14$  intersects the curve  $2x^2 - y^2 = 2xy - 6$  at the points  $A$  and  $B$ . Show that the length of  $AB$  is  $24\sqrt{5}$  units.  
(N2006/P2/5)

18. Solutions to this question by accurate drawing will not be accepted.



The diagram shows an isosceles triangle  $ABC$  in which  $A$  is the point  $(3, 3)$ ,  $B$  is the point  $(6, 3)$  and  $C$  lies below the  $x$ -axis. Given that the area of triangle  $ABC$  is 6 square units,

(i) find the coordinates of  $C$ .

The line  $CB$  is extended to the point  $D$  so that  $B$  is the mid-point of  $CD$ .

(ii) Find the coordinates of  $D$ .

A line is drawn from  $D$ , parallel to  $AC$ , to the point  $E(10, k)$  and  $C$  is joined to  $E$ .

(iii) Find the value of  $k$ .

(iv) Prove that angle  $CED$  is **not** a right angle.

(N2006/P2/12OR)

19. The diagram shows a trapezium  $ABCD$  in which  $A$  is  $(0, 3)$ ,  $C$  is  $(14, 5)$  and angle  $ABC$  is  $90^\circ$ . The point  $D$  lies on the  $x$ -axis and the point  $B$  has coordinates  $(2, p)$ , where  $p$  is a positive constant.

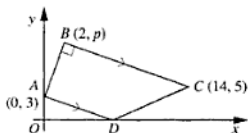
(i) Express the gradient of  $AB$  and the gradient of  $BC$  in terms of  $p$ .

(ii) Hence find the value of  $p$ .

(iii) Find the coordinates of  $D$ .

(iv) Calculate the perimeter of  $ABCD$ , correct to 1 decimal place.

(SP08/P2/8)



# TOPIC 9 GRAPHS OF LINEAR ABSOLUTE VALUE FUNCTIONS

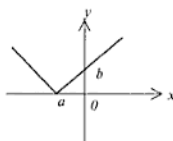
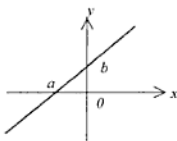
## FORMULAE AND IMPORTANT NOTES

1. To draw the graph of  $y = |mx + b|$ : Draw the graph of  $y = mx + b$ . Draw the reflection image of the portion below the  $x$ -axis. Erase the portion below the  $x$ -axis.

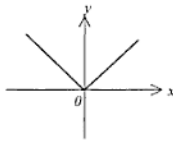
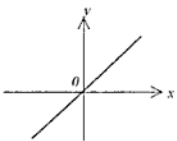
Graph of  $y = mx + b$

Corresponding graph of  $y = |mx + b|$

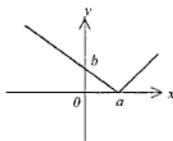
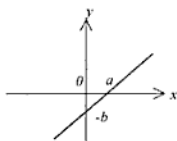
(i)



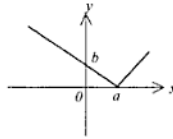
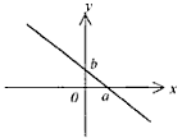
(ii)



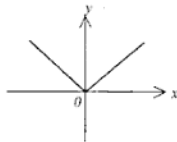
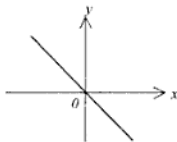
(iii)



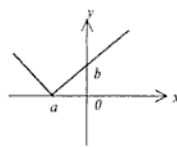
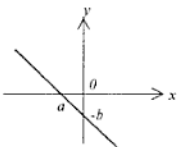
(iv)



(v)



(vi)



2. (i) If  $|f(x)| \geq a$  then either  $f(x) \leq -a$  or  $f(x) \geq a$  where  $a$  is a non-negative constant.  
(ii) If  $|f(x)| > a$  then either  $f(x) < -a$  or  $f(x) > a$ .  
(iii) If  $|f(x)| = a$  then  $f(x) = \pm a$ .  
(iv) If  $|f(x)| < a$  then  $-a < f(x) < a$ .  
(v) If  $|f(x)| \leq a$  then  $-a \leq f(x) \leq a$ .

## PAST EXAMINATION QUESTIONS

1. Sketch the graphs of  $3y = 4x + 2$  and  $3y = |4x - 8|$  on the same diagram. Solve the simultaneous equations  $3y = 4x + 2$ ,  $3y = |4x - 8|$ .  
(N97/P1/17b)
2. (i) Sketch the graph of  $y = \ln x$ .  
(ii) Determine the equation of the straight line which would need to be drawn on the graph of  $y = \ln x$  in order to obtain a graphical solution of the equation  $x^2 e^{x-2} = 1$ .  
(N2002/P2/8)
3. A function  $f$  is defined by  $f: x \mapsto |2x - 3| - 4$ , for  $-2 \leq x \leq 3$ .  
(i) Sketch the graph of  $y = f(x)$ .  
(ii) State the range of  $f$ .  
(iii) Solve the equation  $f(x) = -2$ .  
A function  $g$  is defined by  $g: x \mapsto |2x - 3| - 4$ , for  $-2 \leq x \leq 3$ .  
(iv) State the largest value of  $k$  for which  $g$  has an inverse.  
(v) Given that  $g$  has an inverse, express  $g$  in the form  $g: x \mapsto ax + b$ , where  $a$  and  $b$  are constants.  
(N2005/P1/11)



# TOPIC 10 REDUCTION TO LINEAR EQUATIONS

## FORMULAE AND IMPORTANT NOTES

- Every equation of the first degree in  $x$  and  $y$  is a straight line and vice versa.
- An equation in  $x$  and  $y$  that is not of the first degree can often be reduced to the form of  $Y = mX + c$  where  $X$  and  $Y$  are in terms of  $x$  and  $y$  and  $m$  and  $c$  are constants.
- A graph of  $Y$  against  $X$  is plotted (i.e. the  $Y$ -axis is vertical and the  $X$ -axis is horizontal) by drawing the best-fit line. It should be a straight line.
- $m$  is the gradient of the straight line. Its value can be estimated from  $m = \frac{Y_2 - Y_1}{X_2 - X_1}$ .  $c$  is the vertical intercept.
- From the values of  $m$  and  $c$  the values of all the constant in the original equation of  $x$  and  $y$  can often be estimated.

## PAST EXAMINATION QUESTIONS

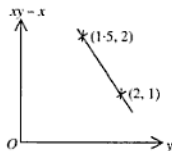
1. (a) The table shows experimental values of two variables  $x$  and  $y$ .

$x$	1.0	1.2	1.4	1.6	1.8
$y$	79.0	56.0	36.5	18.4	0.9

Using the vertical axis for  $xy$  and the horizontal axis for  $x^3$ , plot  $xy$  against  $x^3$  and obtain a straight line graph. Make use of your graph to

- express  $y$  in terms of  $x$ ,
- estimate the value of  $x$  when  $y = \frac{60}{x}$ .

- (b) The equation  $y = \frac{x+c}{x+d}$ , where  $c$  and  $d$  are constant, can be represented by a straight line when  $xy - x$  is plotted against  $y$  as shown in the diagram. Find the value of  $c$  and of  $d$ . (N97/P2/3)



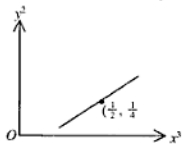
2. (a) The table shows experimental values of two variables,  $x$  and  $y$ .

$y$	0.5	2.2	4.0	5.9	7.8
$x$	-7.5	-4.4	3.1	22.4	68.2

It is known that  $x$  and  $y$  are related by the equation  $y + 10 = Ak^x$ , where  $A$  and  $k$  are constants. Using graph paper, plot  $\lg(y + 10)$  against  $x$  for the above data and use your graph to estimate

- the value of  $A$  and of  $k$ ,
- the value of  $x$  when  $y = 0$ .

- (b) Variables  $x$  and  $y$  are related by the equation  $px^3 + qy^2 = 1$ . The diagram shows the straight-line graph of  $y^2$  against  $x^3$  which passes through the point  $(\frac{1}{2}, \frac{1}{4})$ .
- Given that the gradient of this line is  $\frac{3}{4}$ , calculate the value of  $p$  and of  $q$ .
  - Given also that this line passes through  $(k, 4)$ , find the value of  $k$ . (N98/P2/3)

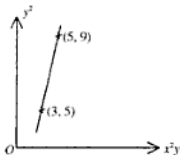


3. (a) The table shows experimental values of  $x$  and  $y$ .

$x$	1.5	2.0	2.5	3.0	3.5
$y$	35	79	176	394	890

It is known that  $x$  and  $y$  are related by an equation of the form  $y = Ak^x$ , where  $A$  and  $k$  are constants.

- (i) Express this equation in a form suitable for drawing a straight line graph.  
 (ii) Draw this graph for the given data and use it to estimate  $A$  and  $k$ .  
 (iii) Estimate the value of  $x$  for which  $y = 600$ .
- (b) The variables  $x$  and  $y$  are related in such a way that, when  $y^2$  is plotted against  $x^2y$ , a straight line is obtained which passes through the points  $(3, 5)$  and  $(5, 9)$ . Find the two values of  $y$  for which  $x = \frac{1}{2}\sqrt{5}$ .  
 (N99/P2/4)

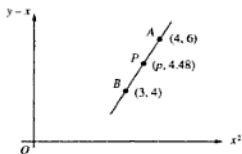


4. (a) The table shows experimental values of two variables,  $x$  and  $y$ .

$x$	1	2	3	4	5
$y$	7.0	6.4	7.7	9.3	11.0

Using the vertical axis for  $xy$  and the horizontal axis for  $x^2$ , plot  $xy$  against  $x^2$  and obtain a straight line graph. Use your graph to

- (i) express  $y$  in terms of  $x$ ,  
 (ii) estimate the value of  $x$  when  $y = \frac{30}{x}$ .
- (b) Variables  $x$  and  $y$  are related in such a way that, when  $y - x$  is plotted against  $x^2$ , a straight line is produced which passes through the points  $A(4, 6)$ ,  $B(3, 4)$  and  $P(p, 4.48)$ , as shown in the diagram. Find  
 (i)  $y$  in terms of  $x$ ,  
 (ii) the value of  $p$ ,  
 (iii) the value of  $x$  and of  $y$  at the point  $P$ .  
 (N2000/P2/4)

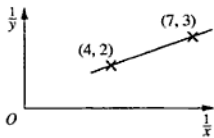


5. (a) The table shows experimental values of two variables  $x$  and  $y$ .

$x$	0.5	1.0	2.0	4.0	10.0
$y$	2.2	3.6	5.1	6.5	7.8

It is known that  $x$  and  $y$  are related by an equation of the form  $y = \frac{ax}{x+b}$ . Using the vertical axis for  $y$  and the horizontal axis for  $\frac{y}{x}$ , draw a straight line graph of  $y$  against  $\frac{y}{x}$  for the given data. Use the graph to estimate

- (i) the value of  $a$  and of  $b$ ,  
 (ii) the value of  $x$  for which  $y = 3x$ .
- (b) The diagram shows part of a straight line graph drawn to represent the equation  $cx + dy = xy$ . Find  
 (i) the value of  $c$  and of  $d$ ,  
 (ii) the value of  $y$  when  $x = 0.2$ .  
 (N01/P2/2)



6. A rectangle of area  $y \text{ m}^2$  has sides of length  $x \text{ m}$  and  $(Ax + B) \text{ m}$ , where  $A$  and  $B$  are constants and  $x$  and  $y$  are variables. Values of  $x$  and  $y$  are given in the table below.

$x$	50	100	150	200	250
$y$	3700	11 000	21 600	36 000	53 500

- (i) Use the data above in order to draw, on graph paper, the straight line graph of  $\frac{y}{x}$  against  $x$ .
- (ii) Use your graph to estimate the value of  $A$  and of  $B$ .
- (iii) On the same diagram, draw the straight line representing the equation  $y = x^2$  and explain the significance of the value of  $x$  given by the point of intersection of the two lines.
- (iv) State the value approached by the ratio of the two sides of the rectangle as  $x$  becomes increasingly large. (N2002/P1/12OR)
7. In order that each of the equations
- (i)  $y = ab^x$ ,
- (ii)  $y = Ax^k$ ,
- (iii)  $px + qy = xy$ ,
- where  $a, b, A, k, p$  and  $q$  are unknown constants, may be represented by a straight line, they each need to be expressed in the form  $Y = mX + c$ , where  $X$  and  $Y$  are each functions of  $x$  and/or  $y$ , and  $m$  and  $c$  are constants. Copy the following table and insert in it an expression for  $Y, X, m$  and  $c$  for each case.

	$Y$	$X$	$m$	$c$
$y = ab^x$				
$y = Ax^k$				
$px + qy = xy$				

(N2004/P2/9)

8. Variables  $x$  and  $y$  are related by the equation  $y^{ax} = a$ , where  $a$  and  $n$  are constants. The table below shows measured values of  $x$  and  $y$ .

$x$	1.5	2	2.5	3	3.5
$y$	7.3	3.5	2.0	1.3	0.9

- (i) On graph paper plot  $\lg y$  against  $\lg x$ , using a scale of 2 cm to represent 0.1 on the  $\lg x$  axis and 1 cm to represent 0.1 on the  $\lg y$  axis. Draw a straight line graph to represent the equation  $yx^n = a$ .
- (ii) Use your graph to estimate the value of  $a$  and of  $n$ .
- (iii) On the same diagram, draw the line representing the equation  $y = x^2$  and hence find the value of  $x$  for which  $x^{n+2} = a$ . (N2005/P1/12EITHER)

9. The variables  $x$  and  $y$  related by the equation  $y = 10^{-x}b^x$ , where  $A$  and  $b$  are constants. The table below shows values of  $x$  and  $y$ .

$x$	15	20	25	30	35	40
$y$	0.15	0.38	0.95	2.32	5.90	14.80

- (i) Draw a straight line graph of  $\lg y$  against  $x$ , using a scale of 2 cm to represent 5 units on the  $x$ -axis and 2 cm to represent 0.5 units on the  $\lg y$ -axis.
- (ii) Use your graph to estimate the value of  $A$  and of  $b$ .
- (iii) Estimate the value of  $x$  when  $y = 10$ .
- (iv) On the same diagram, draw the line representing  $y^x = 10^{-x}$  and hence find the value of  $x$  for which  $10^{A-\frac{x}{b}} = b^x$ . (N2006/P1/12OR)
10. A particle moving in a certain medium, with speed  $v \text{ ms}^{-1}$ , experiences a resistance to motion of  $R$  newtons. It is believed that  $R$  and  $v$  are related by the equation  $R = av^2 + bv$ , where  $a$  and  $b$  are constants.

$v$	5	10	15	20	25
$R$	17	45	80	138	185

The table shows experimental values of the variables  $v$  and  $R$ , but an error has been made in recording one of the values of  $R$ .

- (i) Express the given equation in a form suitable for drawing a straight line graph and, using graph paper, draw the graph for the values given.

Use the graph to

- (ii) correct the reading of  $R$  for which an error has been made,
- (iii) estimate the value of  $a$  and of  $b$ .

In a different medium,  $R$  is directly proportional to  $v$  and  $R = 60$  when  $v = 10$ .

- (iv) Draw a line on your graph to illustrate this second situation and use it to find the value of  $v$  for which the resistance is the same in both cases. (SP08/P1/12)

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# TOPIC 11 QUADRATIC GRAPHS

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## FORMULAE AND IMPORTANT NOTES

1. The general form of a quadratic function is  $y = ax^2 + bx + c$ . One or both  $b$  and  $c$  may be zero.
2. Draw a horizontal line segment. This is the  $x$ -axis. Label it.
3. Put  $y = 0$  and solve for  $x$ . If there is no real roots go to No. 8. Mark and label the roots (or root) on the  $x$ -axis. They are the  $x$ -intercepts.
4. Draw a line segment perpendicular to the  $x$ -axis at the origin to form a cross. This is the  $y$ -axis. Label it.
5. Put  $x = 0$  and solve for  $y$ . Mark the value on the  $y$ -axis. It is the  $y$ -intercept.
6. If  $a$  is positive the graph is in the shape of an upright bowl. If  $a$  is negative the graph is in the shape of an inverted bowl. These curves are called parabolas. A parabolic curve is symmetrical about the vertical line through the vertex of the curve. The vertex is called the turning point. It is a minimum point if  $a > 0$  and a maximum point if  $a < 0$ .
7. Join the points of No. 3 and No. 5 with a smooth parabola. The axis of symmetry must pass through the  $x$ -intercept if there is only one, or halfway between them if there are two. The axis of symmetry needs not be drawn unless asked but the graph must be symmetrical about it. Write the quadratic function above the graph.
8. By 'completing the square' any quadratic function can be expressed as  $a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ . The point  $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$  is a maximum point if  $a < 0$  and a minimum point if  $a > 0$ . The  $y$ -intercept is  $c$ . If  $b^2 - 4ac < 0$  the quadratic equation  $ax^2 + bx + c = 0$  has no real root. When this happens the minimum point and  $y$ -intercept are above the  $x$ -axis and the maximum point and  $y$ -intercept are below the  $x$ -axis.
9. If  $b^2 - 4ac$  is zero there is only one  $x$ -intercept  $-\frac{b}{2a}$ . If it is positive there are two  $x$ -intercepts.
10. To sketch the graph of  $|ax^2 + bx + c|$ : sketch the graph of  $ax^2 + bx + c$ . Draw the reflection image of the portion below the  $x$ -axis, if any. Erase the portion of the graph below the  $x$ -axis.

## PAST EXAMINATION QUESTIONS

1. Find the minimum value of  $(x - 2)^2 - 2$  and the corresponding value of  $x$ . Sketch the graph of  $y = |(x - 2)^2 - 2|$  for  $0 \leq x \leq 4$ . (N99/P1/17b)
2. Draw the graph of  $y = 3 + |x^2 - 5x + 4|$ , for  $0 \leq x \leq 5$ , using a scale of 2 cm to 1 unit along each axis. Use your graph to find the set of values of  $x$  for which  $y \leq 5$ . (N01/P1/13b)
3. The function  $f$  is defined by  $f: x \mapsto |x^2 - 8x + 7|$  for the domain  $3 \leq x \leq 8$ .
  - (i) By first considering the stationary value of the function  $x \mapsto x^2 - 8x + 7$ , show that the graph of  $y = f(x)$  has a stationary point at  $x = 4$  and determine the nature of this stationary point.
  - (ii) Sketch the graph of  $y = f(x)$ .
  - (iii) Find the range of  $f$ .

The function  $g$  is defined by  $g: x \mapsto |x^2 - 8x + 7|$  for the domain  $3 \leq x \leq k$ .

(iv) Determine the largest value of  $k$  for which  $g^{-1}$  exists. (N2004/P2/10)

4. The function  $f$  is defined for the domain  $-3 \leq x \leq 3$  by  $f(x) = 9\left(x - \frac{1}{3}\right)^2 - 11$ .

(i) Find the range of  $f$ .

(ii) State the coordinates and nature of the turning point of

(a) the curve  $y = f(x)$ ,

(b) the curve  $y = |f(x)|$ .

(N2006/P1/7)

# TOPIC 12 QUADRATIC INEQUALITIES

## FORMULAE AND IMPORTANT NOTES

1.  $y = ax^2 + bx + c$

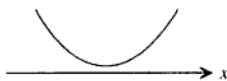
$$= a \left[ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 \right] + c - a \left(\frac{b}{2a}\right)^2$$
$$= a \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

(i) If  $a$  is positive the minimum value of  $y$  is  $\frac{4ac - b^2}{4a}$ . The corresponding value of  $x$  is  $-\frac{b}{2a}$ .

(ii) If  $a$  is negative the maximum value of  $y$  is  $\frac{4ac - b^2}{4a}$ . The corresponding value of  $x$  is  $-\frac{b}{2a}$ .

2.  $y = a(x - \alpha)(x - \beta)$  where  $a > 0$

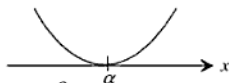
(i)



$y > 0$  for all real values of  $x$ .

$\alpha$  and  $\beta$  are complex numbers.

(ii)



$\alpha = \beta$

$y > 0$  if  $x \neq \alpha$

$y = 0$  if  $x = \alpha$

(iii)



$y \geq 0$  if  $x \leq \alpha$  or  $x \geq \beta$

$y > 0$  if  $x < \alpha$  or  $x > \beta$

$y = 0$  if  $x = \alpha$  or  $x = \beta$

$y < 0$  if  $\alpha < x < \beta$

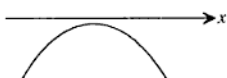
$y \leq 0$  if  $\alpha \leq x \leq \beta$

The converse is also true.

$y$  is a minimum when  $x = \frac{\alpha + \beta}{2}$ .

3.  $y = a(x - \alpha)(x - \beta)$  where  $a < 0$ .

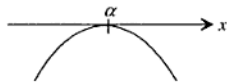
(i)



$y < 0$  for all real values of  $x$

$\alpha$  and  $\beta$  are complex numbers.

(ii)

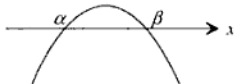


$\alpha = \beta$

$y < 0$  if  $x \neq \alpha$

$y = 0$  if  $x = \alpha$

(iii)



$$y \leq 0 \text{ if } x \leq \alpha \text{ or } \beta \leq x$$

$$y < 0 \text{ if } x < \alpha \text{ or } \beta < x$$

$$y = 0 \text{ if } x = \alpha \text{ or } x = \beta$$

$$y > 0 \text{ if } \alpha < x < \beta$$

$$y \geq 0 \text{ if } \alpha \leq x \leq \beta$$

The converse is also true.

$$y \text{ is a maximum when } x = \frac{\alpha + \beta}{2}.$$

4. (i) If  $[f(x)]^2 \geq a$  then either  $f(x) \leq -\sqrt{a}$  or  $f(x) \geq \sqrt{a}$ , where  $a$  is a non-negative constant.  
 (ii) If  $[f(x)]^2 > a$  then either  $f(x) < -\sqrt{a}$  or  $f(x) > \sqrt{a}$ .  
 (iii) If  $[f(x)]^2 = a$  then  $f(x) = \pm \sqrt{a}$ .  
 (iv) If  $[f(x)]^2 < a$  then  $-\sqrt{a} < f(x) < \sqrt{a}$ .  
 (v) If  $[f(x)]^2 \leq a$  then  $-\sqrt{a} \leq f(x) \leq \sqrt{a}$ .

## PAST EXAMINATION QUESTIONS

- Find the range of values of  $x$  for which  $3x^2 - 5x + 4 > 3 - x^2$ . (N97/P1/4)
- Find the range of values of  $x$  for which  $x(10 - x) \geq 24$ . (N98/P1/3a)
- Find the range of values of  $x$  for which  $3(x + 1)^2 < 16x$ . (N99/P1/2)
- Find the range of values of  $x$  for which  $(2x + 1)(4 - x) > 4$ . (N2000/P1/3b)
- Given that  $f(x) \equiv 2x^2 - 5x - 7$ ,
  - find the value of  $a$ , of  $b$  and of  $c$  for which  $f(x) \equiv a(x - b)^2 - c$ ,
  - state the minimum value of  $f(x)$ ,
  - sketch the graph of  $y = |f(x)|$  for  $-2 \leq x \leq 4.5$ , indicating on your graph the coordinates of the stationary point and of the points where the graph meets the coordinate axes,
  - calculate the values of  $x$  for which  $|f(x)| = 7$ , giving your answers to 2 decimal places where appropriate. (N2000/P1/17)
- Find the range of values of  $x$  for which  $x(2x + 5) > 12$ . (N01/P1/7a)
- Find the  $x$ -coordinate of the point on the line  $y = 5 - 2x$  where  $xy$  is a maximum. (N01/P1/15a)
- Find the set of values of  $x$  for which  $(x - 6)^2 > x$ .  $y = (2x - 1)(x - 2)$  (N2005/P1/1)
- Find the value of  $c$  and of  $d$  for which  $\{x : -5 < x < 3\}$  is the solution set of  $x^2 + cx < d$ . (N2006/P2/7b)



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# TOPIC 13 NATURE OF ROOTS OF QUADRATIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

For any quadratic equation  $y = ax^2 + bx + c = 0$ ,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  •  $b^2 - 4ac$  is the discriminant of the equation.

- If  $b^2 - 4ac < 0$  the equation has no real roots. It has complex roots.  
If  $b^2 - 4ac \geq 0$  the equation has real roots.
  - If  $b^2 - 4ac = 0$  the equation has equal (repeated, identical) roots which are equal to  $-\frac{b}{2a}$ .
  - If  $b^2 - 4ac > 0$  the roots are unequal (different, distinct).
    - If  $a$ ,  $b$  and  $c$  are rational and  $b^2 - 4ac$  is non-zero and a perfect square, then the roots are rational.
    - If  $b^2 - 4ac$  is not a perfect square but positive then the roots are irrational.
- The  $x$ -coordinates of the points of intersection of the straight line  $y = mx + d$  and the quadratic curve  $y = ax^2 + bx + c$  can be obtained from:  
$$ax^2 + bx + c = mx + d$$

Hence  $ax^2 + (b - m)x + (c - d) = 0$   
Discriminant  $= (b - m)^2 - 4a(c - d)$   
If the discriminant is negative, the straight line and the curve have no common point.  
If the discriminant is zero, the straight line touches the curve at one and only one point. It is a tangent to the curve.  
If the discriminant is positive, the straight line intersect the curve at two distinct points.
- If  $y = ax^2 + bx + c$  is either positive or negative for all values of  $x$  then the equation  $y = 0$  has no real roots and  $b^2 - 4ac < 0$ .

## PAST EXAMINATION QUESTIONS

- Find the value of the constant  $c$  for which the line  $y = 2x + c$  is a tangent to the curve  $y = 4x^2 - 6x + 11$ . This tangent meets the  $x$ -axis at  $A$  and the  $y$ -axis at  $B$ . Calculate the area of the triangle  $OAB$ , where  $O$  is the origin. (N97/P1/3)
- Find the value of  $k$  for which  $2y + x = k$  is a tangent to the curve  $y^2 + 4x = 20$ . (N98/P1/3b)
- The straight line  $y = 2p + 1$  intersects the curve  $y = x + \frac{p^2}{x}$  at two distinct points. Find the range of values of  $p$ . (N99/P1/5)
- Find the value of  $k$  for which  $8y = x + 2k$  is a tangent to the curve  $2y^2 = x + k$ . (N2000/P1/3a)
- Find the value of the constant  $k$  for which the equation  $4x^2 + k^2 + 1 = 4(k - 3)x$  has equal roots. (N2001/P1/7b)

6. Find the values of  $m$  for which the line  $y = mx - 9$  is a tangent to the curve  $x^2 = 4y$ .  
(N2002/P1/2)
7. Find the values of  $k$  for which the line  $x + 3y = k$  and the curve  $y^2 = 2x + 3$  do not intersect.  
(N2003/P1/1)
8. Find the values of  $k$  for which the line  $y = x + 2$  meets the curve  $y^2 + (x + k)^2 = 2$ .  
(N2004/P2/4)
9. Find the value of  $m$  for which the line  $y = mx - 3$  is a tangent to the curve  $y = x + \frac{1}{x}$  and find the  $x$ -coordinate of the point at which this tangent touches the curve.  
(N2006/P2/7a)
10. The equation of a straight line is  $y = 5 + kx$ , where  $k$  is a constant. Find the values of  $k$  for which this straight line is a tangent to the curve  $y^2 = 4y + x + 1$ .  
(SP08/P1/4)

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# TOPIC 14    ROOTS AND COEFFICIENTS OF QUADRATIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. A function of  $\alpha$  and  $\beta$  is symmetric if it remains unchanged when  $\alpha$  and  $\beta$  are interchanged.

Examples:  $\alpha + \beta$ ,  $\alpha\beta$ ,  $\alpha^2 + \beta^2$ ,  $\sqrt[3]{\alpha\beta}$  and  $\frac{\alpha}{\beta}$  are symmetric functions.

$\alpha - \beta$ ,  $\frac{\alpha}{\beta}$ ,  $\alpha^2 - \beta^2$  and  $\alpha\beta$  are not symmetric functions unless  $\alpha = \beta$ .

2. (i) If  $\alpha$  and  $\beta$  are the roots of a quadratic equation  $ax^2 + bx + c = 0$ , then  $\alpha + \beta = -\frac{b}{a}$  and  $\alpha\beta = \frac{c}{a}$ . The equation can also be written in the form  $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ .  
(ii) Any symmetric function of  $\alpha$  and  $\beta$  can be expressed in terms of  $a$ ,  $b$  and  $c$ .

Examples:  $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{b}{a} \div \frac{c}{a} = -\frac{b}{c}$

$$\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = \frac{c}{a}(-\frac{b}{a}) = -\frac{bc}{a^2}$$

- (iii) Let  $p$  and  $q$  be the roots of another quadratic equation  $x^2 - (p + q)x + pq = 0$ . If  $p$  and  $q$  are both symmetric equations of  $\alpha$  and  $\beta$ , then the coefficients of the second quadratic equation can be expressed in terms of  $a$ ,  $b$  and  $c$ .

Examples: Find the quadratic equation of  $x$  whose roots are  $\frac{1}{\alpha} + \frac{1}{\beta}$  and  $\alpha^2\beta + \beta^2\alpha$ .

Express its coefficient in terms of  $a$ ,  $b$  and  $c$ .

Working: sum of the roots of the second equation

$$= (\frac{1}{\alpha} + \frac{1}{\beta}) + (\alpha^2\beta + \beta^2\alpha)$$

$$= -\frac{b}{c} + (-\frac{bc}{a^2})$$

$$= -\frac{a^2b + bc^2}{a^2c}$$

product of the roots of the second equation

$$= (\frac{1}{\alpha} + \frac{1}{\beta})(\alpha^2\beta + \beta^2\alpha)$$

$$= (-\frac{b}{c})(-\frac{bc}{a^2})$$

$$= \frac{b^2}{a^2}$$

$\therefore$  The second equation:

$$x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$$

$$x^2 - (-\frac{a^2b + bc^2}{a^2c})x + \frac{b^2}{a^2} = 0$$

$$a^2 - c x^2 + b^2 (a^2 + c^2) x + b^2 c = 0$$

## PAST EXAMINATION QUESTIONS

1. The roots of the quadratic equation  $2x^2 - 4x + 5 = 0$  are  $\alpha$  and  $\beta$ .

(i) State the value of  $\alpha + \beta$  and of  $\alpha\beta$ .

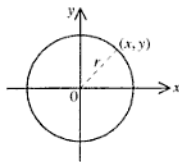
(ii) Find the quadratic equation in  $x$  whose roots are  $\frac{1}{\alpha^2}$  and  $\frac{1}{\beta^2}$ .

(SP08/P2/5)

# TOPIC 15 EQUATIONS OF CIRCLES

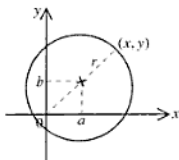
## FORMULAE AND IMPORTANT NOTES

1. Equation of circle with centre at the origin and value  $r$  units:

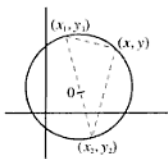


$$x^2 + y^2 = r^2$$

2. Equation of circle with centre at  $(a, b)$  and radius  $r$  units:  
 $(x - a)^2 + (y - b)^2 = r^2$



3. Equation of circle which passes through the end-points of a diameter:



$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

4. The equation of a circle in a quadratic equation in two variables  $x$  and  $y$  such that (i) the coefficients of  $x$  and  $y$  are equal and (ii) there is no  $xy$  term.
5. Find the centre and radius of a circle whose equation is:  $x^2 + 2ax + y^2 + 2by = c$

Working:  $x^2 + 2ax + a^2 + y^2 + 2by + b^2 = c^2 + a^2 + b^2$

$$(x + a)^2 + (y + b)^2 = a^2 + b^2 + c^2$$

$$[x - (-a)]^2 + [y - (-b)]^2 = \sqrt{a^2 + b^2 + c^2}$$

The centre is  $(-a, -b)$  and the radius is  $\sqrt{a^2 + b^2 + c^2}$ .

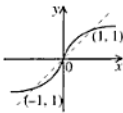
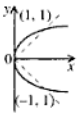
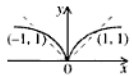
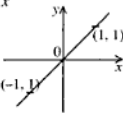
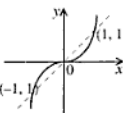
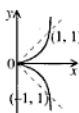

## PAST EXAMINATION QUESTIONS

1. The equation of a circle,  $C$ , is  $x^2 + y^2 - 6x - 8y + 16 = 0$ .
- (i) Find the coordinates of the centre of  $C$  and find the radius of  $C$ .
- (ii) Show that  $C$  touches the  $y$ -axis.
- (iii) Find the equation of the circle which is a reflection of  $C$  in the  $y$ -axis. (SP08/P1/9)

# TOPIC 16 GRAPHS OF $y = x^m$ WHERE $m$ IS RATIONAL

## FORMULAE AND IMPORTANT NOTES

We shall first consider cases where  $m$  is positive. Let  $m = n/d$  where  $n$  and  $d$  are positive integers which are either mutually prime (have no common factor except 1) or  $d$  is a factor of  $n$  ( $n/d$  is a positive integer).

	$n$ and $d$ are both odd $x$ and $y$ have the same sign because $n$ and $d$ are odd	$n$ is odd, $d$ is even $x > 0$ because $d$ is even	$n$ is even, $d$ is odd $y > 0$ because $n$ is even
$n < d$ $y' \rightarrow \pm\infty$ as $x \rightarrow 0$ $y' \rightarrow 0$ as $x \rightarrow \pm\infty$	e.g. $y = x^{3/5}$ 	e.g. $y = x^{3/4}$ 	e.g. $y = x^{2/3}$ 
$n = d$ $y' = 1$ for all $x$ 's	$y = x$ 		
$n > d$ $y' \rightarrow 0$ as $x \rightarrow 0$ $y' \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$	e.g. $y = x^{5/3}$   If $d$ is a factor of $n$ then this is a graph of an odd integral power of $x$ , e.g. $y = x^{5/3} = x^3$ .	e.g. $y = x^{5/4}$ 	e.g. $y = x^{4/3}$   If $d$ is a factor of $n$ then this is a graph of an even integral power of $x$ , e.g. $y = x^{4/3} = x^2$ .

The graph of  $y = x^{-m}$  is the graph of the reciprocal of  $y = x^m$ .

The graph of  $y = kx^m$  if  $k \neq 1$ , is the graph of  $y = x^m$  stretched vertically with the  $x$ -axis as the axis of stretch, and a stretch factor of  $k$ . Similarly for the graph of  $y = kx^{-m}$ .

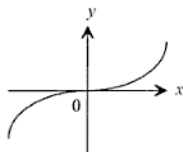
## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

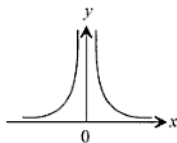
# TOPIC 17 GENERAL GRAPH SKETCHING

## FORMULAE AND IMPORTANT NOTES

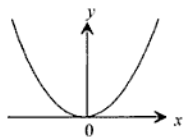
1. The following are the most common graphs of some simple functions  $y=f(x)$ .



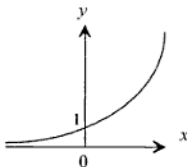
$$y = x^3$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}$$



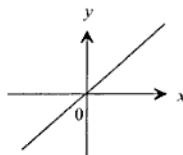
$$y = x^{-2}$$
$$x \in (\mathbb{R}^+ \cup \mathbb{R}^-)$$
$$y \in \mathbb{R}^+$$



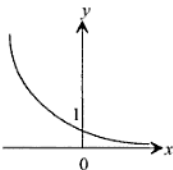
$$y = x^2$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}^+$$



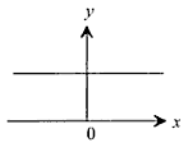
$$y = a^x, a > 1$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}^+$$



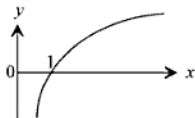
$$y = x$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}$$



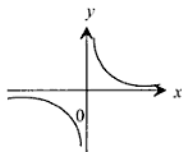
$$y = a^x, 0 < a < 1$$
$$x \in \mathbb{R}$$
$$y \in \mathbb{R}^+$$



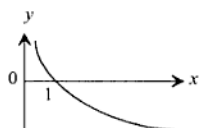
$$y = x^0$$
$$x \in \mathbb{R}$$
$$y = 1$$



$$y = \log_a x, a > 1$$
$$x \in \mathbb{R}^+$$
$$y \in \mathbb{R}$$



$$y = x^{-1}$$
$$x \in (\mathbb{R}^+ \cup \mathbb{R}^-)$$
$$y \in (\mathbb{R}^+ \cup \mathbb{R}^-)$$



$$y = \log_a x, 0 < a < 1$$
$$x \in \mathbb{R}^+$$
$$y \in \mathbb{R}$$

2. To transform the graph of  $f(x)$  into

(1) Graph of  $f(x) + k$  :

Translate the graph of  $f(x)$  vertically  $k$  units upward if  $k$  is positive and  $k$  units downward if  $k$  is negative.

Alternative method: Translate the  $x$ -axis of the graph of  $f(x)$   $k$  units downward if  $k$  is positive and  $k$  units upward if  $k$  is negative. Relabel the  $y$ -axis. (Warning: This will similarly transformed all the other graphs also. The same applies to the following alternative methods.)

(2) Graphs of  $f(x + k)$  :

Translate the graph of  $f(x)$  horizontally  $k$  units to the left if  $k$  is positive and  $k$  units to the right if  $k$  is negative.

Alternative method : Translate the  $y$ -axis of the original graph  $k$  units to the right if  $k$  is positive and  $k$  units to the left if  $k$  is negative. Relabel the  $x$ -axis.

(3) Graph of  $-f(x)$  :

Reflect the graph of  $f(x)$  in the  $x$ -axis.

(4) Graph of  $kf(x)$  :

Stretch the graph of  $f(x)$  vertically. Scale factor is  $k$  and the  $x$ -axis remains invariant. If  $k$  is negative the transformation includes a reflection in the  $x$ -axis.

Alternative method if  $k$  is positive: Multiply the labels on the  $y$ -axis by  $k$ .

(5) Graph of  $f(-x)$ :

Reflect the graph of  $f(x)$  in the  $y$ -axis.

(6) Graph of  $f(kx)$ :

Stretch the graph of  $f(x)$  horizontally. Scale factor is  $\frac{1}{k}$  and the  $y$ -axis remains invariant. If  $k$  is negative the transformation includes a reflection in the  $y$ -axis.

Alternative method if  $k$  is positive: Divide the labels on the original  $x$ -axis by  $k$ .

(7) Graph of  $f^{-1}(x)$ :

Reflect the graph of  $f(x)$  in the line  $y = x$ .

(8) Graph of  $|f(x)|$ :

Reflect about the  $x$ -axis any part of the graph of  $f(x)$  which is below the  $x$ -axis.

(9) Graph of  $\frac{1}{f(x)}$ :

The graph of  $f(x)$  and that of its reciprocal function  $\frac{1}{f(x)}$  are related in the following ways:

- For the same value of  $x$  they have the same sign.
- Both functions equal  $\pm 1$  simultaneously.
- Both functions cannot be zero for any value of  $x$ .
- When one function approaches zero the other approaches infinity.
- When one is decreasing the other is increasing and vice versa.

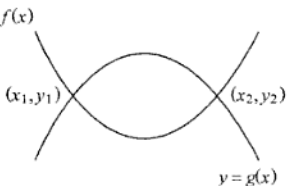
## PAST EXAMINATION QUESTIONS

1. The graph of  $y = h^{-1}(x)$  is a smooth curve passing through the points  $(1, 0)$ ,  $(1.3, 1)$ ,  $(2, 1.7)$  and  $(3, 2)$ . (i) Draw, on graph paper, the graph of  $y = h^{-1}(x)$  using the same scale on each axis. (ii) On the same diagram draw the graph of  $y = h(x)$  for  $0 \leq x \leq 2$ . (N99/P1/17c)

# TOPIC 18 SIMULTANEOUS INEQUALITIES

## FORMULAE AND IMPORTANT NOTES

- Suppose that the graph of  $y = f(x)$  divides the  $xy$ -plane into two half-planes. For every point in the region above the line  $y > f(x)$  and for every point in the region below the line  $y < f(x)$ .
- For these two graphs :
  - $f(x) > g(x)$  if  $x < x_1$  and  $x_2 < x$
  - $f(x) \geq g(x)$  if  $x \leq x_1$  and  $x_2 \leq x$
  - $f(x) = g(x)$  if  $x = x_1$  and  $x = x_2$
  - $f(x) \leq g(x)$  if  $x_1 \leq x \leq x_2$
  - $f(x) < g(x)$  if  $x_1 < x < x_2$
- For any two functions  $f(x)$  and  $g(x)$  :
  - If  $f(x) \cdot g(x) > 0$  or  $\frac{f(x)}{g(x)} > 0$  then  $f(x)$  and  $g(x)$  have the same sign.
  - If  $f(x) \cdot g(x) < 0$  or  $\frac{f(x)}{g(x)} < 0$  then  $f(x)$  and  $g(x)$  have different signs.



## PAST EXAMINATION QUESTIONS

- The speed  $v \text{ ms}^{-1}$  of a particle travelling from  $A$  to  $B$ , at time  $t$  s after leaving  $A$ , is given by  $v = 10t - t^2$ . The particle starts from rest at  $A$  and comes to rest at  $B$ . Show that the particle has a speed of  $5 \text{ ms}^{-1}$  or greater for exactly  $4\sqrt{3}$  s. (N2002/P1/3)
- Given that  $C = \{x : -5 < x < 5\}$ ,  
 $A = \{x : 8 > 2x + 1\}$ ,  
 $B = \{x : x^2 > x + 2\}$ ,  
find the values of  $x$  which define the set  $A \cap B$ . (N2002/P2/4)



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# TOPIC 19 MAPPINGS

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## FORMULAE AND IMPORTANT NOTES

1. An ordered pair consists of two elements, say  $x$  and  $y$ , in which one of them, say  $x$ , is designated as the first element and the other as the second element. It is denoted by  $(x, y)$ .
2. Let  $X$  and  $Y$  be two sets and let  $x$  be an element of  $X$  and  $y$  be an element of  $Y$ . If every  $x$  of  $X$  appears as the first term of at least an ordered pair of  $(x, y)$  we have a mapping from  $X$  to  $Y$ .
3. A function  $f$  is a mapping from  $X$  to  $Y$  ( $f: X \mapsto Y$ ) for which each element of  $X$  has one and only one element of  $Y$  as its image. It is written as  $f: x \mapsto y$  or  $y = f(x)$ .  $y$  is called the image and  $x$  the preimage.  $f(x)$  is called the value of the function  $f$  at  $x$ .  $X$  is the domain of the function and  $Y$  is the codomain. Those elements of  $Y$  which are the images of at least one  $x$  is the range of the function.

## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

# TOPIC 20 COMPOSITION OF FUNCTIONS

## FORMULAE AND IMPORTANT NOTES

- In  $f: X \mapsto Y$ , if every element of  $Y$  is an image of only one element of  $X$  then we can form an inverse function  $f^{-1}: Y \rightarrow X$ . If  $y = f(x)$  then  $x = f^{-1}(y)$  but more usually we first have  $x = f(y)$  and then  $y = f^{-1}(x)$ .
- In the following  $n$  is a constant.  
 $f(x): \quad x \quad x+n \quad x-n \quad n-x \quad nx \quad \frac{x}{n} \quad \frac{n}{x} \quad x^n \quad \sqrt[n]{x}$   
 $f^{-1}(x): \quad x \quad x-n \quad x+n \quad n-x \quad \frac{x}{n} \quad nx \quad \frac{n}{x} \quad \sqrt[n]{x} \quad x^n$
- $f(x) = n$  where  $n$  is a constant, is called a constant function.
- $f(x) = x$  is called an identity function. It is denoted by  $I$ .
- If  $f$  and  $g$  are a pair of functions such that the range of  $f$  is a subset of the domain of  $g$ , then the composite of  $f$  and  $g$  (or the composite of  $f$  followed by  $g$ ) is denoted by  $g \circ f$  or  $gf$  where  $gf(x) = g(f(x))$ .  $gf$  is generally not identical to  $fg$ .
- $f^{-1} \circ f = f \circ f^{-1} = I$  if both  $f$  and  $f^{-1}$  coexist.
- $I \circ f = f \circ I = f$
- $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$
- $f^2(x) = f(f(x))$ . It is usually equal to neither  $(f(x))^2$  nor  $f(x^2)$ .
- $f^{-2} = (f^2)^{-1} = (f^{-1})^2 = f^{-1} \circ f^{-1}$
- $f^n = f^r f^{n-r} = f^{n-r} f^r$  where  $n$  and  $r$  are positive integer with  $n > r$ .
- If  $f^n = I$  then  $f = f^{n+1} = f^{2n+1} = f^{3n+1} = \dots$   
 $f^2 = f^{n+2} = f^{2n+2} = f^{3n+2} = \dots$   
 $\dots$   
 $f^n = f^{2n} = f^{3n} = f^{4n} = \dots = I$
- $f(f^{-1}(a)) = a$  and  $f^{-1}(f(a)) = a$

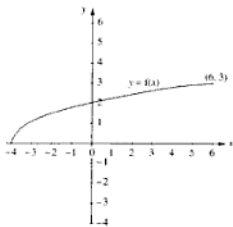
## PAST EXAMINATION QUESTIONS

- A function  $f$  is defined by  $f: x \mapsto 5 - \frac{6}{x}$ ,  $x \neq 0$ .
  - Find  $f^{-1}$  and state the value of  $x$  for which  $f^{-1}$  is undefined.
  - Find the values of  $x$  for which  $f(x) = f^{-1}(x)$ . (N97/P1/7)
- Functions  $f$  and  $g$  are defined by  $f: x \mapsto \frac{3x-1}{x-2}$ ,  $x \neq 2$ ,  $g: x \mapsto \frac{2x-1}{x-3}$ ,  $x \neq 3$ .
  - Show that  $fg: x \mapsto x$ .
  - Evaluate  $f^{-1}(5)$ ,  $g^{-1}(4)$  and  $ffg(7)$ . (N97/P1/17a)

3. (a) The function  $g$  is defined by  $g : x \mapsto 8 - 3x$ . Find (i) an expression for  $g^{-1}(x)$  and for  $g^2(x)$ , (ii) the value of  $x$  for which  $g^{-1}(x) = g^2(x)$ .
- (b) The function  $h$  is defined by  $h : x \mapsto ax + b$ ,  $a \neq -1$ , for the domain  $0 \leq x \leq 5$ . Given that the graph of  $y = h(x)$  passes through the point  $(8, 5)$  and that the graphs of  $y = h(x)$  and  $y = h^{-1}(x)$  intersect at the point whose  $x$ -coordinate is 3, find the value of  $a$  and of  $b$ .  
(N98/P1/16b, c)

4. Functions  $f$  and  $g$  are defined by  $f : x \mapsto \frac{a}{3-x}$ ,  $x \neq 3$ ,  $g : x \mapsto 11 + bx^2$ , where  $x \in \mathbf{R}$ . Given that  $f^2(5) = \frac{4}{3}$  and  $fg(2) = \frac{1}{2}$ , evaluate  $a$  and  $b$ .  
(N99/P1/17a)

5. (a) The diagram shows the graph of  $y = f(x)$ , where  $f$  is a function with domain  $-4 \leq x \leq 6$  and range  $0 \leq f(x) \leq 3$ . Make a sketch of the diagram shown and add a sketch of the graph of  $y = f^{-1}(x)$ . State the range of  $f^{-1}$ .



- (b) The functions  $f$  and  $g$  are defined for real values of  $x$  by  $f : x \mapsto 2x - 1$ ,  $g : x \mapsto \frac{1}{x-1}$ , where  $x \neq 1$ .

Express (i)  $fg(x)$ , (ii)  $g^{-1}(x)$ , in form of a single fraction in terms of  $x$ , stating in each case the value of  $x$  for which the function is not defined.

- (c) The functions  $f$  and  $g$  are defined for real values of  $x$  by  $f : x \mapsto x^2$ ,  $g : x \mapsto x - 3$ .

Express each of the following in terms of  $f$  and  $g$  only.

- (i)  $x \mapsto x - 6$ , (ii)  $x \mapsto x^2 - 6$ , (iii)  $x \mapsto (x + 3)^2$ .  
(N2000/P1/11)

6. Functions  $f$  and  $g$  are defined, for  $x \in \mathbf{R}$ , by  $f : x \mapsto 5x - 2$ ,  $g : x \mapsto \frac{1}{2x-1}$ ,  $x \neq \frac{1}{2}$ .

- (a) Find the value of  $g^2(2)$ .

Find the value of  $x$  for which

(b)  $f(x) = f^{-1}(x)$ ,

(c)  $fg(x) + 3g(x) = 0$ .

(N01/P1/13a)

7. The function  $f$  is given by  $f : x \mapsto 5 - 3e^{\frac{1}{2}x}$ ,  $x \in \mathbf{R}$ .

- (i) State the range of  $f$ .

- (ii) Solve the equation  $f(x) = 0$ , giving your answer correct to two decimal places.

- (iii) Sketch the graph of  $y = f(x)$ , showing on your diagram the coordinates of the points of intersection with the axes.

- (iv) Find an expression for  $f^{-1}$  in terms of  $x$ .

(N2003/P1/10)

## 8. EITHER

Functions  $f$  and  $g$  are defined for  $x \in \mathbf{R}$  by

$$f : x \mapsto 3x - 2, x \neq \frac{4}{3},$$

$$g : x \mapsto \frac{4}{2-x}, x \neq 2.$$

- (i) Solve the equation  $gf(x) = 2$ .

- (ii) Determine the number of real roots of the equation  $f(x) = g(x)$ .

- (iii) Express  $f^{-1}$  and  $g^{-1}$  in terms of  $x$ .

- (iv) Sketch, on a single diagram, the graphs of  $y = f(x)$  and  $y = f^{-1}(x)$ , stating the coordinates of the point of intersection of the two graphs.

**OR**

- (i) Find the value of  $a$  and of  $b$  for which  $1 - x^2 + 6x$  can be expressed in the form  $a - (x + b)^2$ .

A function  $f$  is defined by  $f : x \mapsto 1 - x^2 + 6x$  for the domain  $x \geq 4$ .

- (ii) Explain why  $f$  has an inverse.  
(iii) Find an expression for  $f^{-1}$  in terms of  $x$ .

A function  $g$  is defined by  $g : x \mapsto 1 - x^2 + 6x$  for the domain  $2 \leq x \leq 7$ .

- (iv) Find the range of  $g$ .  
(v) Sketch the graph of  $y = |g(x)|$  for  $2 \leq x \leq 7$ . (N2003/P2/12)

9. The function  $f$  is defined, for  $x \in \mathbb{R}$ , by

$$f : x \mapsto \frac{3x+11}{x-3}, x \neq 3.$$

- (i) Find  $f^{-1}$  in terms of  $x$  and explain what this implies about the symmetry of the graph of  $y = f(x)$ .

The function  $g$  is defined, for  $x \in \mathbb{R}$ , by

$$g : x \mapsto \frac{x-3}{2}.$$

- (ii) Find the values of  $x$  for which  $f(x) = g^{-1}(x)$ .  
(iii) State the value of  $x$  for which  $gf(x) = -2$ . (N2004/P1/8)

10. The functions  $f$  and  $g$  are defined for  $x \in \mathbb{R}$  by  $f : x \mapsto x^3$ ,  
 $g : x \mapsto x + 2$ .

Express each of the following as a composite function, using only  $f$ ,  $g$ ,  $f^{-1}$  and/or  $g^{-1}$ :

- (i)  $x \mapsto x^3 + 2$ ,  
(ii)  $x \mapsto x^3 - 2$ ,  
(iii)  $x \mapsto (x + 2)^{\frac{1}{3}}$ . (N2006/P2/1)

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# TOPIC 21    PARAMETRIC AND CARTESIAN EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. The set of all points whose positions are determined by a set of given conditions (such as those specified by an equation) is called the locus of points subject to those conditions.
2. It is sometimes advantageous to express each of the  $x$  and  $y$  co-ordinates of the general point of a curve in terms of a third variable, called the parameter. The two equations expressing  $x$  and  $y$  in terms of the parameter are called parametric equations. It may be possible to obtain an equation connecting  $x$  and  $y$  by eliminating the parameter. Such an equation is called a cartesian equation, or simply, equation.

## WORKED EXAMPLES

1. Find the equation of the line (i) through the origin  $O$  with gradient  $m$ , (ii) through the point  $(0,2)$  with gradient  $\frac{1}{m}$ . These two lines intersect at  $P$ . Find the co-ordinates of  $P$  in terms of  $m$ . Write down the equation of the locus of  $P$  as  $m$  varies.

$$y = mx \quad \dots(i)$$

$$\frac{y-2}{x-0} = \frac{1}{m} \quad \dots(ii)$$

$$y = \frac{1}{m}x + 2$$

$$\text{At } P, mx = \frac{1}{m}x + 2$$

$$(m - \frac{1}{m})x = 2$$

$$\frac{m^2-1}{m}x = 2$$

$$x = \frac{2m}{m^2-1} \quad \dots(iii)$$

$$y = mx = \frac{2m^2}{m^2-1}$$

$$\therefore P = \left( \frac{2m}{m^2-1}, \frac{2m^2}{m^2-1} \right)$$

$$\text{From (i), } m = \frac{y}{x} \quad \dots(iv)$$

$$\text{From (iii) and (iv), } x = \frac{2\frac{y}{x}}{(\frac{y}{x})^2-1} \times \frac{x^2}{x^2}$$

$$x = \frac{2xy}{y^2-x^2}$$

$$1 = \frac{2y}{y^2-x^2}$$

Equation of the locus of  $P$ :

$$y^2 - x^2 = 2y$$

2. A curve which is defined parametrically by the equations  $x = 2 + t$  and  $y = 8 + 2t - t^2$  meets the line  $y = x + 6$  at the points  $A$  and  $B$ , find the mid-point of  $AB$ .

At  $A$  and  $B$ :  $y = x + 6$

$$8 + 2t - t^2 = 2 + t + 6$$

$$t - t^2 = 0$$

$$t(1 - t) = 0$$

If  $t = 0$ ,  $x = 2 + 0 = 2$ ,  $y = 8 + 2(0) - 0^2 = 8$

If  $t = 1$ ,  $x = 2 + 1 = 3$ ,  $y = 8 + 2(1) - 1^2 = 9$

$\therefore$  Mid-point of  $AB = \left(\frac{2+3}{2}, \frac{8+9}{2}\right) = \left(2\frac{1}{2}, 8\frac{1}{2}\right)$

3. The parametric equations of a curve are  $x = \frac{t+1}{t-1}$ ,  $y = 2t^2$ . Find the cartesian equation of the curve and where it intercepts the  $y$ -axis.

$$x = \frac{t+1}{t-1}$$

$$xt - x = t + 1$$

$$xt - t = x + 1$$

$$t(x - 1) = x + 1$$

$$t = \frac{x+1}{x-1}$$

$$y = 2t^2$$

$$= 2\left(\frac{x+1}{x-1}\right)^2$$

$\therefore$  Cartesian equation of the curve is

$$y(x-1)^2 = 2(x+1)^2$$

If  $x = 0$ ,  $y(0-1)^2 = 2(0+1)^2$

$$y = 2$$

$\therefore$  It intercepts the  $y$ -axis at  $(0, 2)$ .

## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

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# TOPIC 22 IDENTITIES

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## FORMULAE AND IMPORTANT NOTES

1. A mathematical statement that one expression is identical to another for all values of the variables is called an identity.
2. The expressions are joined by the identity symbol " $\equiv$ ". Very often, the equal sign " $=$ " is used instead of " $\equiv$ ".
3. It can be proved that if two polynomials are identical, then the coefficients of the same power of the variables on both sides of the identity sign must be equal.  
i.e.  $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0 \equiv b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_2 x^2 + b_1 x + b_0$  then  $a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} = b_{n-2}, \dots, a_2 = b_2, a_1 = b_1, a_0 = b_0$ .

## WORKED EXAMPLES

1. Given that  $5x^3 + 6x^2 + 8 \equiv (Ax + 1)(x - 2)(x + B) + x - 5C$ , find the value of  $A, B$  and  $C$ .

Coefficient of  $x^3 = 5 = A$

$$\therefore A = 5$$

$$\text{If } x = 2, \quad 5 \times 2^3 + 6 \times 2^2 + 8 = (2A + 1)(2 - 2)(2 + B) + 2 - 5C$$

$$72 = 2 - 5C$$

$$70 = -5C$$

$$5C = -70$$

$$C = -14$$

$$\text{If } x = 0, \quad 0 + 0 + 8 = (1)(-2)(B) + 0 + 70$$

$$2B = 62$$

$$B = 31$$

Answer:  $A = 5, B = 31, C = -14$

2. Given that  $a(x + b)^2 \equiv x^2 + 6x + c$  for all values of  $x$ , find the values of  $a, b$  and  $c$ .

$$a(x + b)^2 \equiv x^2 + 6x + c$$

$$ax^2 + 2abx + b^2 \equiv x^2 + 6x + c$$

$$a = 1$$

$$2ab = 6$$

$$2b = 6$$

$$b = 3$$

$$b^2 = c$$

$$c = 3^2$$

$$= 9$$

Answer:  $a = 1, b = 3, c = 9$

## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

# TOPIC 23 PARTIAL FRACTIONS

## FORMULAE AND IMPORTANT NOTES

In this topic,  $a, b, c, d, A, B$  and  $C$  are constant “ $\equiv$ ” means is identical with. Sometimes the symbol “ $=$ ” is used.

$$1. \frac{ax+b}{(cx+d)(cx+f)} \equiv \frac{A}{cx+d} + \frac{B}{cx+f}$$

$$ax+b \equiv A(cx+f) + B(cx+d)$$

(i) To find  $A$ , let  $x = -\frac{d}{c}$

To find  $B$ , let  $x = -\frac{f}{c}$

(ii) Equating coefficient of  $x$ :  $a = Ae + Bc$

Equating constant:  $b = Af + Bd$

Solve the simultaneous equations for the values of  $A$  and  $B$ .

(iii) Alternative method

To find  $A$ , let  $x = -\frac{d}{c}$  in  $\frac{ax+b}{cx+f}$

To find  $B$ , let  $x = -\frac{f}{c}$  in  $\frac{ax+b}{cx+d}$

$$2. \frac{ax+b}{(cx+d)(cx+f)^2} \equiv \frac{A}{cx+d} + \frac{B}{cx+f} + \frac{C}{(cx+f)^2}$$

$$ax+b \equiv A(cx+f)^2 + B(cx+d)(cx+f) + C(cx+d)$$

To find  $A$ , let  $x = -\frac{d}{c}$

To find  $C$ , let  $x = -\frac{f}{c}$

To find  $B$ , use the values of  $A$  and  $C$  and let  $x = 0$ .

$$3. \frac{ax+b}{(cx+d)(cx+f)(gx+h)} \equiv \frac{A}{cx+d} + \frac{B}{cx+f} + \frac{C}{gx+h}$$

$$ax+b \equiv A(cx+f)(gx+h) + B(cx+d)(gx+h) + C(cx+d)(cx+f)$$

To find  $A$ , let  $x = -\frac{d}{c}$

To find  $B$ , let  $x = -\frac{f}{c}$

To find  $C$ , use the values of  $A$  and  $B$  and let  $x = 0$ .

$$4. \frac{ax+b}{(cx+d)(cx^2+f)} \equiv \frac{A}{cx+d} + \frac{Bx+C}{cx^2+f}$$

$$ax+b \equiv A(cx^2+f) + (Bx+C)(cx+d)$$

To find  $A$ , let  $x = -\frac{d}{c}$

Use the value of  $A$  and let  $x = 0$  to find  $C$ .

Use the value of  $A$  and  $C$  and let  $x = 1$  to find  $B$ .

## PAST EXAMINATION QUESTIONS

No questions on this topic in the last 10 years.



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# TOPIC 24      REMAINDER AND FACTOR THEOREM, SOLUTION OF CUBIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. A polynomial (rational integral function) is a sum of multiples of integral powers of a variable. The general equation for a polynomial of degree  $n$  in the variable  $x$  is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients  $a_n, a_{n-1}, \dots$ , are constants and  $n$  is a positive integer or zero. If  $n$  is 1, 2, 3 then the polynomial  $f(x)$  is a linear, quadratic, or cubic function respectively.

2. Dividend  $\equiv$  Divisor  $\times$  Quotient + Remainder
3. When a polynomial  $f(x)$  of degree  $n$  is divided by a divisor  $(x-a)$ , where  $a$  is a constant, then  $f(x) \equiv (x-a)Q(x) + R$ , where  $Q(x)$  is the quotient, a polynomial of degree  $n-1$ , and  $R$  is the remainder independent of  $x$ . This is an identity which is true for all values of  $x$ .
4. **The Remainder Theorem:** If a polynomial  $f(x)$  is divided by  $(x-a)$ , where  $a$  is any constant, until a constant remainder independent of  $x$  is obtained, this remainder is equal to  $f(a)$ .
5. There are two ways to find the remainder  $f(a)$ :
- Substitute  $a$  for  $x$  in  $f(x)$  and evaluate the resulting expression.
  - Divide  $f(x)$  by  $(x-a)$ . This process can be greatly simplified by the process of synthetic division.
6. **Synthetic Division:** To divide  $f(x)$  by  $(x-a)$ , arrange in order of descending power of  $x$  the coefficients of  $f(x)$ , inserting zero for the coefficient of any missing power of  $x$ . Write  $a$  on the left. Write the first coefficient of  $f(x)$  to the first position on the third line. Multiply this first coefficient by  $a$ , writing the product in the second line under the second coefficient of  $f(x)$ . The sum of this product and second coefficient is written in the third line. Multiply this sum by  $a$ , add the product to the next coefficient of  $f(x)$ , again writing the new sum on the third line, and so on, until a product has been added to the last coefficient of  $f(x)$ . The last sum in the third line is the remainder. The preceding numbers in this line are the coefficients of the powers of  $x$  in the quotient, arranged in descending order.
7. An example of synthetic division:

Divide  $2x^4 - 7x^3 + 12x + 1$  by  $x - 3$ .

$$\begin{array}{r} 3 \quad 2 \quad -7 \quad 0 \quad + \quad 12 \quad + \quad 1 \\ \quad + \quad 6 \quad - \quad 3 \quad - \quad 9 \quad + \quad 9 \\ \hline 2 \quad - \quad 1 \quad - \quad 3 \quad + \quad 3 \quad + \quad 10 \end{array}$$

$$2x^4 - 7x^3 + 12x + 1 = (x-3)(2x^3 - x^2 - 3x + 3) + 10$$

8. **The Factor Theorem:** If  $f(a) = R = 0$  then  $(x-a)$  is a factor of  $f(x)$ . This also means that  $a$  is a root of the equation  $f(x) = 0$ . The other factors and roots, if any, can be found by using the same theorem on  $Q(x)$ . The converse of the factor theorem is also true.

9. If  $(x - a)$  is a factor of  $f(x)$  the constant term of which is  $a_0$ , then  $a$  is a factor of  $a_0$ .

10. Divisor:	$x - a$	$x + a$	$bx - a$	$bx + a$
Remainder:	$f(a)$	$f(-a)$	$f\left(\frac{a}{b}\right)$	$f\left(-\frac{a}{b}\right)$

## WORKED EXAMPLES

1. Solve the equation  $2x^3 + 3x^2 - 11x - 6 = 0$

$$\text{If } x = 2, 2(2)^3 + 3(2)^2 - 11(2) - 6 = 0$$

$$\therefore 2x^3 + 3x^2 - 11x - 6 = (x - 2)(2x^2 + 7x + 3)$$

$$\begin{array}{r} -4x^2 \\ + 7x^2 \\ \hline + 3x^2 \end{array}$$

$$= (x - 2)(2x + 1)(x + 3)$$

$$(x - 2)(2x + 1)(x + 3) = 0$$

$$x = 2, -\frac{1}{2}, -3$$

2. Solve the equation  $x^3 - 3x^2 - 2x + 2 = 0$

$$\text{If } x = -1, (-1)^3 - 3(-1)^2 - 2(-1) + 2 = 0$$

$$\therefore x^3 - 3x^2 - 2x + 2 = (x + 1)(x^2 - 4x + 2)$$

$$\begin{array}{r} +x^2 \\ + -4x^2 \\ \hline -3x^2 \end{array}$$

$$\text{If } x + 1 = 0, x = -1$$

$$\begin{aligned} \text{If } x^2 - 4x + 2 = 0, x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

$$\therefore x = -1, 3.41, 0.586$$

## PAST EXAMINATION QUESTIONS

- (a) The expressions  $x^3 - ax + a^2$  and  $ax^3 + x^2 - 17$  have the same remainder when divided by  $x - 2$ . Find the possible values of (i)  $a$ , (ii) the remainder.

(b) Find the  $x$ -coordinate of each of the three points of intersection of the curves  $y = 6x^2 - 5$  and  $y = 17x - \frac{6}{x}$ .

(c) Find the value of  $k$  for which  $x^2 - 3x + k$  is a factor of  $x^3 - 5x^2 + 12$ . (N97/P2/1)
- Find the value of  $k$  for which  $x^2 + (k - 1)x + k^2 - 16$  is exactly divisible by  $x - 3$  but not divisible by  $x + 4$ . (N98/P2/1a)
- (a) Find the remainder when  $3x^3 - x^2 - 5x + 2$  is divided by  $3x + 2$ .

(b) Solve the equation  $2x^3 + 13x^2 + x - 70 = 0$ .

- (c) Given that  $x^2 + 2x - 3$  is factor of  $f(x)$ , where  $f(x) = x^4 + 6x^3 + 2ax^2 + bx - 3a$ , find  
 (i) the value of  $a$  and of  $b$ , (ii) the other quadratic factor of  $f(x)$ . (N99/P2/1)
4. (a) The curve whose equation is  $y = (2x^2 + 3x - 9)(x - k)$ , where  $k$  is a constant, has a turning point where  $x = -1$ .  
 (i) Calculate the value of  $k$ .  
 (ii) Calculate the value of  $x$  at the other turning point on the curve.  
 (iii) Draw a rough sketch of the curve and find the set of values of  $x$  for which  $y > 0$ .
- (b)  $f(x)$  and  $g(x)$  are given by  $f(x) = x^4 + 3x^3 - 12x^2 + 2x + 4$ ,  $g(x) = x^4 + 2x^3 - 8x^2 + x - 2$ .  
 (i) Solve completely the equation  $f(x) - g(x) = 0$   
 $f(x)$  and  $g(x)$  have a common factor  $x - \alpha$ .  
 (ii) Find the value of  $\alpha$ . (N2000/P2/1)
5. (a) The expression  $2x^4 - 5x^2 + ax + b$  has a factor of  $x + 2$  and leaves a remainder of 6 when divided by  $x - 1$ . Calculate the value of  $a$  and of  $b$ .  
 (b) The straight line  $y = 11x - 6$  intersects the curve  $y = x^2(2x - 3)$  at three points. Calculate the  $x$ -coordinate of each point of intersection.  
 (c) Find the value of  $k$  for which  $a - 3b$  is a factor of  $a^4 - 7a^2b^2 + kb^4$ . Hence, for this value of  $k$ , factorise  $a^4 - 7a^2b^2 + kb^4$  completely. (N01/P2/1)
6. The cubic polynomial  $f(x)$  is such that the coefficient of  $x^3$  is  $-1$  and the roots of the equation  $f(x) = 0$  are 1, 2 and  $k$ . Given that  $f(x)$  has a remainder of 8 when divided by  $x - 3$ , find  
 (i) the value of  $k$ ,  
 (ii) the remainder when  $f(x)$  is divided by  $x + 3$ . (N2002/P2/6)
7. The expression  $x^3 + ax^2 + bx - 3$ , where  $a$  and  $b$  are constants, has a factor of  $x - 3$  and leaves a remainder of 15 when divided by  $x + 2$ . Find the value of  $a$  and of  $b$ . (N2003/P1/3)
8. Given that  $6x^3 + 5ax - 12a$  leaves a remainder of  $-4$  when divided by  $x - a$ , find the possible values of  $a$ . (N2004/P2/7)
9. The function  $f(x) = x^3 - 6x^2 + ax + b$ , where  $a$  and  $b$  are constants, is exactly divisible by  $x - 3$  and leaves a remainder of  $-55$  when divided by  $x + 2$ .  
 (i) Find the value of  $a$  and of  $b$ .  
 (ii) Solve the equation  $f(x) = 0$ . (N2005/P2/9)
10. The cubic polynomial  $f(x)$  is such that the coefficient of  $x^3$  is 1 and the roots of  $f(x) = 0$  are 1,  $k$  and  $k^2$ . It is given that  $f(x)$  has a remainder of 7 when divided by  $x - 2$ .  
 (i) Show that  $k^3 - 2k^2 - 2k - 3 = 0$ .  
 (ii) Hence find a value for  $k$  and show that there are no other real values of  $k$  which satisfy this equation. (N2006/P1/10)
11. Find the value of  $a$  and of  $b$  for which  $2x^2 + x - 1$  is a factor of  $2x^3 + ax^2 + x + b$ . (SP08/P1/5)

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# TOPIC 25 PERMUTATIONS AND COMBINATIONS

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## FORMULAE AND IMPORTANT NOTES

1. Suppose that an operation can be performed in  $r$  sets of ways and that these  $r$  sets have no way in common. If there are  $n_1$  ways in the first set,  $n_2$  ways in the second set, and so forth, then there are  $n_1 + n_2 + \dots + n_r$  ways to perform the operation.
2. Fundamental Principle of Countings: If an operation can be carried out in  $n_1$  ways and after this is done another operation can be carried out in  $n_2$  ways and after this is done another operation can be carried out in  $n_3$  ways, and so forth, then the total number of ways in which all operations can be carried out is  $n_1 \times n_2 \times n_3 \times \dots$ .
3. If each of  $r$  operations can be carried out in  $n$  different ways, there are  $n^r$  different ways to carry out all the operations. This is a special case of the Fundamental Principle of Counting.
4. (i) An arrangement of all members of a set of  $n$  objects in a given order is called a permutation of the  $n$  objects taken all at a time.  
(ii) The number of permutations of all of  $n$  objects is denoted by  ${}^n P_n$  or  $n!$ , called factorial  $n$ .  
(iii)  $n! = n(n-1)(n-2) \dots 3 \times 2 \times 1$   
(iv)  $0! = 1$
5. (i) An arrangement of any  $r$  of  $n$  objects, where  $0 \leq r \leq n$  is called a permutation of  $n$  objects taken  $r$  at a time. The number of ways of doing so is denoted by  ${}^n P_r$ .  
(ii)  ${}^n P_r = n(n-1)(n-2) \dots [n-(r-2)][n-(r-1)]$   
$$= \frac{n!}{(n-r)!}$$
  
(iii)  ${}^n P_n = {}^n P_{n-1}$
6. (i) A combination of  $n$  objects taken  $r$  ( $0 \leq r \leq n$ ) at a time is any subset of  $r$  elements from a set of  $n$  elements. It is any choice of  $r$  of the  $n$  elements where order does not matter.  
(ii) The number of ways of combinations of  $n$  objects taken  $r$  at a time is denoted variously by  $\binom{n}{r}$ ,  ${}^n C_r$ ,  $C_r^n$ ,  $C(n, r)$  etc. We shall use  $\binom{n}{r}$ .  
(iii)  $\binom{n}{r} = \frac{{}^n P_r}{r!} = \frac{n!}{r!(n-r)!}$   
(iv)  $\binom{n}{r} = \binom{n}{n-r}$   
(v)  $\binom{n}{n} = \binom{n}{0} = 1$

## WORKED EXAMPLES

1. In how many ways can 6 different Chinese books, 3 different English books, and 4 different Malay books be arranged on a shelf so that all the books in the same language come together?

The 6 Chinese books can be arranged in  $6!$  ways.

The 3 English books can be arranged in  $3!$  ways.

The 4 Malay books can be arranged in  $4!$  ways.

Books of 3 different languages can be arranged in  $3!$  ways.

The number of ways the books can be arranged on the shelf  $= 6! \times 3! \times 4! \times 3!$   
 $= 622\,080$  ways

2. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. How many ways can this be done?

Number of ways if the committee contains 4 men and 2 women  $= \binom{10}{4} \times \binom{7}{2} = 4410$

Number of ways if the committee contains 3 men and 3 women  $= \binom{10}{3} \times \binom{7}{3} = 4200$

Number of ways the committee can be formed  $= 4410 + 4200 = 8610$

3. In how many ways can 4 boys and 4 girls be arranged in a line so that boys and girls are placed alternatively?

Each group of 4 can be arranged in  $4!$  ways.

If the arrangement is B G B G B G B G, there are  $4! \times 4! = 576$  ways.

The arrangement G B G B G B G B also has 576 ways

$\therefore$  The number of ways the children can be arranged  $= 576 \times 2 = 1152$  ways

4. There are 4 pigeon holes marked  $W$ ,  $X$ ,  $Y$  and  $Z$ . In how many ways can I arrange 14 different books so that 5 of them are in  $W$ , 4 of them are in  $X$ , 3 of them are in  $Y$  and the rest in  $Z$ ? The order of books in each hole does not matter.

5 books can be chosen from 14 for  $W$  in  $\binom{14}{5}$  ways.

$14 - 5 = 9$ . 4 books can be chosen from 9 for  $X$  in  $\binom{9}{4}$  ways.

$9 - 4 = 5$ . 3 books can be chosen from 5 for  $Y$  in  $\binom{5}{3}$  ways.

The two books have to be put in  $Z$ .

The total number of ways of arranging the books  $= \binom{14}{5} \binom{9}{4} \binom{5}{3} \times 1 = 2\,522\,520$

## PAST EXAMINATION QUESTIONS

- An art gallery has 9 paintings by a famous artist. 9 selection of 4 of there are to be shown in an exhibition. Calculate the number of different selections that can be taken if
  - there are no restriction,
  - one special painting must be included.  
4 of these paintings, including the special one, are selected and exhibited in a line. Find the number of arrangements of these 4 paintings in which the special painting does come at either end. (N99/P2/24c)
- The producer of a play requires a total cast of 5, of which 3 are actors and 2 are actresses. He auditions 5 actors and 4 actresses for the cast. Find the total number of ways in which the cast can be obtained.
  - Find how many different odd 4-digit numbers less than 4000 can be made from the digits 1, 2, 3, 4, 5, 6, 7 if no digit may be repeated. (N2002/P2/5)
- A garden centre sells 10 different varieties of rose bush. A gardener wishes to buy 6 rose bushes, all of different varieties.
  - Calculate the number of ways she can make her selection.  
Of the 10 varieties, 3 are pink, 5 are red and 2 are yellow. Calculate the number of ways in which her selection of 6 rose bushes could contain
  - no pink rose bush,
  - at least one rose bush of each colour. (N2003/P2/8)
- Find the number of different arrangements of the 9 letters of the word SINGAPORE in which S does **not** occur as the first letter.
  - 3 students are selected to form a chess team from a group of 5 girls and 3 boys. Find the number of possible teams that can be selected in which there are more girls than boys. (N2004/P1/7)
- Each day a newsagent sells copies of 10 different newspapers, one of which is *The Times*. A customer buys 3 different newspapers. Calculate the number of ways the customer can select his newspapers
    - if there is no restriction,
    - if 1 of the 3 newspapers is *The Times*.
  - Calculate the number of different 5-digit numbers which can be formed using the digits 0, 1, 2, 3, 4 without repetition and assuming that a number cannot begin with 0. How many of these 5-digit numbers are even? (N2005/P2/11)
- How many different four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if no digit may be repeated?
  - In a group of 13 entertainers, 8 are singers and 5 are comedians. A concert is to be given by 5 of these entertainers. In the concert there must be at least 1 comedian and there must be more singers than comedians. Find the number of different ways that the 5 entertainers can be selected. (N2006/P2/10)

# TOPIC 26 BINOMIAL THEOREM : POSITIVE INTEGRAL INDEX

## FORMULAE AND IMPORTANT NOTES

In the following  $n$  and  $r$  are non-negative integers and  $n \geq r$ .

$$1. \quad n! = n(n-1)(n-2)\dots 3 \times 2 \times 1$$

$$0! = 1$$

$$2. \quad \binom{n}{r} = \frac{n(n-1)(n-2)\dots [n-(r-1)]}{r!} = \binom{n}{n-r}$$

On the keyboard of calculators,  $\binom{n}{r}$  is denoted by  $nCr$ . Notice that the numerator has  $r$  factors.

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$3. \quad (x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= x^n + nx^{n-1}y + \frac{n(n-1)}{1 \times 2}x^{n-2}y^2 + \dots + \frac{n(n-1)}{1 \times 2}x^2y^{n-2} + nxy^{n-1} + y^n$$

The expansion has  $(n+1)$  terms.

The general term is  $u_{r+1} = \binom{n}{r}x^{n-r}y^r = \binom{n}{n-r}x^{n-r}y^r$ .

$$4. \quad (x-y)^n = \binom{n}{0}x^n - \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 - \dots + \binom{n}{n-2}x^2(-y)^{n-2} + \binom{n}{n-1}x(-y)^{n-1} + \binom{n}{n}(-y)^n$$

$$= x^n - nx^{n-1}y + \frac{n(n-1)}{1 \times 2}x^{n-2}y^2 - \dots + \frac{n(n-1)}{1 \times 2}x^2(-y)^{n-2} + n(-y)^{n-1} + (-y)^n$$

The expansion has  $(n+1)$  terms which alternate in sign.

The general term is  $u_{r+1} = \binom{n}{r}x^{n-r}(-y)^r = (-1)^r \binom{n}{r}x^{n-r}y^r$

$$= (-1)^r \binom{n}{n-r}x^{n-r}y^r$$

$(-1)^r$  equals 1 if  $r$  is even and  $-1$  if  $r$  is odd.

5. Pascal's Triangle: The following two triangles are equivalent.

$\binom{0}{0}$	1
$\binom{1}{0}\binom{1}{1}$	1 1
$\binom{2}{0}\binom{2}{1}\binom{2}{2}$	1 2 1
$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$	1 3 3 1
$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$	1 4 6 4 1
etc.	etc.

## PAST EXAMINATION QUESTIONS

- Find the coefficient of  $x^3$  in the expansion of  $(9 + 8x)1 - \frac{x}{3})^9$ . (N97/P1/5)
- Find the coefficient of  $x$  in the expansion of  $(x^2 - \frac{3}{x})^5$ .
  - Obtain the first 4 terms in the expansion of  $(1 + p)^7$  in ascending powers of  $p$ . Hence find the coefficient of  $x^3$  in the expansion of  $(1 + x + 2x^2)^7$ . (N98/P1/5)
- Find the term independent of  $x$  in the expansion of  $(2x - \frac{1}{2x})^9$ . (N99/P1/4)
- Find the coefficient of  $x^3$  in the expansion of  $(10 - 7x)(1 + \frac{x}{3})^8$ . (N2000/P1/4)
- In the expansion of  $(1 - 2x)^n$  the sum of the coefficients of  $x$  and  $x^2$  is 16. Given that  $n$  is positive, find the value of (i)  $n$ , (ii) the coefficient of  $x^3$ . (N2001/P1/5)
- Find the first three terms in the expansion, in ascending powers of  $x$ , of  $(2 + x)^6$  and hence obtain the coefficient of  $x^2$  in the expansion of  $(2 + x - x^2)^6$ . (N2002/P2/2)
- Obtain
  - the first 3 terms in the expansion, in descending powers of  $x$ , of  $(3x - 1)^5$ ,
  - the coefficient of  $x^4$  in the expansion of  $(3x - 1)^5(2x + 1)$ . (N2003/P2/5)
- Given that the expansion of  $(a + x)(1 - 2x)^n$  in ascending powers of  $x$  is  $3 - 41x + bx^2 + \dots$ , find the values of the constants  $a$ ,  $n$  and  $b$ . (N2004/P1/5)
- The binomial expansion of  $(1 + px)^n$ , where  $n > 0$ , in ascending powers of  $x$  is  $1 - 12x + 28p^2x^2 + qx^3 + \dots$ . Find the value of  $n$ , of  $p$  and of  $q$ . (N2005/P2/5)
- Given that the coefficient of  $x^2$  in the expansion of  $(k + x)(2 - \frac{x}{2})^6$  is 84, find the value of the constant  $k$ . (N2006/P1/6)
- Write down the first three terms in the expansion, in ascending powers of  $x$ , of  $(1 + px)^5$ , where  $p$  is a constant.
  - The first three terms in the expansion of  $(1 + qx)(1 + px)^5$ , where  $p$  and  $q$  are integers, are  $1 - 10x + 15x^2$ . Obtain two equations in  $p$  and  $q$  and hence find their values. (SP08/P1/8)



# TOPIC 27 MATRICES

## FORMULAE AND IMPORTANT NOTES

1. A matrix is a rectangular array of numbers enclosed by a pair of brackets. It is designated by an upper-case alphabet in bold point. Plural for 'matrix' is 'matrices'.
2. Each number in a matrix is called an **element**.
3. (i) A  $m \times n$  or " $m$  by  $n$ " matrix has  $m$  horizontal rows and  $n$  vertical columns.  
(ii) A  $m \times 1$  matrix has  $m$  rows and only one column. It is a column matrix.  
(iii) A  $1 \times n$  matrix has only one row and  $n$  columns. It is a row matrix.  
(iv) A  $m \times m$  matrix has  $m$  rows and  $m$  columns. It is a square matrix.
4. (i) Matrices with the same number of rows and columns (same shape) belong to the same **order (type, dimension)**.  
(ii) Matrices with all corresponding elements equal are called **equal matrices**.
5. **QUICK REFERENCE**

$$(i) \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

$$(i) \quad \mathbf{A} + \mathbf{B} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$(ii) \quad \mathbf{A} - \mathbf{B} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

$$(iii) \quad k\mathbf{A} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \text{ This is called } \mathbf{scalar multiplication}. \text{ } k \text{ is the scalar.}$$

$$(iv) \quad \frac{1}{k}\mathbf{A} = \frac{\mathbf{A}}{k} = \mathbf{A} \div k = \begin{pmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{c}{k} & \frac{d}{k} \end{pmatrix}$$

6. The division of a number by a matrix and the division of a matrix by another matrix are not defined.

## 7. QUICK REFERENCE

$$(i) \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(ii) \quad \mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$(iii) \quad k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$$

$$(iv) \quad k(\mathbf{A} - \mathbf{B}) = k\mathbf{A} - k\mathbf{B}$$

$$(v) \quad (h \pm k)\mathbf{A} = h\mathbf{A} \pm k\mathbf{A}, \text{ where } h \text{ and } k \text{ are scalars.}$$

$$(vi) \quad hk\mathbf{A} = h(k\mathbf{A}) = (hk)\mathbf{A}$$

8. The **multiplication** of a matrix is possible only when **the number of rows in the right-hand matrix is equal to the number of columns in the left-hand matrix.**

$$(m \times n \text{ matrix}) \times (n \times p \text{ matrix}) = m \times p \text{ matrix}$$

**Examples of matrix multiplication:**

$$(i) \quad (a)(e) = (ae)$$

$$(ii) \quad (a) \begin{pmatrix} e & f \end{pmatrix} = \begin{pmatrix} ae & af \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} = (ae + bg)$$

$$(iv) \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \end{pmatrix}$$

$$(v) \quad \begin{pmatrix} a \\ c \end{pmatrix} (e) = \begin{pmatrix} ae \\ ce \end{pmatrix}$$

$$(vi) \quad \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} e & f \end{pmatrix} = \begin{pmatrix} ae & af \\ ce & cf \end{pmatrix}$$

$$(vii) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} ae + bg \\ ce + dg \end{pmatrix}$$

$$(viii) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

9. (i) If **AB** is possible, **BA** may or may not be possible.  
 (ii) If both are possible, **AB** may or may not be equal to **BA**.

$$(iii) \quad \mathbf{A B C} = (\mathbf{A B})\mathbf{C} = \mathbf{A}(\mathbf{B C})$$

10. Let **A** be a square matrix, and **m** and **n** be positive integers.

$$(i) \quad \mathbf{A}^m = \mathbf{A A A} \dots \text{to } m \text{ factors}$$

$$(ii) \quad \mathbf{A}^m \mathbf{A}^n = \mathbf{A}^{m+n}$$

$$(iii) \quad (\mathbf{A}^m)^n = \mathbf{A}^{mn}$$

11. A matrix in which each element is zero is called a **null matrix** or **zero matrix**. It is an additive identity denoted by **O**.

$$(i) \quad \mathbf{AO} = \mathbf{OA} = \mathbf{O}, \text{ where } \mathbf{A} \text{ and } \mathbf{O} \text{ must be of the same order.}$$

$$(ii) \quad \mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$$

$$(iii) \quad \mathbf{A} - \mathbf{O} = \mathbf{A}$$

12. A square matrix in which each element of the diagonal from left to right (called the **principal diagonal**) is unity and every other element is zero is called the **unit matrix** or **multiplicative identity matrix**. It is represented by **I**.

For a  $2 \times 2$  matrix,  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .  $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$  for all  $\mathbf{A}$ 's.

13. Let  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\mathbf{B} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$ .

(i) It can be shown that  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ .

(ii)  $ad - bc$  is called the **determinant** of matrix  $\mathbf{A}$ . It is denoted by "det  $\mathbf{A}$ ".

(iii)  $\mathbf{B} = \mathbf{A}^{-1}$  is called the **multiplicative inverse** of  $\mathbf{A}$ .

$\mathbf{A} = \mathbf{B}^{-1}$  is called the **multiplicative inverse** of  $\mathbf{B}$ .

Matrices which have inverses are **non-singular**.

(iv) If  $ad - bc = 0$ , then  $\frac{1}{ad-bc}$  is not possible. In such a case matrix  $\mathbf{A}$  has no inverse. It is said to be **singular**.

14. (i)  $\mathbf{AB} = \mathbf{O}$  does not necessarily imply that one of them ( $\mathbf{A}$  or  $\mathbf{B}$ ) must be a null matrix  $\mathbf{O}$ .

(ii)  $\mathbf{AB} = \mathbf{AC}$  does not necessarily imply that  $\mathbf{A} = \mathbf{O}$  or  $\mathbf{B} = \mathbf{C}$ .

15. Study the following before proceeding on to No.16.

If  $\mathbf{AC} = \mathbf{B}$  and  $\mathbf{A}$  is non-singular,

$$\text{then } \mathbf{A}^{-1} \mathbf{AC} = \mathbf{A}^{-1} \mathbf{B}$$

$$\mathbf{IC} = \mathbf{A}^{-1} \mathbf{B}$$

$$\mathbf{C} = \mathbf{A}^{-1} \mathbf{B}$$

16. Consider a pair of simultaneous linear equations:

$$ax + by = e$$

$$cx + dy = f$$

From them we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

$$\text{where } \begin{pmatrix} g \\ h \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

$\therefore x = g$  and  $y = h$

This is possible only if  $ad - bc \neq 0$ . If  $ad - bc = 0$ , then step two, which involves division by zero, is impossible.

If  $ad - bc = 0$ , the graphs of the two linear equations are either parallel or they coincide. They have no unique solution.

## PAST EXAMINATION QUESTIONS

1. A company produces 4 types of central heating radiator, known as types *A*, *B*, *C* and *D*.  
 A builder buys radiators for all the houses on a new estate.  
 There are 20 small houses, 30 medium-sized houses and 15 large houses.  
 A small house needs 3 radiators of type *A*, 2 of type *B* and 2 of type *C*.  
 A medium-sized house needs 2 radiators of type *A*, 3 of type *C* and 3 of type *D*.  
 A large house needs 1 radiator of type *B*, 6 of type *C* and 3 of type *D*.  
 The costs of the radiators are \$30 for type *A*, \$40 for *B*, \$50 for *C* and \$80 for *D*.  
 Using matrix multiplication twice, find the total cost to the builder of all the radiators for the estate. (N2002/P1/5)

2. Write down the inverse of the matrix  $\begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$  and use this to solve the simultaneous equations  $4x + 3y + 7 = 0$ ,  
 $7x + 6y + 16 = 0$ . (N2002/P2/1)

3. A small manufacturing firm produces four types of product, *A*, *B*, *C* and *D*. Each product requires three processes – assembly, finishing and packaging. The number of minutes required for each type of product for each process and the cost, in \$ per minute, of each process are given in the following table.

Process \ Type	Number of minutes				Cost per minute (\$)
	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	
Assembly	8	6	6	5	0.60
Finishing	5	4	3	2	0.20
Packaging	3	3	2	2	0.50

The firm receives an order for 40 of type *A*, 50 of type *B*, 50 of type *C* and 60 of type *D*. Write down three matrices such that matrix multiplication will give the total cost of meeting this order. Hence evaluate this total cost. (N2003/P1/7)

4. Given  $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$ , write down the inverse of  $\mathbf{A}$  and of  $\mathbf{B}$ .  
 Hence find  
 (i) the matrix  $\mathbf{C}$  such that  $2\mathbf{A}^{-1} + \mathbf{C} = \mathbf{B}$ ,  
 (ii) the matrix  $\mathbf{D}$  such that  $\mathbf{BD} = \mathbf{A}$ . (N2003/P2/7)

5. Given that  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix}$ , find  $\mathbf{A}^{-1}$  and hence solve the simultaneous equations  $2x + 3y + 4 = 0$ ,  
 $-5x + 4y + 13 = 0$ . (N2004/P2/1)

6. It is given that  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix}$  and that  $\mathbf{A} + \mathbf{A}^{-1} = k\mathbf{I}$ , where  $p$  and  $k$  are constants and  $\mathbf{I}$  is the identity matrix. Evaluate  $p$  and  $k$ . (N2005/P2/6)

7. A large airline has a fleet of aircraft consisting of 5 aircraft of type *A*, 8 of type *B*, 4 of type *C* and 10 of type *D*. The aircraft have 3 classes of seat known as Economy, Business and First. The table below shows the number of these seats in each of the 4 types of aircraft.

Type of aircraft \ Class of seat	Economy	Business	First
	<i>A</i>	300	60
<i>B</i>	150	50	20
<i>C</i>	120	40	0
<i>D</i>	100	0	0

- (i) Write down two matrices whose product shows the total number of seats in each class.  
 (ii) Evaluate this product of matrices.

On a particular day, each aircraft made one flight. 5% of the Economy seats were empty, 10% of the Business seats were empty and 20% of the First seats were empty.

- (iii) Write down a matrix whose product with the matrix found in part (ii) will give the total number of empty seats on that day.  
 (iv) Evaluate this total. (N2006/P1/5)

8. Given that  $A = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ , use the inverse matrix of **A** to

(i) solve the simultaneous equations  $y - 4x + 8 = 0$ ,  
 $2y - 3x + 1 = 0$ ,

(ii) find the matrix **B** such that  $BA = \begin{pmatrix} -2 & 3 \\ 9 & -1 \end{pmatrix}$ . (N2006/P2/8)

# TOPIC 28 TRIGONOMETRIC FUNCTIONS FOR THE GENERAL ANGLE

## IMPORTANT NOTES AND FORMULAE

- In the first quadrant all the trigonometric functions are positive.  
In the second quadrant only sine and cosecant are positive.  
In the third quadrant only tangent and cotangent are positive.  
In the fourth quadrant only cosine and secant are positive.
- Any trigonometric function of  $(n \cdot 90^\circ \pm x)$ , where  $n$  is an even integer, is equal to the same functions of  $x$ , with the sign depending upon the quadrant in which the angle lies.  
Any trigonometric function of  $(n \cdot 90^\circ \pm x)$ , where  $n$  is an odd integer, is equal to the co-function of  $x$ , with the sign depending upon the quadrant in which the angle lies.

Trigonometric Functions	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$360^\circ$
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cosecant	$\mp \infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\mp \infty$	-1	$\mp \infty$
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$\mp \infty$	-1	$\mp \infty$	1
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\mp \infty$	0	$\mp \infty$	0
cotangent	$\mp \infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\mp \infty$	0	$\mp \infty$

- The ratio  $\frac{\text{Length of arc}}{\text{Radius of circle}}$  is always the same for a given central angle. This ratio is the circular measure of the angle. Its unit is a radian.  
 $2\pi$  radians =  $360^\circ$ ,  $\pi$  radians =  $180^\circ$ ,  $r$  radians =  $\frac{r}{\pi} 180^\circ$ ,  $d$  degrees =  $\frac{d}{180} \pi$  radians.

## WORKED EXAMPLES

- State the limits between which  $B$  must lie if  $B$  is acute and  $\cos 3B$  is negative.

$$B \text{ is acute} \rightarrow 0^\circ < B < 90^\circ \quad \dots(i)$$

$$\text{and } 0^\circ < 3B < 270^\circ$$

$$\cos 3B \text{ is negative} \rightarrow 90^\circ < 3B < 270^\circ$$

$$\text{and } 30^\circ < B < 90^\circ \quad \dots(ii)$$

$$\text{From (i) and (ii), } 30^\circ < B < 90^\circ$$

2. Find the value of  $\tan 1020^\circ$  without using tables or calculators.

$$\begin{aligned}\tan 1020^\circ &= \tan (990^\circ + 30^\circ) \\ &= \tan (11 \times 90^\circ + 30^\circ) \\ &= -\cot 30^\circ\end{aligned}$$

Since 11 is an odd integer and  $1020^\circ$  is in the fourth quadrant (or the "twelfth" quadrant).

$$\tan 1020^\circ = -\sqrt{3}$$

3. Given that  $\sin A$  is 0.6,  $\tan A$  is negative and  $0^\circ < A < 360^\circ$ , find the value of  $\cos \frac{1}{2}A$ .

$$\sin A > 0 \text{ and } \tan A < 0, \text{ therefore } 90^\circ < A < 180^\circ$$

$$\text{From the calculator } A = \sin^{-1}(0.6) = -36.9^\circ + 180^\circ = 143.1^\circ$$

$$\cos \frac{1}{2}A = \cos \frac{1}{2}(143.1^\circ) = 0.316$$

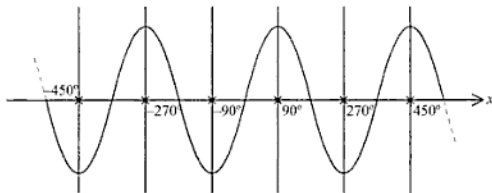
## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

- (ii) Every vertical line through  $x = \pm 90^\circ, \pm 270^\circ, \pm 450^\circ, \dots$  (or  $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$ ) is an axis of reflective symmetry.

With these values of  $x$ ,  $\sin(x + \alpha) = \sin(x - \alpha)$  for any  $\alpha$ .

$$y = \sin x^\circ$$

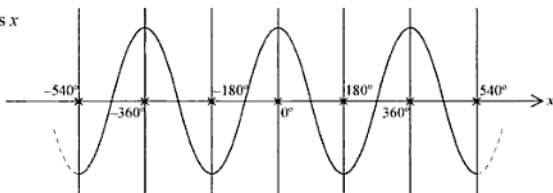


### 3. Symmetry of Cosine Curve

- (i) Every vertical line through  $x = 0^\circ, \pm 180^\circ, \pm 360^\circ, \dots$  (or  $0, \pm\pi, \pm 2\pi, \dots$ ) is an axis of reflective symmetry.

For these values of  $x$ ,  $\cos(x + \alpha) = \cos(x - \alpha)$  for any  $\alpha$ .

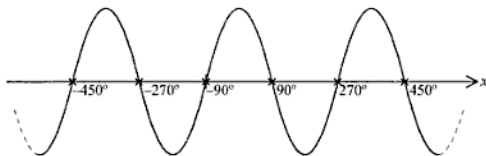
$$y = \cos x$$



- (ii) Every point on the  $x$ -axis with  $x = \pm 90^\circ, \pm 270^\circ, \pm 450^\circ, \dots$  (or  $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$ ) is a centre of point symmetry.

For these values of  $x$ ,  $\cos(x - \alpha) = -\cos(x + \alpha)$  for any  $\alpha$ .

$$y = \cos x$$



### 4. To transform the graph of $f(x)$ , where $f(x) = \sin x$ or $\cos x$ , into the graph of

- (i)  $f(x) + b$ :

Translate the graph of  $f(x)$  vertically  $b$  units upward if  $b$  is positive and  $b$  units downward if  $b$  is negative. The amplitude and period are unchanged.

Alternative method: Translate the  $x$ -axis of the graph of  $f(x)$   $b$  units in the opposite direction. Relabel the  $y$ -axis.



(ii)  $f(x + b)$  :

Translate the graph of  $f(x)$  horizontally  $b$  units to the left if  $b$  is positive and  $b$  units to the right if  $b$  is negative.

The amplitude and period remain unchanged.

Alternative method : Translate the  $y$ -axis of the graph of  $f(x)$   $b$  units in the opposite direction. Relabel the  $x$ -axis.

(iii)  $-f(x)$  :

Reflect the graph of  $f(x)$  in the  $x$ -axis. The amplitude and period remain unchanged.

(iv) Graph of  $bf(x)$  :

Sketch the graph of  $f(x)$  vertically. Scale factor is  $b$  and the  $x$ -axis remains invariant. The amplitude is now  $b$  but the period remains unchanged.

Alternative method if  $b$  is positive: Multiply the label of the  $y$ -axis of  $f(x)$  by  $b$ .

(v) Graph of  $f(-x)$  :

Reflect the graph of  $f(x)$  in the  $y$ -axis. The amplitude and period remain unchanged.

(vi) Graph of  $f(bx)$  :

Stretch the graph of  $f(x)$  horizontally. Scale factor is  $\frac{1}{b}$  and the  $y$ -axis remains unchanged. The amplitude remains unchanged. The period is  $\frac{2\pi}{b}$  radian or  $\frac{360^\circ}{b}$ .

Alternative method if  $b$  is positive : Divide the label of the  $x$ -axis of  $y = f(x)$  by  $b$ .

(vii) Graph of  $|f(x)|$  :

Reflect about the  $x$ -axis any part of the graph of  $f(x)$  which is below the  $x$ -axis.

(viii) Graph of  $\frac{1}{f(x)}$  :

The graph of  $f(x)$  and that of reciprocal function  $\frac{1}{f(x)}$  are related in the following ways:

(a) For the same value of  $x$  they have the same sign.

(b) Both functions equal  $\pm 1$  simultaneously.

(c) Both functions cannot be zero for any value of  $x$ .

(d) When one function approaches zero the other approaches infinity.

(e) When one function decreases the other increases.

## PAST EXAMINATION QUESTIONS

1. The function  $f$  is defined, for  $0 < x < \pi$ , by  $f(x) = 5 + 3 \cos 4x$ . Find

(i) the amplitude and the period of  $f$ ,

(ii) the coordinates of the maximum and minimum points of the curve  $y = f(x)$ .

(N2004/P1/6)

2. The function  $f$  is given by  $f : x \mapsto 2 + 5 \sin 3x$  for  $0^\circ \leq x \leq 180^\circ$ .

(i) State the amplitude and period of  $f$ .

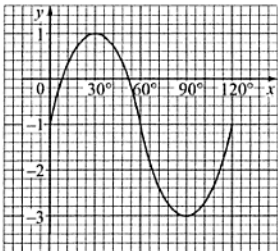
(ii) Sketch the graph of  $y = f(x)$ .

(N2005/P2/4)

3. The diagram shows part of the graph of  $y = a \sin(bx) + c$ .

State the value of

- (i)  $a$ ,
- (ii)  $b$ ,
- (iii)  $c$ .



(N2006/P1/2)

4. The function  $f$  is defined, for all values of  $x$ , by  $f(x) = 2 \cos\left(\frac{x}{2}\right) - 1$ .

(i) State the amplitude and period of  $f$ .

The function  $g$  is defined, for  $0^\circ \leq x \leq 360^\circ$ , by  $g(x) = 2 \cos\left(\frac{x}{2}\right) - 1$ .

- (ii) Find the  $x$ -coordinates of the point where the graph of  $y = g(x)$  crosses the  $x$ -axis.
- (iii) Sketch the graph of  $y = g(x)$ .
- (iv) Sketch the graph of  $y = |g(x)|$ .

(SP08/P2/2)

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# TOPIC 30 TRIGONOMETRIC IDENTITIES AND EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

1. Reciprocal Identities:

$$\sin x \operatorname{cosec} x = 1 \quad \cos x \sec x = 1 \quad \tan x \cot x = 1$$

2. Quotient Identities:

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

3. Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \operatorname{cosec}^2 x$$

4. Addition And Subtraction Identities:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad \equiv \cos(y - x)$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

5. Double -angle Identities:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\equiv 1 - 2 \sin^2 x$$

$$\equiv 2 \cos^2 x - 1$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

6. Treble-angle Identities:

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

7. Half-angle Identities:

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad \cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

8.  $a \sin x + b \cos x \equiv \sqrt{a^2 + b^2} \left( \frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x \right)$   
 $\equiv \sqrt{a^2 + b^2} (\sin \alpha \sin x + \cos \alpha \cos x)$  where  $\alpha = \tan^{-1}(\frac{a}{b})$   
 $\equiv \sqrt{a^2 + b^2} \cos(x - \alpha)$

$$\text{Maximum value of } a \sin x + b \cos x = \sqrt{a^2 + b^2}$$

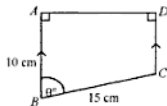
This happens when  $x - \alpha = \text{multiples of } 360^\circ$ .

$$\text{Minimum value of } a \sin x + b \cos x = -\sqrt{a^2 + b^2}$$

This happens when  $x - \alpha = \text{odd multiples of } 180^\circ$ .

## PAST EXAMINATION QUESTIONS

- Given that  $\sin \beta = p$  where  $\beta$  is an acute angle measured in degrees, obtain an expression, in terms of  $p$ , for (i)  $\tan \beta$ , (ii)  $\sin(90^\circ - \beta)$ , (iii)  $\sin(180^\circ + \beta)$ . (N98/P1/8)
- Prove the identity  $(1 + \operatorname{cosec} \theta)(1 - \sin \theta) \equiv \cos \theta \cot \theta$ . (N98/P1/9)
- Given that  $\frac{\cos(A-B)}{\cos(A+B)} = -\frac{9}{7}$ , find the value of  $\tan A$  and  $\tan B$ . (N98/P2/4c)
- The parametric equations of a curve are  $x = 3 \sin \alpha + \cos \alpha$ ,  $y = \sin \alpha - 2 \cos \alpha$ . Express each of  $\sin \alpha$  and  $\cos \alpha$  in terms of  $x$  and  $y$ . Hence obtain the cartesian equation of the curve.
  - The cartesian equation of a curve is  $(y - 3)^2 = 1 + x^2$ . Given that  $x$  may be defined parametrically by  $x = \cot \theta$ , and that  $y = 1$  when  $\theta = \frac{\pi}{6}$ , express  $y$  in terms of  $\operatorname{cosec} \theta$ . (N98/P2/8b, c)
- Show that  $(\tan \theta + \sin \theta)(\tan \theta - \sin \theta) = \tan^2 \theta \sin^2 \theta$ . (N99/P1/6)
- Show that  $4 \sin A \cos^3 A - 4 \cos A \sin^3 A = \sin 4A$ .
  - The trapezium  $ABCD$  is right angled at  $A$  and at  $D$ , and  $AB$  is parallel to  $DC$ . The acute angle  $ABC$  is  $\theta^\circ$ ,  $AB = 10$  cm and  $BC = 15$  cm.
    - Express  $AD$  and  $DC$  in terms of  $\theta$ .
    - Find the value of  $\theta$  for which the perimeter is 45 cm.
- Given that  $\frac{\sin(A-B)}{\sin(A+B)} = 2$ , find the value of  $\tan A \cot B$ . (N99/P2/5)
- Find the value of each of the constants  $a$  and  $b$  for which  $\sin x \cos x (5 \tan x + 2 \cot x) \equiv a + b \sin^2 x$ . (N2000/P1/9)
- Prove that  $\cot A + \tan A = \sec A \operatorname{cosec} A$ . (N01/P1/4)
- Given that  $\sin(A+B) = 2 \sin(A-B)$ , express  $\tan A$  in terms of  $\tan B$ . (N01/P2/5c)
- Show that  $\cos \theta \left( \frac{1}{1 - \sin \theta} - \frac{1}{1 + \sin \theta} \right)$  can be written in the form  $k \tan \theta$  and find the value of  $k$ . (N2003/P2/2)
- Prove the identity  $\cos x \cot x + \sin x \equiv \operatorname{cosec} x$ . (N2006/P2/2)
- Given that  $p = \cos A + \sin A$ ,  $q = \cos A - \sin A$  and that  $A$  is measured in degrees,
  - find the value of  $p^2 + q^2$ ,
  - show that  $\frac{p}{q} = \tan(45^\circ + A)$ . (SP08/P1/2)



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# TOPIC 31 SOLUTION OF TRIGONOMETRIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

- If  $\sin(ax + b) = y$  and let  $\theta = \sin^{-1}y$  be the basic angle obtained from the inverse trigonometric function key of a calculator, then  $ax + b = \theta + n360^\circ$  or  $(-\theta + 180^\circ) + n360^\circ$  where  $n$  is an integer.
- Similarly for  $\cos(ax + b) = y$   
 $ax + b = \theta + n360^\circ$  or  $(-\theta + 360^\circ) + n360^\circ$
- If  $\tan(ax + b) = y$   
 $ax + b = \theta + n180^\circ$
- If solutions are required in radians,  $\pi$  and  $2\pi$  must replace  $180^\circ$  and  $360^\circ$  respectively.
- If  $0^\circ \leq x \leq 360^\circ$ ,  $\sin(ax + b) = \pm y$  and  $\theta = \sin^{-1}y$  is the basic angle, then  $ax + b = \theta, -\theta + 180^\circ, \theta + 180^\circ, -\theta + 360^\circ$ . The same is true for  $\cos(ax + b)$  and  $\tan(ax + b)$ .
- Solve the equation  $a \sin x + b \cos x = c$  where  $0^\circ \leq x \leq 360^\circ$ .

Solution:  $a \sin x + b \cos x = c$

$$\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\sin \alpha \sin x + \cos \alpha \cos x = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\text{where } \alpha = \tan^{-1}\left(\frac{a}{b}\right)$$

$$\therefore \cos(x - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}$$

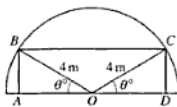
$$x = \cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) + \alpha$$

Notice that there are two values of  $\cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right)$ .

## PAST EXAMINATION QUESTIONS

- Find all the angles, between  $0^\circ$  and  $360^\circ$ , which satisfy the equation (i)  $16 \sin x - 8 \sin^2 x = 5 \cos^2 x$ , (ii)  $4 \sin y \cos y - 3 \cos^2 y = 0$ , (iii)  $\sec\left(\frac{3x}{2} - 18^\circ\right) + 2 = 0$ . (N97/P1/14)
- Find all the angles between  $0^\circ$  and  $360^\circ$  which satisfy the equation (i)  $2 \cos 2x = 4 \sin x + 3$ , (ii)  $\sin(y + 30^\circ) = 3 \cos y$ . (N97/P2/5a)
- (a) Find all the angles between  $0^\circ$  and  $360^\circ$  which satisfy the equation (i)  $2 \sin 2x + 1 = 0$ , (ii)  $\sec y(1 + \tan y) = 6 \operatorname{cosec} y$ .  
(b) Find all the values of  $t$  between 0 and 10, for which  $\cos\left(\frac{\pi t}{5}\right) = 0.6$ , where  $\frac{\pi t}{5}$  is measured in radians. (N98/P1/12)
- Solve the equation  $8x^3 - 2x^2 - 5x - 1 = 0$ . Hence find the values of  $\theta$ , between  $0^\circ$  and  $180^\circ$ , which satisfy the equation  $8 \tan^2 \theta - 2 \tan \theta - 5 = \cot \theta$ . (N98/P2/1b)

12. (a) Solve, for  $0^\circ < x < 360^\circ$ , the equation  $4 \tan^2 x + 8 \sec x = 1$ .  
 (b) Given the  $y < 4$ , find the largest value of  $y$  such that  $5 \tan(2y + 1) = 16$ . (N2003/P1/9)
13. (a) Solve, for  $0^\circ \leq x \leq 360^\circ$ , the equation  $\sin^2 x = 3 \cos^2 x + 4 \sin x$ .  
 (b) Solve, for  $0 < y < 4$ , the equation  $\cot 2y = 0.25$ , giving your answers in radians correct to 2 decimal places. (N2004/P1/9)
14. Given that  $x = 3\sin\theta - 2\cos\theta$  and  $y = 3\cos\theta + 2\sin\theta$ ,  
 (i) find the value of the acute angle  $\theta$  for which  $x = y$ ,  
 (ii) show that  $x^2 + y^2$  is constant for all values of  $\theta$ . (N2004/P2/6)
15. (a) Find all the angles between  $0^\circ$  and  $360^\circ$  which satisfy the equation  $3 \cos x = 8 \tan x$ .  
 (b) Given that  $4 \leq y \leq 6$ , find the value of  $y$  for which  $2\cos\left(\frac{2y}{3}\right) + \sqrt{3} = 0$ . (N2005/P1/9)
16. (a) Solve, for  $0^\circ \leq x \leq 360^\circ$ , the equation  $2\cot x = 1 + \tan x$ .  
 (b) Given that  $y$  is measured in radians, find the two smallest positive values of  $y$  such that  $6\sin(2y + 1) + 5 = 0$ . (N2006/P1/11)
17. Find all the angles between  $0^\circ$  and  $180^\circ$  which satisfy the equation  $2 \cos x - 3 \sin 2x = 0$ . (SP08/P1/1)
18. The diagram shows a rectangle  $ABCD$  inside a semicircle, centre  $O$  and radius 4 m, such that angle  $BOA =$  angle  $COD = \theta^\circ$ . The perimeter of the rectangle is  $P$  m.  
 (i) Show that  $P = 16\cos\theta^\circ + 8\sin\theta^\circ$ .  
 (ii) Express  $P$  in the form  $R \cos(\theta^\circ - \alpha^\circ)$ .  
 (iii) Find the maximum value of  $P$  and the corresponding value of  $\theta$ .  
 (iv) Find the value of  $\theta$  for which  $P = 15$ .



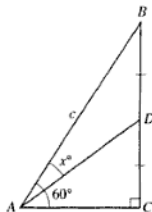
(SP08/P2/11)

## PAST EXAMINATION QUESTIONS

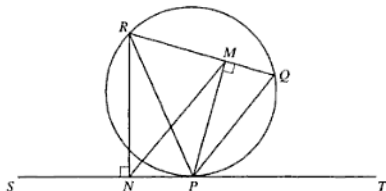
1. The diagram shows a triangle  $ABC$  in which  $AB = c$ , angle  $BAC = 60^\circ$  and angle  $BCA = 90^\circ$ . The mid-point of  $BC$  is  $D$  and angle  $BAD = x^\circ$ . Show that

(i) the length of  $AD$  is exactly  $\frac{\sqrt{3}c}{4}$ ,

(ii)  $x^\circ = 60^\circ - \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ . (SP08/P1/6)



2.



In the diagram above  $SPT$  is a tangent to a circle at the point  $P$ . The points  $Q$  and  $R$  lie on the circle. The line  $PM$  is perpendicular to the chord  $QR$  and the line  $RN$  is perpendicular to the tangent  $SPT$ .

- (i) By considering  $QP$  as a chord of the circle, find, with explanation, an angle equal to angle  $QPT$ .
- (ii) Explain why a circle with  $PR$  as diameter passes through  $M$  and  $N$ .
- (iii) Prove that the lines  $MN$  and  $QP$  are parallel. (SP08/P2/7)

# TOPIC 33 CIRCLES, SECTORS AND SEGMENTS

## FORMULAE AND IMPORTANT NOTES

$r$  = radius

$d$  = diameter =  $2r$

$R$  = radius

$p$  = perimeter

$c$  = circumference

$a$  = arc

$\theta$  = angle at the centre

$A$  = area

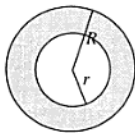
1. Circle



$$c = p = \pi d = 2\pi r$$

$$A = \pi r^2 = \frac{1}{4}\pi d^2$$

2. Annulus



$$c = 2\pi R + 2\pi r$$

$$= 2\pi(R + r)$$

$$A = \pi R^2 - \pi r^2$$

$$= \pi(R^2 - r^2)$$

$$= \pi(R + r)(R - r)$$

3. Sector



$$p = 2r + a$$

$\theta$  measured in degrees :

$$a = \frac{\theta}{360} 2\pi r = \frac{\theta}{180} \pi r$$

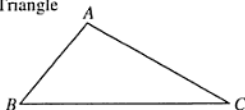
$$A = \frac{\theta}{360} \pi r^2 = \frac{1}{2} ar$$

$\theta$  measured in radians :

$$a = r\theta$$

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} ar$$

4. Triangle



$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

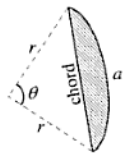
$$= \frac{1}{2} AB \times AC \times \sin A$$

$$= \frac{1}{2} AB \times BC \times \sin B$$

$$= \frac{1}{2} BC \times CA \times \sin C$$



5. Minor segment



$$p = a + \text{chord}$$

$$= a + 2r \times \sin \frac{\theta}{2}$$

$A$  = Area of Sector – Area of triangle

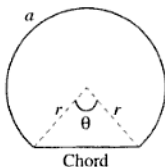
$\theta$  measured in degrees :

$$A = \frac{\theta}{360} \pi r^2 - \frac{1}{2} r^2 \times \sin \theta$$

$\theta$  measured in radians :

$$A = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \times \sin \theta = \frac{1}{2} r^2 (\theta - \sin \theta)$$

6. Major segment



$\theta$  measured in degrees:

$$p = a + \text{chord}$$

$$= \frac{360 - \theta}{180} \pi r + 2r \sin \frac{\theta}{2}$$

$A$  = area of major sector + area of triangle

$$= \frac{1}{2} r^2 \sin \theta + \frac{360 - \theta}{360} \pi r^2$$

$\theta$  measured in radians:

$$P = (2\pi - \theta) r + 2r \sin \frac{\theta}{2}$$

$$= r (2\pi - \theta + 2 \sin \frac{\theta}{2})$$

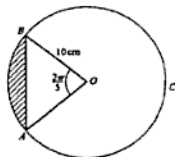
$$A = \frac{1}{2} r^2 \sin \theta + \frac{1}{2} (2\pi - \theta) r^2$$

$$= \frac{1}{2} r^2 (\sin \theta + 2\pi - \theta)$$

## PAST EXAMINATION QUESTIONS

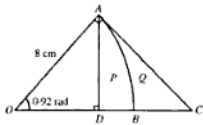
1. The figure shows a circle, centre  $O$ , radius 10 cm, and a chord  $AB$  such that angle  $AOB = \frac{2\pi}{5}$  radians. Calculate

- (i) the length of the major arc  $ACB$ ,  
 (ii) the area of the shaded region. (N97/P1/9)

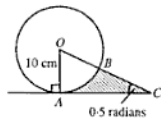


2. In the diagram,  $OAB$  is a sector of a circle, centre  $O$ , of radius 8 cm and angle  $AOB = 0.92$  radians. The line  $AD$  is the perpendicular from  $A$  to  $OB$ . The line  $AC$  is perpendicular to  $OA$  and meets  $OB$  produced at  $C$ . Find

- (i) the perimeter of the region  $ADB$ , marked  $P$ ,  
 (ii) the area of the region  $ABC$ , marked  $Q$ . (N98/P1/10)

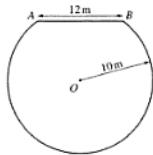


3. The diagram shows a circle, centre  $O$ , of radius 10 cm. The line  $AC$  is perpendicular to the radius  $OA$ , and the line  $OC$  intersects the circle at  $B$ . Given that angle  $OCA$  is 0.5 radians, calculate (i) the length of  $AC$ , (ii) the area of the shaded region, (iii) the perimeter of the shaded region. (N99/P1/10)



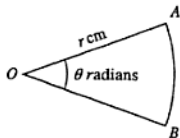
4. The diagram shows a major segment of a circle, centre  $O$ , radius 10 m. The chord  $AB$  is of length 12 m. Calculate

- (i) the perimeter of the segment,  
 (ii) the area of the segment. (N2000/P1/6)



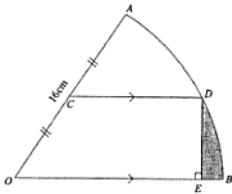
5. The diagram shows a sector,  $AOB$ , of a circle, centre  $O$ , radius  $r$  cm, where the acute angle  $AOB$  is  $\theta$  radians. Given that the perimeter of the sector is 14 cm and that the area of the sector is  $10 \text{ cm}^2$ , evaluate  $r$  and  $\theta$ .

(N01/P1/6)



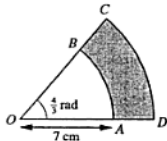
6. In the diagram,  $OAB$  is a sector of a circle, centre  $O$  and radius 16 cm, and the length of the arc  $AB$  is 19.2 cm. The mid-point of  $OA$  is  $C$  and the line through  $C$  parallel to  $OB$  meets the arc  $AB$  at  $D$ . The perpendicular from  $D$  to  $OB$  meets  $OB$  at  $E$ .

- (i) Find angle  $AOB$  in radians.  
 (ii) Find the length of  $DE$ .  
 (iii) Show that angle  $DOE$  is approximately 0.485 radians.  
 (iv) Find the area of the shaded region. (N2003/P2/10)



7. The diagram shows a sector  $COD$  of a circle, centre  $O$ , in which angle  $COD = \frac{4}{3}$  radians. The points  $A$  and  $B$  lie on  $OD$  and  $OC$  respectively, and  $AB$  is an arc of a circle, centre  $O$ , of radius 7 cm. Given that the area of the shaded region  $ABCD$  is  $48 \text{ cm}^2$ , find the perimeter of this shaded region.

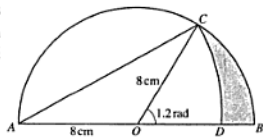
(N2004/P1/4)



8. The diagram shows a semicircle, centre  $O$ , of radius 8 cm. The radius  $OC$  makes an angle of 1.2 radians with the radius  $OB$ . The arc  $CD$  of a circle has centre  $A$  and the point  $D$  lies on  $OB$ . Find the area of

- (i) sector  $COB$ ,  
 (ii) sector  $CAD$ ,  
 (iii) the shaded region.

(N2005/P1/12OR)



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# TOPIC 34 VECTOR GEOMETRY

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## FORMULAE AND IMPORTANT NOTES

1. A scalar quantity is one which is completely determined by a single number, its magnitude (size). It has no direction in space.
2. A vector quantity has magnitude (modulus), in the ordinary algebraic sense, as well as direction in space. It can be represented by a directed line segment whose direction is that of the vector and whose length is proportional to its magnitude.
3. Vectors are usually printed in boldface type:

**a** : the vector **a**

**AB** : the vector represented in magnitude and direction by the directed line segment **AB**.

The magnitude of vectors are denoted by modulus signs :

$|\mathbf{a}|$  : the magnitude of **a**

$|\mathbf{AB}|$  : the magnitude of **AB**

In writing and printing,  $\vec{a}$ ,  $\vec{a}$ ,  $\vec{a}$ ,  $\vec{AB}$ ,  $\vec{AB}$ ,  $\vec{AB}$  may be used to represent vector and  $a$ ,  $AB$ ,  $|a|$ ,  $|\vec{AB}|$  may be used to represent magnitudes.

4. A zero vector (null vector) has zero magnitude and no specific direction. It is represented by a point and symbolised by **0**. When it is added to any other vector the vector is unchanged.  $\mathbf{AB} + \mathbf{0} = \mathbf{AB}$ .
5. (i) Triangle Law : The sum (resultant) of any two vectors (called components) of the same kind represented by **AB** and **BC** is a vector represented by **AC**.  
(ii) Parallelogram Law: The sum of two component vectors **AB** and **AD** drawn from any point **A** is represented by the diagonal **AC** of the parallelogram **ABCD**.  
(iii) Resultant by rectangular resolution : If  $A = (0, 0)$ ,  $B = (x_1, y_1)$ ,  $C = (x_2, y_2)$  then **AD**, the sum of **AB** and **AC** has its terminal point (terminus) at  $(x_1 + x_2, y_1 + y_2)$ .
6. The Polygon Law: A set of three or more vectors may be added by forming a sequence in which the vectors are placed tip-to-tail in any order. The vector joining the unattached tail to the unattached tip is the resultant.
7. A vector can be resolved into any number of  $n$  components, where  $n$  is a positive integer, in infinite number of ways, by drawing a polygon of  $(n + 1)$  sides according to the Polygon Law.
8. Addition of vectors is both commutative and associative :  
(i)  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ ,  
(ii)  $\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ .
9. A vector which has the same magnitude and parallel to **AB** (or  $a$ ) but is of the opposite sense is called the negative of **AB**. It is designated as  $-\mathbf{AB}$  (or  $-\mathbf{a}$ ).  $\mathbf{AB} + (-\mathbf{AB}) = \mathbf{0}$ .  $-\mathbf{AB} = \mathbf{BA}$ .
10. The subtraction of vectors is defined as the addition of their negatives :  
(i)  $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ ,  
(ii)  $\mathbf{a} - \mathbf{a} = \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ .

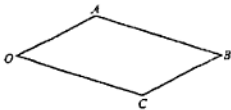
## PAST EXAMINATION QUESTIONS

1. The diagram shows the parallelogram  $OABC$ . Given that

$$\vec{OA} = 3\mathbf{i} + \mathbf{j} \text{ and that } \vec{OC} = 4\mathbf{i} - 2\mathbf{j},$$

(i) find  $\vec{OB}$ .

- (ii) use a scalar product to find the acute angle between the diagonals of the parallelogram. (N97/P1/8)



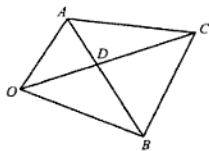
2. The position vectors of the points  $A$ ,  $B$  and  $C$ , relative to an origin  $O$ , are  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{a} + 2\mathbf{b}$  respectively.  $AB$  and  $OC$

meet at  $D$ , where  $\frac{AD}{AB} = p$  and  $\frac{OD}{OC} = q$ . Express  $\vec{OD}$  in terms of (i)  $\mathbf{a}$ ,  $\mathbf{b}$  and  $p$ , (ii)  $\mathbf{a}$ ,  $\mathbf{b}$  and  $q$ . Hence evaluate  $p$

and  $q$ . Given that  $\mathbf{a} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ ,  $k > 0$  and that  $\mathbf{b} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ ,

find the value of  $k$  for which angle  $ODA = 90^\circ$ .

(N97/P1/16)



3. (a) Given that  $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$  and  $\mathbf{c} = \begin{pmatrix} p \\ p+2 \end{pmatrix}$ , find (i) the value of  $\mathbf{a} \cdot \mathbf{b}$ , (ii) the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , (iii) the value of  $p$  for which  $\mathbf{b}$  and  $\mathbf{c}$  are perpendicular.

- (b) The position vectors of the points  $L$ ,  $M$  and  $N$ , relative to an origin  $O$ , are  $\mathbf{d}$ ,  $\mathbf{e}$  and  $2\mathbf{d} + 2\mathbf{e}$  respectively. The point  $P$  lies on  $LM$  and is such that  $\vec{LP} = \frac{2}{3}\vec{LM}$ . The line  $OP$  is produced to meet the line  $LN$  at  $Q$ . Given that  $\vec{OQ} = \lambda\vec{OP}$  and that  $\vec{LQ} = \mu\vec{LN}$ , express  $\vec{OQ}$  in terms of (i)  $\lambda$ ,  $\mathbf{d}$  and  $\mathbf{e}$ , (ii)  $\mu$ ,  $\mathbf{d}$  and  $\mathbf{e}$ . Hence determine the value of  $\lambda$  and of  $\mu$ .

(N98/P1/15)

4. Relative to an origin  $O$  the position vectors of the points  $P$  and  $Q$  are  $3\mathbf{i} + \mathbf{j}$  and  $7\mathbf{i} - 15\mathbf{j}$  respectively. Given that  $R$  is the point such that  $3\vec{PR} = \vec{RQ}$ , find a unit vector in the direction  $\vec{OR}$ . (N99/P1/3)

5. The position vectors of points  $P$  and  $Q$  relative to an origin  $O$  are  $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} k \\ 4 \end{pmatrix}$  respectively. Given that  $\vec{QP}$  is perpendicular to  $\vec{OP}$ , use a scalar product to find (i) the value of  $k$ , (ii) the angle  $OQP$ . The position vector of the point  $R$  relative to the origin  $O$  is  $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$ .

Given that  $X$  is a point on  $RP$  such that  $\vec{RX} = \lambda\vec{RP}$ , (iii) express  $\vec{OX}$  as a column vector in terms of  $\lambda$ . Given also that  $\vec{OX} = \mu\vec{OQ}$ , (iv) find the value of  $\lambda$  and of  $\mu$ . (N99/P1/16)

6. (a) Find the positive value of  $p$  for which  $0.6\mathbf{i} + p\mathbf{j}$  is a unit vector.  
 (b) The vector  $3\mathbf{i} + 4\mathbf{j}$  is parallel to the vector  $\mathbf{a} + 3\mathbf{b}$ , where  $\mathbf{a} = q\mathbf{i} - \mathbf{j}$  and  $\mathbf{b} = \mathbf{i} + q\mathbf{j}$ . Find the value of  $q$ . (N2000/P1/16)

7. Solutions to this question by accurate drawing will not be accepted.

- (a) The four points  $O$ ,  $A$ ,  $B$  and  $D$  are such that  $\vec{OA} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ ,  $\vec{OB} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$  and

$$\vec{OD} = \begin{pmatrix} d \\ 3d \end{pmatrix}.$$

- (i) Show that  $O$ ,  $A$  and  $B$  lie on the same straight line.  
 (ii) Find the value of  $d$  given that  $|\vec{AD}| = |\vec{BD}|$ .
- (b) The position vectors of the points  $P$  and  $Q$  relative to an origin  $O$  are  $\mathbf{p}$  and  $\mathbf{q}$  respectively. The point  $R$  is such that  $\vec{OR} = \frac{3}{4}\vec{OP}$  and the point  $S$  is mid-point of  $PQ$ .

- (i) Express  $\vec{RQ}$  and  $\vec{OS}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ .

$RS$  is produced to meet  $OQ$  produced at the point  $T$  so that  $\vec{ST} = \lambda\vec{RS}$  and  $\vec{QT} = \mu\vec{OQ}$ .

Express  $\vec{ST}$  in terms of (i)  $\lambda$ ,  $\mathbf{p}$  and  $\mathbf{q}$ , (ii)  $\mu$ ,  $\mathbf{p}$  and  $\mathbf{q}$ .

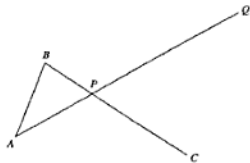
Hence find the value of  $\lambda$  and of  $\mu$ .

(N2000/P1/16)

8. (a) Points  $A$ ,  $B$  and  $C$  have positive vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  respectively, relative to an origin  $O$ . The point  $P$  lies on  $BC$  such that  $BP : PC = 1 : 2$ . The point  $Q$  lies on  $AP$  produced such that  $AP : PQ = 1 : 2$ . Find, in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ ,

(i)  $\vec{OP}$ ,

(ii)  $\vec{OQ}$ .



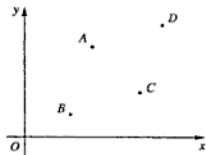
Show that  $\vec{CQ}$  is parallel to  $\vec{AB}$ .

- (b) The point  $A$ ,  $B$ ,  $C$  and  $D$ , shown in diagram, are the vertices of a parallelogram  $ABCD$ . The positive vectors of  $A$ ,  $B$  and  $C$ , relative to  $O$ , are  $3\mathbf{i} + 4\mathbf{j}$ ,  $2\mathbf{i} + \mathbf{j}$  and  $5\mathbf{i} + 2\mathbf{j}$  respectively. Find

(i)  $\vec{AC}$ ,

(ii)  $\vec{BD}$ ,

(iii)  $\vec{OD}$ .



The point  $A'$  is the reflection of  $A$  in the  $x$ -axis.

- (iv) Show that the points  $A'$ ,  $C$  and  $D$  are collinear and find the ratio  $A'C : CD$ .

(N01/P1/14)

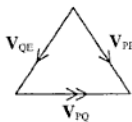
# TOPIC 35 VELOCITY VECTORS

## FORMULAE AND IMPORTANT NOTES

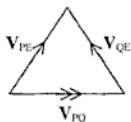
- In elementary problems on velocity vectors, the earth is assumed to be stationary.
- The velocity of a moving object  $P$  relative to the earth can be denoted by  $\mathbf{V}_{PE}$ , or simply,  $\mathbf{V}_P$ . It is the "actual" velocity of  $P$  as measured by an observer who is not moving relative to the earth.
- (i)  $\mathbf{V}_{PE}$  is a vector quantity. It must be specified by its direction and magnitude (speed). The magnitude of  $\mathbf{V}_{PE}$  is  $V_{PE}$ . In written form they can be denoted by  $\vec{V}_{PE}$  and  $|\vec{V}_{PE}|$  respectively.  
(ii) Example:  $\mathbf{V}_{PE}$  — on a bearing of  $060^\circ$   
 $35 \text{ km h}^{-1}$   
 $\mathbf{V}_{PE}$  —  $35 \text{ km h}^{-1}$

- $\mathbf{V}_{PQ}$  is the velocity of  $P$  relative to  $Q$ . It is the velocity of  $P$  as measured by an observer at  $Q$ .

- (i)



$$\mathbf{V}_{PQ} = \mathbf{V}_{PE} - \mathbf{V}_{QE}$$

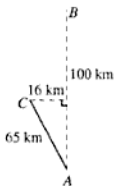


$$\mathbf{V}_{PQ} = \mathbf{V}_{PE} - \mathbf{V}_{QE}$$

- (ii)  $\mathbf{V}_{PQ} = -\mathbf{V}_{QP}$  or  $\mathbf{V}_{QP} = -\mathbf{V}_{PQ}$
- $\mathbf{V}_{PQ} + \mathbf{V}_{QR} = \mathbf{V}_{PR}$   
The sum of the velocity of  $P$  relative to  $Q$  and the velocity of  $Q$  relative to  $R$  is the velocity of  $P$  relative to  $R$ .
- (i) The position vector of  $Q$  relative to  $P$  is the vector  $\mathbf{PQ}$  (maybe written as  $\vec{PQ}$ ). Its magnitude is  $PQ$  (maybe written as  $|\vec{PQ}|$ ).  
(ii)  $\mathbf{PQ} = -\mathbf{QP}$
- Do not confuse position vector with velocity vector. Do not confuse distance or displacement with speed.
- (i) If  $\mathbf{V}_{PQ}$  is in the same direction as  $\mathbf{PQ}$ , or  $\mathbf{V}_{QP}$  is in the same direction as  $\mathbf{QP}$ , then and only then will the two moving object  $P$  and  $Q$  intercept each other.  
(ii) If  $\mathbf{V}_{PQ}$  is in the same direction as  $\mathbf{QP}$ , or  $\mathbf{V}_{QP}$  is in the same direction as  $\mathbf{PQ}$ , then the two moving objects  $P$  and  $Q$  are moving away from each other.

## PAST EXAMINATION QUESTIONS

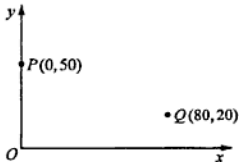
1. An aircraft leaves  $A$  to fly to  $B$  which is 100 km due North of  $A$ . The pilot sets a course due North but after 15 minutes he realises that, owing to a wind blowing from due East, the plane is at a point  $C$ , where  $C$  is 65 km from  $A$  and 16 km West of  $A$ . Find (i) the speed, in  $\text{km h}^{-1}$ , of the wind, (ii) the speed, in  $\text{km h}^{-1}$ , of the aircraft in still air, (iii) the course the pilot should have set from  $A$  order to have arrived directly at  $B$ . Find also the course the pilot should set from  $C$  in order to fly directly to  $B$  and, in this case, the time, to the nearest minute, taken for the journey from  $A$  to  $B$ . (N97/P1/13)



2. Aberdeen is 750 km due north of London. A plane, whose speed in still air is  $480 \text{ km h}^{-1}$ , flies directly from London to Aberdeen in a time of  $1\frac{1}{2}$  hours. There is a wind of  $80 \text{ km h}^{-1}$  blowing from a direction whose bearing is  $(\theta + 180)^\circ$ , where  $0 < \theta < 90$ . Find  
 (i) the value of  $\theta$ ,  
 (ii) the direction in which the pilot heads the plane.  
 On the return journey next day, the wind speed is still  $80 \text{ km h}^{-1}$  but its direction is now from the north-east. The plane leaves Aberdeen at 0800. Assuming that the speed in still air is again  $480 \text{ km h}^{-1}$ , find  
 (iii) the direction in which the pilot must head the plane,  
 (iv) the expected time of arrival in London. (N2000/P2/10)

3. A plane, whose speed in still air is  $500 \text{ km h}^{-1}$ , flies from a point  $A$  to a point  $B$ , 1560 km due north of  $A$ . Because of the action of a constant wind the plane must head in a direction whose bearing is  $009^\circ$ . Given that the flight takes 3 hours,  
 (i) show that the speed of the wind is approximately  $82.5 \text{ km h}^{-1}$ ,  
 (ii) find the bearing of the direction from which the wind is blowing.  
 On another occasion the wind, whose speed is now  $90 \text{ km h}^{-1}$ , is blowing from a direction whose bearing is  $120^\circ$ . A second plane, whose speed in still air is  $375 \text{ km h}^{-1}$ , flies from  $A$  to a point  $C$  which is due east of  $A$ . Given that this flight also takes 3 hours,  
 (iii) find the distance  $AC$ . (N2001/P2/10)

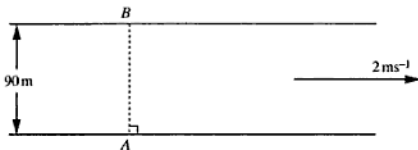
4. At 1200 hours, ship  $P$  is at the point with position vector  $50\mathbf{j}$  km and ship  $Q$  is at the point with position vector  $(80\mathbf{i} + 20\mathbf{j})$  km, as shown in the diagram. Ship  $P$  is travelling with velocity  $(20\mathbf{i} + 10\mathbf{j}) \text{ km h}^{-1}$  and ship  $Q$  is travelling with velocity  $(-10\mathbf{i} + 30\mathbf{j}) \text{ km h}^{-1}$ .  
 (i) Find an expression for the position vector of  $P$  and of  $Q$  at time  $t$  hours after 1200 hours.  
 (ii) Use your answers to part (i) to determine the distance apart of  $P$  and  $Q$  at 1400 hours.  
 (iii) Determine, with full working, whether or not  $P$  and  $Q$  will meet. (N2002/P1/10)



5. In this question,  $\mathbf{i}$  is a unit vector due east and  $\mathbf{j}$  is a unit vector due north. A plane flies from  $P$  to  $Q$ . The velocity, in still air, of the plane is  $(280\mathbf{i} - 40\mathbf{j}) \text{ km h}^{-1}$  and there is a constant wind blowing with velocity  $(50\mathbf{i} - 70\mathbf{j}) \text{ km h}^{-1}$ . Find  
 (i) the bearing of  $Q$  from  $P$ ,  
 (ii) the time of flight, to the nearest minute, given that the distance  $PQ$  is 273 km. (N2003/P1/6)

6. A motor boat travels in a straight line across a river which flows at  $3 \text{ ms}^{-1}$  between straight parallel banks  $200 \text{ m}$  apart. The motor boat, which has a top speed of  $6 \text{ ms}^{-1}$  still water, travels directly from a point  $A$  on one bank to a point  $B$ ,  $150 \text{ m}$  downstream of  $A$ , on the opposite bank. Assuming that the motor boat is travelling at top speed, find, to the nearest second, the time it takes to travel from  $A$  to  $B$ . (N2004/P2/8)

7.



The diagram shows a river  $90 \text{ m}$  wide, flowing at  $2 \text{ ms}^{-1}$  between parallel banks. A ferry travels in a straight line from a point  $A$  to a point  $B$  directly opposite  $A$ . Given that the ferry takes exactly one minute to cross the river, find

- (i) the speed of the ferry in still water,  
(ii) the angle to the bank at which the ferry must be steered. (N2006/P2/4)



# TOPIC 37 QUOTIENT RULE

## FORMULAE AND IMPORTANT NOTES

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

## WORKED EXAMPLES

1. Calculate the gradient of the curve  $y = \frac{x^2+2}{2x+1}$  at the point where  $x = 3$ .

$$\frac{d}{dx} \frac{x^2+2}{2x+1} = \frac{(2x+1)(2x) - (x^2+2)2}{(2x+1)^2}$$

$$\begin{aligned} \text{If } x = 3 \quad \frac{dy}{dx} &= \frac{(6+1)(6) - (9+2)2}{(6+1)^2} \\ &= \frac{20}{49} \end{aligned}$$

2. Differentiate with respect to  $x$ :  $\frac{\sqrt{x}}{\sqrt{x}+1}$ .

$$\begin{aligned} \frac{d}{dx} \frac{\sqrt{x}}{\sqrt{x}+1} &= \frac{(\sqrt{x}+1) \times \frac{1}{2\sqrt{x}} - \sqrt{x} \left( \frac{1}{2\sqrt{x}} \right)}{(\sqrt{x}+1)^2} \\ &= \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2}}{(\sqrt{x}+1)^2} \\ &= \frac{1}{2\sqrt{x}(\sqrt{x}+1)^2} \end{aligned}$$

3. Differentiate with respect to  $x$ ,  $\frac{5x-2}{3x^2-1}$ .

$$\begin{aligned} \frac{d}{dx} \frac{5x-2}{3x^2-1} &= \frac{(3x^2-1)5 - (5x-2)(6x)}{(3x^2-1)^2} \\ &= \frac{15x^2-5-30x^2+12x}{(3x^2-1)^2} \\ &= -\frac{15x^2-12x+5}{(3x^2-1)^2} \end{aligned}$$

## PAST EXAMINATION QUESTIONS

1. Find the value of  $k$  for which  $\frac{d}{dx} \left( \frac{2x-3}{x+5} \right) = \frac{k}{(x+5)^2}$ .

(N97/P2/6b)

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# TOPIC 41 IMPLICIT DIFFERENTIATION

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## FORMULAE AND IMPORTANT NOTES

$$\frac{d}{dx}xy = x \frac{dy}{dx} + y \quad \text{or} \quad (xy)' = xy' + y$$

$$\frac{d}{dx}y^n = ny^{n-1} \times \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) \quad \text{or} \quad y'' = (y')'$$

$$\frac{d}{dx} \left( x \frac{dy}{dx} \right) = x \frac{d^2y}{dx^2} + \frac{dy}{dx} \quad \text{or} \quad (xy')' = xy'' + y'$$

## PAST EXAMINATION QUESTIONS

1. Find the gradient of the curve  $y^2 = x^2 + 2xy + 8$  at each of the points where  $x = 2$ . (N97/P2/6c)

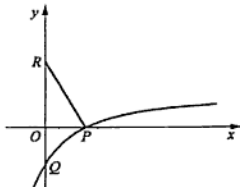
# TOPIC 42 TANGENTS AND NORMALS

## FORMULAE AND IMPORTANT NOTES

1. The equation of the tangent to the curve  $f(x)$  at the point  $(x_1, y_1)$  is  $y - y_1 = f'(x_1) \times (x - x_1)$ .
2. The equation of the normal to the curve at  $(x_1, y_1)$  is  $(y - y_1) \times f'(x_1) = -(x - x_1)$ .

## PAST EXAMINATION QUESTIONS

1. The gradient at any point on a particular curve is given by the expression  $x^2 + \frac{16}{x^2}$ , where  $x > 0$ . Given that the curve passes through the point  $P(4, 18)$ , find (i) the equation of the normal to the curve at  $P$ , (ii) the equation of the curve. Find the coordinates of the point on the curve when the gradient is a minimum and calculate this minimum value. (N97/P1/12)
2. A curve has the equation  $y = \frac{6}{1-2x}$ . Find an expression for  $\frac{dy}{dx}$ . Hence find (i) the equation of the normal to the curve at the point where  $x = 2$ , (ii) the approximate increase in  $y$  as  $x$  increases from 2 to  $2 + p$ , where  $p$  is small. (N98/P1/4)
3. Find the equation of the tangent to the curve  $y^2 = x^2y + 6x$  at the point  $(2, 6)$ . (N98/P2/6b)
4. The gradient at any point  $(x, y)$  on a particular curve is given by  $\frac{dy}{dx} = 1 + \frac{1}{2x^2}$ . The equation of the tangent at the point  $P$  on the curve is  $y = 3x + 1$ . Given that the  $x$ -coordinate of  $P$  is positive, find (i) the coordinates of  $P$ , (ii) the equation of the curve. (N99/P1/7)
5. Find the equation of the tangent to the curve  $y = \sqrt{x^2 - 6x + 25}$  at the point  $(0, 5)$ . (N99/P2/6b)
6. The equation of a curve is  $y = \frac{10}{1+x^2}$ . Find the equation of the normal to the curve at the point where  $x = 3$ . (N2000/P1/8)
7. Find the equation of the tangent to the curve  $xy + x^2 = 2y$  at the point on the curve where  $x = 1$ . (N2000/P2/8a)
8. The point  $P$  lies on the curve  $y = x^2 - 3x + c$ , where  $c$  is a constant. The equation of the tangent to the curve at  $P$  is  $y = 5x + 3$ . Find the equation of the normal to the curve at  $P$ . (N01/P1/10)
9. Find the equation of the normal to the curve  $y = \frac{2x+4}{x-1}$  at the point where the curve meets the  $x$ -axis. (N01/P2/3b)
10. The diagram shows part of the curve  $y = \frac{2x-6}{x+2}$  crossing the  $x$ -axis at  $P$  and the  $y$ -axis at  $Q$ . The normal to the curve at  $P$  meets the  $y$ -axis at  $R$ .
  - (i) Given that  $\frac{dy}{dx} = \frac{k}{(x+2)^2}$ , evaluate  $k$ .
  - (ii) Find the length of  $RQ$ . (N2002/P1/11)



11. A curve has the equation  $y = \frac{2x-4}{x+3}$ .

(i) Obtain an expression for  $\frac{dy}{dx}$  and hence explain why the curve has no turning points.

The curve intersects the  $x$ -axis at the point  $P$ . The tangent to the curve at  $P$  meets the  $y$ -axis at the point  $Q$ .

(ii) Find the area of the triangle  $POQ$ , where  $O$  is the origin.

(N2006/P1/9)

$$\text{If } x = 0.9$$

$$\begin{aligned}y' &= (0.9 - 1)^2(4 \times 0.9 - 10) \\ &= (-0.1)^2(-6.4) \\ &< 0\end{aligned}$$

$$\text{If } x = 1.1$$

$$\begin{aligned}y' &= (1.1 - 1)^2(4 \times 1.1 - 10) \\ &= 0.1^2(-5.6) \\ &< 0\end{aligned}$$

$\therefore (1, 0)$  is a point of inflexion on falling curve.

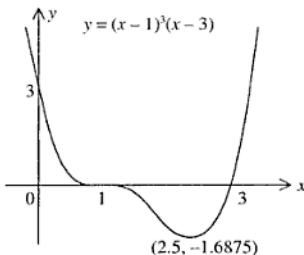
$$\text{If } y' = 0$$

$$\text{and } x = 2.5$$

$$\begin{aligned}y &= (2.5 - 1)^3(2.5 - 3) \\ &= 1.5^3(-0.5) \\ &= -1.6875\end{aligned}$$

$$\begin{aligned}y'' &= 4[(x - 1)^2(1) + (x - 2.5)^2(x - 1)] \\ &= 4[(x - 1)^2 + 2(x - 1)(x - 2.5)] \\ &= 4[(2.5 - 1)^2 + 2(2.5 - 1)(2.5 - 2.5)] \\ &> 0\end{aligned}$$

$\therefore (2.5, -1.6875)$  is a minimum turning point.



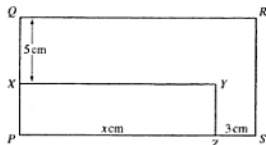
## PAST EXAMINATION QUESTIONS

- A circular cylinder, open at one end, is constructed of thin sheet metal whose area is  $432\pi \text{ cm}^2$ . The cylinder has a radius of  $r \text{ cm}$  and a height of  $h \text{ cm}$ . (i) Show that the volume,  $V \text{ cm}^3$ , contained by the cylinder is given by  $V = \frac{\pi}{2}(432r - r^3)$ . Given that  $r$  can vary, (ii) find the value of  $r$  for which  $V$  is stationary, (iii) evaluate the stationary value of  $V$  and determine, with working, whether this value is a maximum or a minimum. (N97/P1/13)
- Find the coordinates of the stationary points on the curve  $y = 27 + 12x + 3x^2 - 2x^3$  and deduce the nature of each of these points.
  - A hollow closed rectangular tank is made from sheet metal of negligible thickness. The tank has length  $2x \text{ m}$  width  $x \text{ m}$  and a total external surface area of  $48 \text{ m}^2$ . Express, in terms of  $x$ , (i) the height of the tank, (ii) the volume of the tank. Given that  $x$  can vary, find the dimensions of the tank for which the volume is a maximum. (N98/P1/14)

3. Find the range of the function  $f: x \mapsto \frac{18}{x} + 8x$  for the domain  $1 \leq x \leq 3$ . (N98/P1/16a)
4. Two flower beds, one a circle of radius  $r$  m, the other a square of side  $x$  m, are planned for a large garden. To protect the young plants the two beds are to be surrounded by wire netting; the total length of wire netting to be used is 40 m. (i) Express  $x$  in terms of  $r$  and  $\pi$ . The combined area of the two flower beds is  $A$  m<sup>2</sup>. (ii) Show that  $A = \frac{\pi}{4}(4 + \pi)r^2 - 10\pi r + 100$ . Given that  $r$  may vary, (iii) find the value of  $r$  corresponding to the stationary value of  $A$ , (iv) show that, when  $A$  is stationary, the side of the square is equal in length to the diameter of the circle, (v) determine whether the stationary value of  $A$  is a maximum or a minimum. (N99/P1/11)

5. Show that the curve  $y = e^x - e^{-x}$  has no turning points. Tabulate the values of  $e^x - e^{-x}$  for values of  $x$  from  $x = 0$  to  $x = 2$  at intervals of 0.5. Hence draw, on graph paper, the curve  $y = e^x - e^{-x}$  for  $0 \leq x \leq 2$ . Use your graph to solve the equation  $e^{2x} - 1 = 3e^x$ . (N99/P2/3c)
6. (a) A circular cylinder, open at one end, has radius  $r$  cm and external surface area  $27\pi$  cm<sup>2</sup>.  
 (i) Show that the volume of the cylinder,  $V$  cm<sup>3</sup>, is given by  $V = \frac{\pi}{2}(27r - r^3)$ .  
 (ii) Given that  $r$  can vary, find the stationary value of  $V$  and determine whether this value is a maximum or a minimum.

- (b) The diagram shows the rectangles  $PQRS$  and  $PXYZ$ , where  $XQ = 5$  cm,  $PZ = x$  cm,  $ZS = 3$  cm and the area of  $PXYZ$  is 60 cm<sup>2</sup>.



- (i) Show that the area,  $A$  cm<sup>2</sup>, of  $PQRS$  is given by  $A = 5x + 75 + \frac{180}{x}$ . Given that  $x$  can vary,  
 (ii) find an expression for  $\frac{dA}{dx}$ ,  
 (iii) show that when  $A$  takes its minimum value the rectangles  $PQRS$  and  $PXYZ$  are similar. (N2000/P1/14)

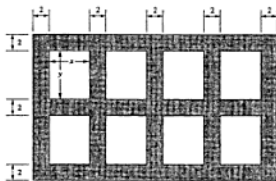
7. The diagram, where all dimensions are in metres, shows a rectangular wall with 8 rectangular windows, each  $x$  m by  $y$  m, set 2 m apart and 2 m from the boundaries of the wall.

Given that the total area of the 8 windows is 240 m<sup>2</sup>,

- (i) show that the area,  $A$  m<sup>2</sup>, of the brickwork, shaded in the diagram, is given by  
 $A = 60 + \frac{600}{x} + 24x$ .

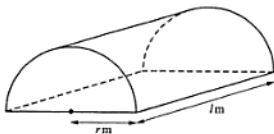
Given also that  $x$  and  $y$  vary, find

- (ii) the value of  $x$  and of  $y$  for which  $A$  is a minimum.  
 (iii) the minimum value of  $A$ . (N01/P1/15b)



## 8. EITHER

The diagram shows a greenhouse standing on a horizontal rectangular base. The vertical semicircular ends and the curved roof are made from polythene sheeting. The radius of each semicircle is  $r$  m and the length of the greenhouse is  $l$  m. Given that 120 m<sup>2</sup> of polythene sheeting is used for the greenhouse, express  $l$  in terms of  $r$  and show that the volume,  $V$  m<sup>3</sup>, of the greenhouse is given by  $V = 60r - \frac{\pi r^3}{2}$ .



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# TOPIC 44 DERIVATIVES AND GRAPH SKETCHING

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## FORMULAE AND IMPORTANT NOTES

To sketch the graph of  $f(x, y) = 0$  :

1. Put  $y = 0$  and solve for  $x$ . Plot the  $x$ -intercepts.
2. Put  $x = 0$  and solve for  $y$ . Plot the  $y$ -intercepts.
3. Find where the graph is above the  $x$ -axis ( $y > 0$ ) or below the  $x$ -axis ( $y < 0$ ).
4. If  $\frac{dy}{dx} > 0$  the graph is a rising curve. If  $\frac{dy}{dx} < 0$  the graph is a falling curve.
5. If  $\frac{dy}{dx} = 0$  the point is either a turning point or a point of inflection on horizontal tangent.
6. Where the gradient is infinite, the tangent at that point is parallel to the  $y$ -axis.
7. Mark any point whose co-ordinates are either given or can be calculated if necessary. Now join this points with a smooth curve but do not cross points of discontinuity.

## PAST EXAMINATION QUESTIONS

*No questions on this topic in the last 10 years.*

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# TOPIC 45 DERIVATIVES AND PARAMETRIC EQUATIONS

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## FORMULAE AND IMPORTANT NOTES

- Let the parametric equations of a curve be  $x = f(t)$  and  $y = g(t)$ . The gradient of the curve at any point with parameter  $t$  is  $\frac{dy}{dx} = \frac{d}{dt}g(t) \div \frac{d}{dt}f(t)$ .
- The equation of the tangent to the curve at this point is  $y - g(t) = \frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)} \times [x - f(t)]$ .
- The equation of the normal to the curve at this point is  $[y - g(t)] \times \frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)} = -[x - f(t)]$ .

## PAST EXAMINATION QUESTIONS

- The parametric equations of a curve are  $x = 4t + \frac{9}{t}$ ,  $y = 2t - 5$ . Find (i) an expression for  $\frac{dy}{dx}$  in terms of  $t$ , (ii) the coordinates of the point at which the tangent to the curve at  $(13, -3)$  meets the  $x$ -axis, (iii) the coordinates of each of the points on the curve at which the tangent to the curve is parallel to the  $y$ -axis. Find also (iv) the value of  $t$  at each of the points of intersection of the curve with the line  $x + y = 16$ , (v) the cartesian equation of the curve. (N97/P2/8)
- In terms of the parameter  $t$ , the equations of a curve are  $x = t^2 - t$ ,  $y = 2t + 1$ . (i) Find the value of  $t$  at the point  $P$  on the curve where the gradient is  $\frac{2}{3}$ . (ii) Show that the equation of the normal at  $P$  is  $2y + 5x = 44$ . (iii) Find the value of  $t$  at the point where the normal at  $P$  again intersects the curve. (N98/P2/8a)
- (a) The parametric equations of a curve are  $x = t^3 + 6t^2 + 10t$ ,  $y = 3t^2 + 12t + 5$ . (i) Show that the normal to the curve, at the point where  $t = 0$ , passes through the point  $(6, 0)$ . (ii) Find the value of  $t$  at each of the points where the line  $x - 2y + 10 = 0$  intersects the curve. (b) The parametric equations of a curve are  $x = 2 \sin \theta - 1$ ,  $y = 2 \cos 2\theta$ , for  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . (i) Find the gradient of this curve at the point where this curve intersects the  $y$ -axis. (ii) Find the value of  $a$  and of  $b$  for which the cartesian equation of this curve is  $y = 1 + ax + bx^2$ . (N99/P2/8)
- The parametric equations of a curve are  $x = 4t + 1$ ,  $y = 1 + t^3$ . The line  $y = x$  intersects the curve at points  $A$ ,  $B$  and  $C$ . The coordinates of  $A$  are negative and  $B$  lies between  $A$  and  $C$ . (i) Find the coordinates of  $A$ ,  $B$  and  $C$ . (ii) Show that  $AB = BC$ . (iii) Obtain an expression for  $\frac{dy}{dx}$  in terms of  $t$ . (iv) Show that the tangent to the curve at  $A$  is parallel to the tangent at  $C$  and find the equation of each of these tangents. (v) Given that the tangent at  $A$  meets the curve again at the point  $D$ , find the value of  $t$  at  $D$ . (vi) Obtain the cartesian equation of the curve. (N2000/P2/2)
- The parametric equations of a curve are  $x = t^2 - 4t + 5$ ,  $y = t^2 + 4$ . (i) Express  $\frac{dy}{dx}$  in terms of  $t$ .  $A$  is the point on the curve where  $t = 1$ . The tangent to the curve at  $A$  meets the  $x$ -axis at  $B$ . (ii) Find the area of triangle  $AOB$ , where  $O$  is the origin.



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# TOPIC 46 SMALL INCREMENTS AND APPROXIMATIONS

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## FORMULAE AND IMPORTANT NOTES

- Let  $\Delta x$  and  $\Delta y$  be the change in the value of  $x$  and the change in the value of  $y$  respectively. They are to be positive if the change is an increase and negative if the change is a decrease. They may also be written as  $\delta x$  and  $\delta y$ .
- If  $\Delta x$  is small then (i)  $\Delta y \approx \frac{dy}{dx} \times \Delta x$   
(ii)  $\frac{\Delta y}{y} \approx \frac{dy}{dx} \times \frac{\Delta x}{x}$   
(iii)  $f(x + \Delta x) \approx f(x) + \frac{dy}{dx} \times \Delta x$
- (i) A percentage change of  $p\%$  in  $x$  means  $\Delta x = p\% x$  or  $\frac{\Delta x}{x} = p\%$ .  
(ii) A percentage change of  $q\%$  in  $y$  means  $\Delta y = q\% y$  or  $\frac{\Delta y}{y} = q\%$ .  
(iii) If a small change of  $p\%$  in  $x$  causes a change of  $q\%$  in  $y$ , then, from No.2 (ii):  $q\% \approx \frac{dy}{dx} \times \frac{p\% x}{y}$ .

## PAST EXAMINATION QUESTIONS

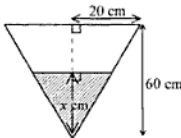
- A curve has the equation  $y = \frac{c}{(1+2x)^2}$  where  $c$  is constant. (i) Obtain an expression for  $\frac{dy}{dx}$ . (ii) When  $x$  increases from 1 to  $1+p$ , where  $p$  is small, the corresponding change in  $y$  is approximately  $-\frac{8p}{3}$ . Find the value of  $c$ . (N97/P2/6d)
- Use calculus to determine, in terms of  $p$ , the approximate change in the radius of a circle when the area of circle increases from  $900\pi$  to  $(900+p)\pi$ , where  $p$  is small. (N99/P1/8)
- Given that  $y = x^3 + 9x$ , use calculus to find, in terms of  $p$ , the approximate percentage increase in  $y$  when  $x$  increase from 3 to  $3+p$ , where  $p$  is small. (N2000/P1/13a)
- Given that  $y = x \ln x - x$ , find an expression for  $\frac{dy}{dx}$ . Hence find, in terms of  $p$ , the approximate change in  $y$  when  $x$  changes from  $e^2$  to  $e^2 + p$ , where  $p$  is small. (N2000/P2/8c)
- Two variables,  $x$  and  $y$ , are related by the equation  $y = x^2 + \frac{6}{x}$ . (i) Obtain an expression for  $\frac{dy}{dx}$ . (ii) Use your expression to find the approximate change in the value of  $y$  when  $x$  changes from 2 to 2.05. (N01/P2/8)
- (i) Given that  $y = 1 + \ln(2x - 3)$ , obtain an expression for  $\frac{dy}{dx}$ . (ii) Hence find, in terms of  $p$ , the approximate value of  $y$  when  $x = 2 + p$ , where  $p$  is small. (N2005/P2/3)
- The equation of a curve is  $y = \frac{8}{(3x-4)^2}$ . (i) Find the gradient of the curve where  $x = 2$ . (ii) Find the approximate change in  $y$  when  $x$  increases from 2 to  $2+p$ , where  $p$  is small. (N2006/P1/3)

# TOPIC 47 CONNECTED RATES OF CHANGE

## FORMULAE AND IMPORTANT NOTES

1. If a quantity  $x$  is increasing with the passage of time its rate of change  $\frac{dx}{dt}$  is positive.
2. If  $x$  is decreasing as time increases  $\frac{dx}{dt}$  is negative.
3.  $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$  or  $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

## PAST EXAMINATION QUESTIONS

1. Given that  $y = \frac{(3x-2)^{10}}{6}$ , find the value of  $\frac{dy}{dx}$  when  $x = \frac{1}{3}$ . The rate of increase of  $x$ , when  $x = \frac{1}{3}$ , is 2 units per second. Calculate the corresponding rate of change of  $y$ . (N97/P1/6)
2. The mass,  $m$  grams, of radioactive substance, present at time  $t$  days after first being observed, is given by the formula  $m = 24e^{-0.02t}$ . Find (i) the value of  $m$  when  $t = 30$ , (ii) the value of  $t$  when the mass is half of its value at  $t = 0$ , (iii) the rate at which the mass is decreasing when  $t = 50$ . (N97/P2/4a)
3. The diagram shows a vertical cross-section of a container in the form of an inverted cone of height 60 cm and base radius 20 cm. The circular base is held horizontal and uppermost. Water is poured into the container at a constant rate of  $40 \text{ cm}^3 \text{ s}^{-1}$ . (i) Show that, when the depth of water in the container is  $x$  cm, the volume of water in the container is  $\frac{\pi x^3}{27} \text{ cm}^3$ . (ii) Find the rate of increase of  $x$  at the instant when  $x = 2$ . (N98/P1/6)  

4. Given that  $y = \frac{7x-2}{2x+3}$ , where  $x \neq -1.5$ , show that  $\frac{dy}{dx}$  is always positive. Use your expression for  $\frac{dy}{dx}$  to find the approximate increase in  $y$  as  $x$  increases from 36 to 36.045. Given that  $x$  and  $y$  vary with time  $t$ , find the values of  $x$  for which  $\frac{dy}{dt} = 4 \frac{dx}{dt}$ . (N99/P2/6c)
5. Liquid is poured into a bucket at a rate of  $60 \text{ cm}^3 \text{ s}^{-1}$ . The volume,  $V \text{ cm}^3$ , of the liquid in the bucket, when the depth of liquid is  $x$  cm, is given by  $V = 0.01x^3 + 2.2x^2 + 200x$ . Find (i) the rate of increase in the depth of liquid when  $x = 10$ , (ii) the depth of liquid when the rate of increase in the depth is  $0.2 \text{ cm s}^{-1}$ . (N2000/P1/13b)
6. The velocity,  $v \text{ ms}^{-1}$ , of a particle, travelling in the straight line, at time  $t$  s after leaving a fixed point  $O$ , given by  $v = 3t^2 - 18t + 32$ , where  $t \geq 0$ . Find (i) the value of  $t$  for which the acceleration is zero, (ii) the distance of the particle from  $O$  when its velocity is a minimum. (N01/P1/9)
7. Two variables,  $x$  and  $y$ , are related by the equation  $y = \frac{3}{4}(\frac{x}{12} - 1)^6$ . Given that both  $x$  and  $y$  vary with time, find the value of  $y$  when the rate of change of  $y$  is 12 times the rate of change of  $x$ . (N01/P1/16b)

# TOPIC 48 INDEFINITE INTEGRALS

## FORMULAE AND IMPORTANT NOTES

- If  $\frac{d}{dx}F(x) = f(x)$  then  $\int f(x)dx = F(x) + c$  where  $c$  is a constant.
  - $\int n dx = nx + c$ , where  $n$  is a constant
  - $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ ,  $n \neq -1$
  - $\int \frac{1}{x^n} dx = \frac{x^{1-n}}{1-n} + c$ ,  $n \neq 1$
  - $\int \sqrt{x} dx = \frac{n}{\frac{n}{2}+1} x^{\frac{n}{2}+1} + c$ ,  $n \neq \pm 1$
  - $\int \frac{1}{\sqrt{x}} dx = \frac{n}{n-1} x^{1-\frac{1}{n}} + c$ ,  $n \neq \pm 1$
  - $\int e^x dx = e^x + c$
  - $\int \frac{1}{x} dx = \ln x + c$ , if  $x > 0$
  - $\int \sin x dx = -\cos x + c$
  - $\int \cos x dx = \sin x + c$
  - $\int \sec^2 x dx = \tan x + c$
- If  $\frac{d}{dx}F(x) = f(x)$  then  $\int f(mx+b)dx = \frac{1}{m}F(mx+b) + c$  where  $m$ ,  $b$  and  $c$  are constants and  $m \neq 0$ .
  - $\int (mx+b)^n dx = \frac{1}{m(n+1)}(mx+b)^{n+1} + c$ ,  $n \neq -1$
  - $\int \frac{1}{(mx+b)^n} dx = \frac{1}{m(1-n)}(mx+b)^{1-n} + c$ ,  $n \neq 1$
  - $\int \sqrt{mx+b} dx = \frac{n}{m(\frac{n}{2}+1)}(mx+b)^{\frac{n}{2}+1} + c$ ,  $n \neq \pm 1$
  - $\int \frac{1}{\sqrt{mx+b}} dx = \frac{n}{m(n-1)}(mx+b)^{1-\frac{1}{n}} + c$ ,  $n \neq \pm 1$
  - $\int e^{mx+b} dx = \frac{1}{m}e^{mx+b} + c$
  - $\int \frac{1}{mx+b} dx = \frac{1}{m}\ln(mx+b) + c$ ,  $mx+b > 0$
  - $\int \sin(mx+b) dx = -\frac{1}{m}\cos(mx+b) + c$
  - $\int \cos(mx+b) dx = \frac{1}{m}\sin(mx+b) + c$
  - $\int \sec^2(mx+b) dx = \frac{1}{m}\tan(mx+b) + c$
- $\int mf(x) dx = m \int f(x) dx$
  - $\int \{f(x) \pm g(x)\} dx = \int f(x) dx \pm \int g(x) dx$
- $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}(x - \frac{\sin 2x}{2}) + c = \frac{1}{2}(x - \sin x \cos x) + c$
  - $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}(x + \frac{\sin 2x}{2}) + c = \frac{1}{2}(x + \sin x \cos x) + c$
  - $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + c$
  - $\int \operatorname{cosec}^2 x dx = -\cot x + c$
  - $\int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1) dx = -\cot x - x + c$
  - $\int \sin^3 x dx = \int \frac{1}{4}(3 \sin x - \sin 3x) dx = \frac{1}{4}(\frac{\cos 3x}{3} - 3 \cos x) + c$
  - $\int \cos^3 x dx = \int \frac{1}{4}(\cos 3x + 3 \cos x) dx = \frac{1}{4}(\frac{\sin 3x}{3} + 3 \sin x) + c$

## PAST EXAMINATION QUESTIONS

- Find  $\int \sqrt{4x+5} \, dx$ . (N97/P2/7a)
- Find the equation of the curve which passes through the point (3, 6) and for which  $\frac{dy}{dx} = 2x(x-3)$ . (N98/P1/2)
- A curve is such that  $\frac{dy}{dx} = 3x^2 + \frac{2}{x}$ . Given that the curve passes through the point (1, 3), find the equation of the curve. (N98/P2/7a)
- Find  $\int (7x-2)^{-2} dx$ . (N99/P2/7a)
- The gradient at any point (x, y) on a curve is given by  $6x^2 + 6x - 5$ . Given that the curve passes through the point (2, 12), find the equation of the curve. (N2000/P1/5)
- The gradient of a curve at any point is given by  $\frac{dy}{dx} = 2 - \frac{y^3}{8}$ . The curve intersects the x-axis at the point P. Given that the gradient of the curve at P is 1, find the equation of the curve. (N01/P1/16a)
- A curve is such that  $\frac{dy}{dx} = \frac{6}{(2x-3)^2}$ . Given that the curve passes through the point (3, 5), find the coordinates of the point where the curve crosses the x-axis. (N2002/P1/6)
- A curve has the equation  $y = x^3 \ln x$ , where  $x > 0$ .
  - Find an expression for  $\frac{dy}{dx}$ .  
Hence
  - calculate the value of  $\ln x$  at the stationary point of the curve,
  - find the approximate increase in y as x increases from e to e + p, where p is small,
  - find  $\int x^2 \ln x \, dx$ . (N2004/P1/10)
- A curve is such that  $\frac{d^2y}{dx^2} = 6x - 9$ . The gradient of the curve at the point (2, -9) is 3.
  - Express y in terms of x.
  - Show that the gradient of the curve is never less than  $-\frac{16}{3}$ . (N2005/P2/10)

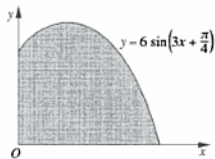
# TOPIC 49 DEFINITE INTEGRALS

## FORMULAE AND IMPORTANT NOTES

1. If  $\frac{d}{dx}F(x) = f(x)$  then  $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$ .
2.  $\int_a^a f(x) dx = 0$
3.  $\int_a^b f(x) dx = -\int_b^a f(x) dx$
4.  $\int_a^b cf(x) dx = c \int_a^b f(x) dx$ , where  $c$  is a constant
5.  $\int_a^b \{f(x) \pm g(x)\} dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
6.  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

## PAST EXAMINATION QUESTIONS

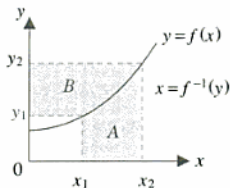
1. Show that  $\frac{d}{dx}(2x + \sin 2x) = 4 \cos^2 x$ . Hence, or otherwise, evaluate  $\int_{\frac{\pi}{2}}^{\pi} \cos^2 x dx$ . (N98/P2/7b)
2. Using the identity  $\cos 3x = 4 \cos^3 x - 3 \cos x$ , find the value of  $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3 x dx$ . (N99/P2/7b)
3. (a) Evaluate (i)  $\int_{\frac{\pi}{2}}^{\pi} 3 \cos 2x dx$ , (ii)  $\int_0^2 \sqrt{4x+1} dx$ .  
(b) Find  $\frac{d}{dx}(\frac{1}{9-4x^2})$  and hence evaluate  $\int_0^1 \frac{x}{(9-4x^2)^2} dx$ . (N2000/P2/7a, b)
4. (a) Evaluate (i)  $\int_3^4 \frac{1}{(x-2)^3} dx$ , (ii)  $\int_0^1 \sin 2x dx$ .  
(b) Find the value of  $k$  for which  $\int_4^k \frac{1}{2x-5} dx = \ln 2$ . (N2001/P2/4a, b)
5. (i) Differentiate  $x \sin x$  with respect to  $x$ .  
(ii) Hence evaluate  $\int_0^{\frac{\pi}{2}} x \cos x dx$ . (N2002/P2/7)
6. The diagram shows part of the curve  $y = 6 \sin(3x + \frac{\pi}{4})$ . Find the area of the shaded region bounded by the curve and the coordinate axes. (N2003/P1/5)



# TOPIC 50 PLANE AREAS BY INTEGRATION

## FORMULAE AND IMPORTANT NOTES

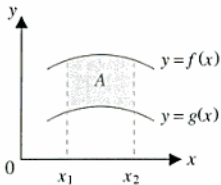
1.



$$\text{Area of region } A = \int_{x_1}^{x_2} y \, dx = \int_{x_1}^{x_2} f(x) \, dx$$

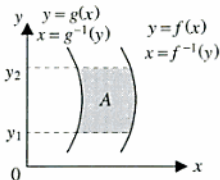
$$\text{Area of region } B = \int_{y_1}^{y_2} x \, dy = \int_{y_1}^{y_2} f^{-1}(y) \, dy$$

2.



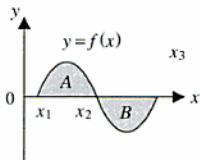
$$\text{Area of region } A = \int_{x_1}^{x_2} \{f(x) - g(x)\} \, dx$$

3.



$$\text{Area of region } A = \int_{y_1}^{y_2} \{f^{-1}(y) - g^{-1}(y)\} \, dy$$

4.



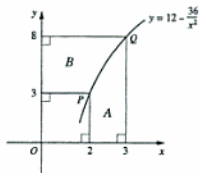
$$\int_{x_1}^{x_2} y \, dx > 0$$

$$\int_{x_2}^{x_3} y \, dx < 0$$

$\int_{x_1}^{x_3} y \, dx =$  algebraic difference between areas of region A and region B

$$\text{Absolute areas of region A and region B} = \int_{x_1}^{x_2} y \, dx + \int_{x_2}^{x_3} y \, dx$$

6. (a) The diagram shows part of the curve  $y = 12 - \frac{36}{x^2}$ , passing through the points  $P(2, 3)$  and  $Q(3, 8)$ . Find the area of (i) the region A, (ii) the region B.



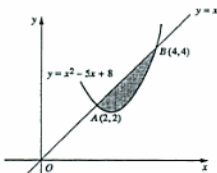
- (b) The line  $y = x$  intersects the curve  $y = x^2 - 5x + 8$  at  $A(2, 2)$  and  $B(4, 4)$ . The diagram shows the shaded region bounded by the line and the curve.

(i) Find the area of the shaded region.

The curve  $x = y^2 - 5y + 8$  also passes through A and B.

(ii) Find the area enclosed by the two curves.

(N01/P1/17)



7. Use the formula  $\cos 2A = 2 \cos^2 A - 1$  to express  $\cos^2 2x$  in terms of  $\cos 4x$ . Hence find, to 2 decimal places, the area of the region enclosed by the curve  $y = 6\cos^2 2x$ , the  $x$ -axis and the lines  $x = 0$  and  $x = \frac{\pi}{8}$ .

(N01/P2/4c)

8. (a) Differentiate  $xe^{2x}$  with respect to  $x$ .

(b) Find the  $x$ -coordinate of the stationary point of the curve  $y = xe^{2x}$ .

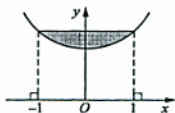
(c) Using your answer from part (a) show that  $\int 4xe^{2x} dx = 2xe^{2x} - e^{2x} + c$ , where  $c$  is a constant.

Hence find the area of the region enclosed by the curve  $y = 4xe^{2x}$ , the  $x$ -axis and the lines  $x = 1$  and  $x = 2$ .

(N01/P2/7b)

9. The diagram shows part of the curve  $y = e^x + e^{-x}$  for  $-1 \leq x \leq 1$ . Find, to 2 decimal places, the area of the shaded region.

(N2002/P1/4)



10. A curve has the equation  $y = e^{\frac{1}{2}x} + 3e^{-\frac{1}{2}x}$ .

(i) Show that the exact value of the  $y$ -coordinate of the stationary point of the curve is  $2\sqrt{3}$ .

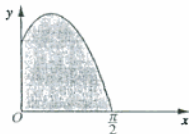
(ii) Determine whether the stationary point is a maximum or a minimum.

(iii) Calculate the area enclosed by the curve, the  $x$ -axis and the lines  $x = 0$  and  $x = 1$ .

(N2004/P1/12OR)

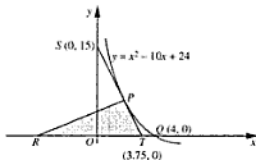
11. The diagram shows part of the curve  $y = 3\sin 2x + 4\cos x$ . Find the area of the shaded region, bounded by the curve and the coordinate axes.

(N2004/P2/3)

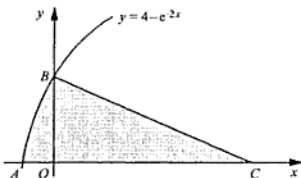


12. Each member of a set of curves has an equation of the form  $y = ax + \frac{b}{x^2}$ , where  $a$  and  $b$  are integers.
- (i) For the curve where  $a = 3$  and  $b = 2$ , find the area bounded by the curve, the  $x$ -axis and the lines  $x = 2$  and  $x = 4$ .
- Another curve of this set has a stationary point at  $(2, 3)$ .
- (ii) Find the value of  $a$  and of  $b$  in this case and determine the nature of the stationary point. (N2004/P2/12OR)

13. The diagram, which is not drawn to scale, shows part of the curve  $y = x^2 - 10x + 24$  cutting the  $x$ -axis at  $Q(4, 0)$ . The tangent to the curve at the point  $P$  on the curve meets the coordinate axes at  $S(0, 15)$  and at  $T(3.75, 0)$ .
- (i) Find the coordinates of  $P$ .
- The normal to the curve at  $P$  meets the  $x$ -axis at  $R$ .
- (ii) Find the coordinates of  $R$ .
- (iii) Calculate the area of the shaded region bounded by the  $x$ -axis, the line  $PR$  and the curve  $PQ$ . (N2005/P2/12EITHER)



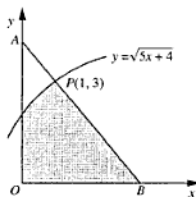
14.



The diagram shows part of the curve  $y = 4 - e^{-2x}$  which crosses the axes at  $A$  and at  $B$ .

- (i) Find the coordinates of  $A$  and of  $B$ .
- The normal to the curve at  $B$  meets the  $x$ -axis at  $C$ .
- (ii) Find the coordinates of  $C$ .
- (iii) Show that the area of the shaded region is approximately 10.3 square units. (N2006/P1/12EITHER)

15. In the diagram, 1 unit represents 1 kilometre along each axis. The triangle  $OAB$  represents a park. The development of a new road will result in the park being reduced to the shaded region shown. One of the boundaries of this shaded region is represented by the curve  $y = \sqrt{5x + 4}$ . The side  $AB$  is normal to the curve at the point  $P(1, 3)$ .
- (i) Find the equation of the line  $AB$ .
- (ii) Show that the length of  $OB$  is 3.5 units and find the length of  $OA$ .
- (iii) Show that the development of the new road will reduce the park to approximately 85.5% of its original size. (SP08/P2/10)





# TOPIC 52 KINEMATICS (CALCULUS)

## FORMULAE AND IMPORTANT NOTES

The syllabus covers only the movement of a particle in a straight line. Let  $t$  be the time,  $a$  the acceleration,  $v$  the velocity,  $s$  the displacement and  $x$  the distance travelled.

$$1. \quad a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d^2x}{dt^2}$$

$$2. \quad v = \frac{dx}{dt} = \int a \, dt$$

$$3. \quad s = \int v \, dt$$

4. (i)  $v$  never changes sign for any  $t$  where  $t_1 < t < t_2$ :

$$\text{Displacement of the particle from } t_1 \text{ to } t_2 = s = \int_{t_1}^{t_2} v \, dt$$

$$\text{Distance travelled by the particle from } t_1 \text{ to } t_2 = x = \left| \int_{t_1}^{t_2} v \, dt \right|$$

$$\therefore x = |s|$$

- (ii)  $v$  changes sign only when  $t = t_3$  and  $t_1 < t_3 < t_2$ :

$$\text{Displacement of the particle from } t_1 \text{ to } t_2 = s = \int_{t_1}^{t_2} v \, dt$$

$$\text{Distance travelled by the particle from } t_1 \text{ to } t_2 = x = \left| \int_{t_1}^{t_3} v \, dt \right| + \left| \int_{t_3}^{t_2} v \, dt \right|$$

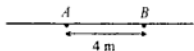
$$\therefore x > |s|$$

## PAST EXAMINATION QUESTIONS

1. A particle moves in a straight line so that,  $t$  seconds after passing through a fixed point  $O$ , its velocity,  $v \text{ m s}^{-1}$ , is given by  $v = 5t^2 + t(1 - 3p) + p$ , where  $p$  is a constant. (i) Find an expression for the acceleration of the particle in terms of  $t$  and  $p$ . (ii) Given that the acceleration of the particle is  $3 \text{ m s}^{-2}$  when  $t = 2$ , find the value of  $p$ . (iii) Using your value of  $p$ , find the values of  $t$  when the particle is at instantaneous rest. (N97/P1/10)

2. A particle moves in a straight line so that, at time  $t$  seconds after passing through a fixed point  $O$ , its velocity,  $v \text{ m s}^{-1}$ , is given by  $v = 8 \cos\left(\frac{t}{4}\right)$ . Find (i) the value of  $t$  at which the particle first comes to rest, (ii) the distance travelled by the particle in the first 4 seconds after passing through  $O$ . (N97/P2/7b)

3. The diagram shows two points  $A$  and  $B$  on a straight line, where  $AB = 4 \text{ m}$ . A particle  $P$  moves along the line so that its velocity,  $v \text{ m s}^{-1}$ , is given by  $v = t^2 - 4t - 5$ ,  $t \geq 0$ ,



where  $t$  is the time in seconds after leaving  $B$ . Initially particle  $P$  is at  $B$ , moving towards  $A$ . Find an expression, in terms of  $t$ , for (i) the acceleration of  $P$ , (ii) the distance of  $P$  from  $A$ . Find (iii) the distance from  $A$  of the point where  $P$  comes instantaneously to rest, (iv) the total distance travelled by  $P$  in the time interval  $t = 0$  to  $t = 10$ . (N98/P1/7)

4. A particle moves in a straight line so that, at time  $t$  seconds after leaving a fixed point  $O$ , its velocity,  $v \text{ ms}^{-1}$ , is given by  $v = 20e^{-t}$ . (i) Sketch the velocity-time curve. (ii) Find the value of  $t$  when  $v = 10$ . (iii) Find the acceleration of the particle when  $v = 10$ . (iv) Obtain an expression, in terms of  $t$ , for the displacement from  $O$  of the particle at time  $t$  seconds.

(N98/P2/5c)

5. A particle  $A$  moves in a straight line so that its displacement,  $s$  m, from a point  $O$  at time  $t$  s, where  $t \geq 0$ , is given by  $s = t^3 - 4t^2 - 3t + 5$ . Find (i) the value of  $t$  when the particle is instantaneously at rest and the distance the particle has then travelled, (ii) the value of  $t$  to two decimal places, when the particle has returned to its initial position. A particle  $B$  moves on a parallel straight line so that its acceleration,  $a \text{ m s}^{-2}$ , at time  $t$  s is given by  $a = 2t + 1$ . Given that  $A$  and  $B$  have the same velocity when  $t = 5$ , obtain an expression, in terms of  $t$ , for the velocity of  $B$ .

(N99/P1/13)

6. A particle moves in a straight line so that,  $t$  seconds after leaving a fixed point  $O$ , its displacement,  $s$  metres from  $O$ , is given by  $s = 9t^2 + 6t^3 - 2t^4$ . Find  
(i) the positive value of  $t$  for which the particle is instantaneously at rest,  
(ii) the total distance travelled by the particle from  $t = 0$  to  $t = 4$ ,  
(iii) the acceleration of the particle when  $t = 1$ .

(N2000/P1/7)

7. The velocity,  $v \text{ ms}^{-1}$ , of a particle, travelling in a straight line, at time  $t$  s after leaving a fixed point  $O$ , is given by

$$v = 3t^2 - 18t + 32, \text{ where } t \geq 0.$$

Find

- (i) the value of  $t$  for which the acceleration is zero.  
(ii) the distance of the particle from  $O$  when its velocity is a minimum. (N2001/P1/9)
8. A particle moves so that,  $t$  s after passing through a fixed point  $O$ , its velocity,  $v \text{ ms}^{-1}$ , is given by  $v = Ae^{-kt}$ , where  $A$  and  $k$  are constants. Given that when  $t = 0$  the velocity is  $5 \text{ ms}^{-1}$  and that when  $t = 10$  the velocity is  $3 \text{ ms}^{-1}$ , find  
(i) the value of  $A$  and  $k$ ,  
(ii) the acceleration of the particle when  $t = 10$ . (N2001/P2/8c)

9. A particle travels in a straight line so that,  $t$  s after passing a fixed point  $A$ , its speed,  $v \text{ ms}^{-1}$ , is given by  $v = 40(e^{-t} - 0.1)$ .

The particle comes to instantaneous rest at  $B$ . Calculate the distance  $AB$ . (N2003/P2/6)

10. A particle, travelling in a straight line, passes a fixed point  $O$  on the line with a speed of  $0.5 \text{ ms}^{-1}$ . The acceleration,  $a \text{ ms}^{-2}$ , of the particle,  $t$  s after passing  $O$ , is given by  $a = 1.4 - 0.6t$ .

- (i) Show that the particle comes instantaneously to rest when  $t = 5$ .  
(ii) Find the total distance travelled by the particle between  $t = 0$  and  $t = 10$ .

(N2004/P2/12EITHER)

11. A particle moves in a straight line so that,  $t$  seconds after leaving a fixed point  $O$ , its velocity,  $v \text{ ms}^{-1}$ , is given by  $v = 16 + 6t - t^2$ . Find

- (i) the velocity of the particle when its acceleration is zero.  
(ii) the value of  $t$  when the particle is instantaneously at rest,  
(iii) the distance from  $O$  at which the particle is instantaneously at rest,  
(iv) the total distance travelled by the particle in the interval  $t = 0$  to  $t = 12$ . (SP08/P2/9)

1 Prove the identity  $\frac{1}{1 + \tan^2 A} = (1 + \sin A)(1 - \sin A)$ . [3]

- 2 A company supplies 4 garden centres – Allseed, Budwise, Croppers and Digwell – with bags of compost, which are sold in 3 sizes – large 145 litres, medium 75 litres and small 20 litres. The number of bags of compost supplied to each garden centre in one delivery is shown in the following table.

	Large	Medium	Small
Allseed	200	500	200
Budwise	300	600	–
Croppers	–	400	300
Digwell	–	700	–

Over a six-month period Allseed received 5 such deliveries, Budwise 6, Croppers 8 and Digwell 7. Write down three matrices such that matrix multiplication will give the total amount of compost supplied over the six-month period and hence find this total. [4]

- 3 The line  $2x + 3y = 12$  meets the curve  $y^2 = 4x - 8$  at the points  $P$  and  $Q$ . Find the length of the line  $PQ$ . [5]

- 4 Find the coefficient of  $x^3$  in the binomial expansion of

(i)  $(1 - 2x)^7$ , [2]

(ii)  $(1 - 7x^2)(1 - 2x)^7$ . [3]

- 5 (i) Differentiate  $\tan(2x + 1)$  with respect to  $x$ . [2]

- (ii) Explain why the curve  $y = \tan(2x + 1)$  has no stationary points. [1]

- (iii) Find, in terms of  $p$ , the approximate change in  $\tan(2x + 1)$  as  $x$  increases from 1 to  $1 + p$ , where  $p$  is small. [2]

- 6 In this question  $\mathbf{i}$  is a unit vector due east and  $\mathbf{j}$  is a unit vector due north.

A plane flies from  $A$  to  $B$ , where  $B$  is 900 km due east of  $A$ . The velocity, in still air, of the plane is  $(270\mathbf{i} - 50\mathbf{j}) \text{ km h}^{-1}$  and there is a wind blowing with a constant velocity of  $(p\mathbf{i} + q\mathbf{j}) \text{ km h}^{-1}$ .

(i) Find the value of  $q$ . [1]

(ii) Given that the journey takes 3 hours, show that  $p = 30$ . [3]

The plane returns from  $B$  to  $A$  with the same wind blowing and the velocity, in still air, of the plane is now  $(-270\mathbf{i} - 50\mathbf{j}) \text{ km h}^{-1}$ .

(iii) Calculate the time taken for the return journey. [2]

- 7 Solve the equation

(i)  $\log_x 72 = 3 - \log_x 3$ . [3]

(ii)  $3 \log_5 y - \log_{25} y = 10$ . [4]

- 8 (a) Each of seven cards has on it one of the digits 1, 2, 3, 4, 5, 6, 7; no two cards have the same digit. Four of these cards are selected and arranged to form a 4-digit number.

(i) How many different 4-digit numbers can be formed in this way? [1]

(ii) How many of these 4-digit numbers begin and end with an even digit? [2]

- (b) 4 people are selected to form a debating team from a group of 5 men and 2 women.

(i) Find the number of possible teams that can be selected. [1]

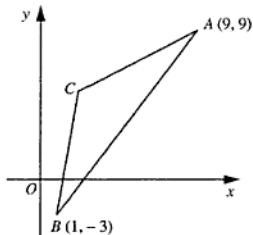
(ii) How many of these teams contain at least 1 woman? [3]

9 (a) Solve the equation  $2 \cos^2 x + 5 \sin x + 1 = 0$  for  $0^\circ \leq x \leq 360^\circ$ . [4]

(b) Solve the equation  $\tan y (1 + \cot y) + 2 = 0$  for  $0 \leq y \leq 2\pi$  radians. [4]

- 10 (a) A function  $f$  is defined by  $f: x \mapsto x^2 + 2x + c$  for  $x \in \mathbb{R}$ . Find the value of the constant  $c$  for which the range of  $f$  is given by  $f(x) \geq 3$ . [4]
- (b) A function  $g$  is defined by  $g: x \mapsto x^2 + 2x + 5$  for  $x \geq k$ , where  $k$  is a constant.
- (i) Express  $x^2 + 2x + 5$  in the form  $(x + a)^2 + b$ , where  $a$  and  $b$  are constants. [1]
- Given that  $g$  has an inverse,
- (ii) state the smallest possible value of  $k$ , [1]
- (iii) find an expression for  $g^{-1}$ . [2]

11 Solutions to this question by accurate drawing will not be accepted.



The diagram, which is not drawn to scale, shows a triangle  $ABC$  in which the point  $A$  is  $(9, 9)$  and the point  $B$  is  $(1, -3)$ . The point  $C$  lies on the perpendicular bisector of  $AB$  and the equation of the line  $BC$  is  $y = 8x - 11$ . Find

- (i) the equation of the perpendicular bisector of  $AB$ , [4]
- (ii) the coordinates of  $C$ . [2]
- The point  $D$  is such that  $ACBD$  is a rhombus.
- (iii) Find the coordinates of  $D$ . [2]
- (iv) Show that  $AB = 2CD$ . [2]

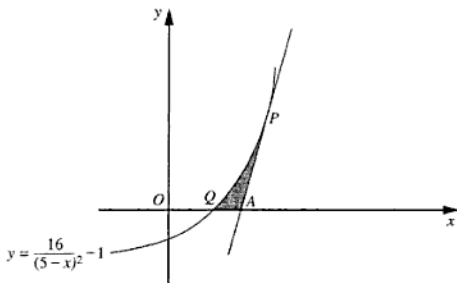
- 12 Answer only one of the following two alternatives.

**EITHER**

A particle starts from a fixed point  $A$  and travels in a straight line. The velocity,  $v \text{ ms}^{-1}$ , of the particle,  $t$ s after leaving  $A$ , is given by  $v = 1 + t - \sqrt{4t+9}$ .

- (i) Find the acceleration of the particle when it is at instantaneous rest. [7]
- (ii) Obtain an expression, in terms of  $t$ , for the displacement, from  $A$ , of the particle  $t$ s after leaving  $A$ . [5]

**OR**



The diagram shows part of the curve  $y = \frac{16}{(5-x)^2} - 1$ , cutting the  $x$ -axis at  $Q$ . The tangent at the point  $P$  on the curve cuts the  $x$ -axis at  $A$ . Given that the gradient of this tangent is 4, calculate

- (i) the coordinates of  $P$ , [5]
- (ii) the area of the shaded region  $PQA$ . [7]



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**SPECIMEN PAPERS  
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LATEST EXAMINATION PAPERS**

ISBN-13 978-981-274-282-7  
ISBN 981-274-282-4



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# ANSWERS

## O-LEVEL ADDITIONAL MATHEMATICS SPECIMEN PAPER 2008

### PAPER 1

1.  $x = 19.5^\circ$  or  $90^\circ$

2. (i) 2

$$\begin{aligned} \text{(ii)} \quad \frac{p}{q} &= \frac{\cos A + \sin A}{\cos A - \sin A} \\ &= \frac{\frac{\cos A}{\cos A} + \frac{\sin A}{\cos A}}{\frac{\cos A}{\cos A} - \frac{\sin A}{\cos A}} \\ &= \frac{1 + \tan A}{1 - \tan A} \\ &= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} \\ &= \tan(45^\circ + A) \end{aligned}$$

3. 3

4.  $k = \frac{1}{10}$  or  $\frac{1}{2}$

5.  $a = 5, b = -2$

6. (i) In  $\triangle ABC, BC = AB \sin 60^\circ$

$$\begin{aligned} &= \frac{\sqrt{3}}{2}c \\ CD &= \frac{1}{2}BC \\ &= \frac{\sqrt{3}}{4}c \end{aligned}$$

$$\begin{aligned} AC &= AB \cos 60^\circ \\ &= \frac{c}{2} \end{aligned}$$

In  $\triangle ADC, AD^2 = CD^2 + AC^2$  Pythagoras'

$$\begin{aligned} &= \left(\frac{\sqrt{3}}{4}c\right)^2 + \left(\frac{c}{2}\right)^2 \\ &= \frac{3}{16}c^2 + \frac{c^2}{4} \\ &= \frac{7}{16}c^2 \\ \therefore AD &= \frac{\sqrt{7}}{4}c \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \tan CAD &= \frac{CD}{AC} \\ &= \frac{\frac{\sqrt{3}}{4}c}{\frac{c}{2}} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} x^\circ &= \angle BAC - \angle CAD \\ &= 60^\circ - \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \end{aligned}$$

7.  $x = 2\frac{1}{4}$

8. (i)  $1 + 5px + 10p^2x^2$

$$\begin{aligned} \text{(ii)} \quad 5p + q &= -10; 10p^2 + 5pq = 15; \\ p &= -3, q = 5 \end{aligned}$$

9. (i) Centre of  $C$  is  $(3, 4)$ .

Radius of  $C$  is 3.

$$\begin{aligned} \text{(ii)} \quad \text{Distance of } C \text{ from the } y\text{-axis} &= x\text{-coordinate of } C \\ &= 3 \\ &= \text{radius} \end{aligned}$$

$\therefore C$  touches the  $y$ -axis.

$$\text{(iii)} \quad x^2 + y^2 + 6x - 8y + 16 = 0$$

10. (i)  $\frac{1}{2}$

(ii)  $8 > 0$  and  $(x+2)^2 > 0$

$$\therefore \frac{dy}{dx} = \frac{8}{(x+2)^2} \neq 0 \text{ whatever the value of } x.$$

This means the curve has no stationary point.

(iii) The corresponding rate of change is 0.005 units/second.

$$\begin{aligned} \text{11. (i)} \quad PS &= \frac{600}{PQ} = \frac{600}{x} \\ l &= 3PQ + 2PS \\ &= 3x + 2\left(\frac{600}{x}\right) \\ l &= 3x + \frac{1200}{x} \end{aligned}$$

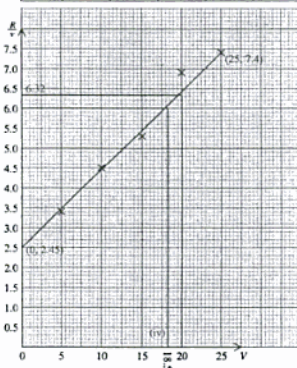
(ii)  $x = 20$

(iii) The value of  $l$ , which is 120 is a minimum value.

12. (i)  $\frac{R}{V} = av + b$

A straight line can be drawn by plotting  $\frac{R}{V}$  against  $V$ .

$V$	5	10	15	20	25
$R$	17	45	80	138	185
$\frac{R}{V}$	3.4	4.5	5.3	6.9	7.4



(ii)  $R = 126$

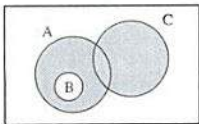
(iii)  $a = 0.20, b = 2.45$

(iv) From the graph the required value of  $v = 18.5$



## TOPIC 1 SETS

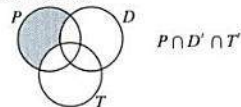
1. (i)



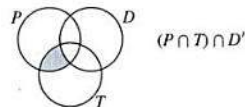
(ii)  $A \cap B' \cap C'$

(iii)  $B \cup (A \cap C)$

2. (i)



(ii)

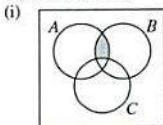


3. (a)

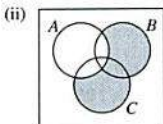
(i)  $A' \cap B$

(ii)  $A' \cup (A \cap B)$

(b)



(ii)



4. (i)

$x \notin A$

(ii)  $n(B') = 16$

(iii)  $C \cap D = \phi$  or  $n(C \cap D) = 0$

## TOPIC 2 IRRATIONAL ROOTS (SURDS)

1. (a)

0.0231

(correct to 4 decimal places)

(b) 70.5

(c) 22.2

(d)  $\frac{d^2}{dt^2} - 1.63e^{-0.0231t}$

2.  $a = 2, b = 4; a = 6\frac{2}{3}, b = 1\frac{1}{3}$

3. (i)

$2 + \sqrt{3}$

(ii)  $2\sqrt{3}$

4.  $2\sqrt{3} + 1\sqrt{3}$

5.  $a = 19, b = -8$

6.  $(3 + 2\sqrt{3})m$

7.  $(2 - \sqrt{5})^2 - \frac{8}{3 - \sqrt{5}}$

$$= (4 - 4\sqrt{5} + 5) - \frac{8(3 + \sqrt{5})}{3^2 - (\sqrt{5})^2}$$

$$= 9 - 4\sqrt{5} - \frac{8(3 + \sqrt{5})}{4}$$

$$= 9 - 4\sqrt{5} - 2(3 + \sqrt{5})$$

$$= 9 - 4\sqrt{5} - 6 - 2\sqrt{5}$$

$$= 3 - 6\sqrt{5}$$

$$= p + q\sqrt{5}$$

$$p = 3, q = -6$$

8.  $(5 - 2\sqrt{2}) \text{ cm}$

## TOPIC 3 INDICES

1.  $x = 1.74$

2. (a)  $a + b^2$

(b) 0.8

3.  $x = 1.6$

4.  $p = 6$

$q = -3$

5.  $x = 7, y = 2$

6.  $6^x = \frac{2}{3}$

## TOPIC 4 LOGARITHMS

1.  $x = 5\frac{1}{3}$

2.  $x = 1.58$

3.  $x = 18$

4.  $x = 1.74$

5. (a)  $y = \frac{\sqrt{10}}{x}$

(b) (i)  $-p$

(ii)  $3 + 2p$

6. (i)  $x = \frac{5}{8}$

(ii)  $y = 256$

7. (a) (i)  $\approx 12\ 100$  (to 3 figures)

(ii) In the year 2006

(b) 1.63

8. (a)  $y = 2$

(b)  $\frac{3}{4}$

9. (i)  $k = 0.0330$

(ii)  $V = 371$

10. (a)  $x = \frac{8}{11}$

(b)  $c = 9 - 3b$

11.  $x = 2\frac{1}{4}$

12. (i) \$48 700

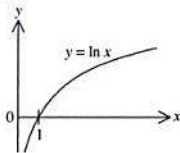
(ii) 56 months

### TOPIC 5 SIMULTANEOUS LINEAR EQUATIONS

- (i)  $2\frac{1}{4} \text{ ms}^{-2}$   
(ii) 2.41 m

### TOPIC 6 SOLUTION OF QUADRATIC EQUATIONS

- $a = 2$
- 0.315
- (i)



- (ii)  $y = \frac{2-x}{2}$
- $x = 8$
- $x = \frac{2}{3}$
- (i)  $(2^x)^2 - 2(2^x) - 3 = 0$   
(ii)  $x = 1.58$

### TOPIC 7 SIMULTANEOUS LINEAR AND NON-LINEAR EQUATIONS

- The points of intersection are (5, 2.5) and (6, 2).
- $x = 7, y = 4$
- $x = \frac{5}{6}, y = \frac{3}{4}; x = -1, y = -2$
- $(1\frac{2}{3}, 6\frac{1}{3}); (-3, -3)$
- (5, 4)
- $x = 5, y = 9$

### TOPIC 8 LINEAR GRAPHS

- (i)  $B = (0, 4)$   
(ii)  $C = (6, 2)$
- (i)  $c = (4, 3); E(2\frac{2}{3}, 5\frac{2}{3})$   
(ii)  $y = x - 1; F = (2\frac{2}{3}, 1\frac{2}{3})$   
(iii)  $14\frac{2}{3} \text{ units}^2$
- (i) A and B are (-1, 3) and (1, 2)  
(ii)  $AB = \sqrt{5}$
- (i)  $y + 2x = 18$   
(ii)  $X = (5\frac{1}{5}, 7\frac{1}{5})$   
(iii)  $C = (18, 14)$   
 $D = (24, 10)$   
(iv) 112 units<sup>2</sup>

- $4x - 2y - 7 = 0$
- $(-2, 3), (5, 4)$
- (i) (6, 7)  
(ii)  $(4\frac{1}{2}, 5)$   
(iii) (3.5, 2), 2.5 units<sup>2</sup>
- $4x + 3y = 39$
- (i)  $AB^2 + AC^2 = BC^2$   
 $\therefore$  The triangle is right angled at A.  
(ii) 25 units<sup>2</sup>
- (i)  $p = (2, 5)$   
(ii)  $y = 9 - 2x$   
(iii)  $Q = (4\frac{1}{2}, 0)$
- $4x + 6y + 5 = 0$
- (i)  $C = (6, 3), B = (8, 9), D = (4, 2)$   
(ii) 26.1 units
- 25 units<sup>2</sup>
- $A = (5, 15)$   
 $B = (0, 16\frac{2}{3})$   
 $C = (-3\frac{1}{3}, 6\frac{2}{3})$

$$15. (i) AB = \sqrt{(0-2)^2 + (10-16)^2}$$

$$= \sqrt{4+36}$$

$$= \sqrt{40} \text{ units}$$

$$BC = \sqrt{(2-8)^2 + (16-14)^2}$$

$$= \sqrt{36+4}$$

$$= \sqrt{40}$$

$$\therefore AB = BC$$

$\triangle ABC$  is isosceles.

- (ii)  $D = (10, 0)$   
(iii)  $\frac{1}{3}$
- (11, -1)
- If  $2x + y = 14$ , then  $y = 14 - 2x$   

$$2x^2 - y^2 = 2xy - 6$$

$$2x^2 - (14 - 2x)^2 = 2x(14 - 2x) - 6$$

$$2x^2 - 196 + 56x - 4x^2 = 28x - 4x^2 - 6$$

$$2x^2 + 28x - 190 = 0$$

$$x^2 + 14x - 95 = 0$$

$$(x + 19)(x - 5) = 0$$

If  $x + 19 = 0$                       If  $x - 5 = 0$   
 $x = -19$                                $x = 5$   
 $y = 14 - 2(-19)$                    $y = 14 - 2(5)$   
 $= 52$                                        $= 4$

 $\therefore A$  and  $B$  are  $(-19, 52)$  and  $(5, 4)$ .  

$$AB = \sqrt{(-19-5)^2 + (52-4)^2}$$

$$= \sqrt{576 + 2304}$$

$$= \sqrt{2880}$$

$$= \sqrt{576 \times 5}$$

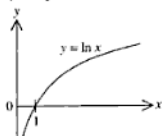
$$= \sqrt{24^2 \times 5}$$

$$= 24\sqrt{5} \text{ units}$$

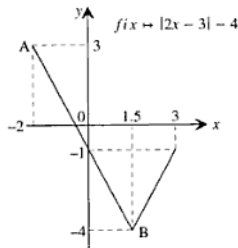
18. (i)  $C = (4.5, -1)$   
 (ii)  $D = (7.5, 7)$   
 (iii)  $k = \frac{1}{3}$   
 (iv) Gradient of  $CE \times$  Gradient of  $DE$   
 $= \frac{\frac{7}{10} - (-1)}{10 - 4.5} \times (-\frac{8}{3})$  from 3rd step of (iii)  
 $= \frac{8}{33} \times (-\frac{8}{3})$   
 $= -\frac{64}{9}$   
 $\neq 1$   
 $\therefore \angle CED$  is not a right angle.
19. (i)  $AB = \frac{p-3}{2}; BC = \frac{5-p}{12}$   
 (ii)  $p = 9$   
 (iii)  $D = (9, 0)$   
 (iv) 35.5 units

**TOPIC 9 - GRAPHS OF LINEAR ABSOLUTE VALUE FUNCTIONS**

1.  $x = \frac{3}{4}, y = 1\frac{1}{2}$   
 2. (i)



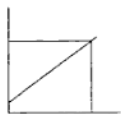
- (ii)  $y = \frac{2-x}{2}$   
 3. (i)



- (ii)  $x = \frac{1}{2}$  or  $2\frac{1}{2}$   
 (iii)  $x = \frac{1}{2}$  or  $2\frac{1}{2}$   
 (iv)  $k = 1.5$   
 (v)  $g: x \mapsto -2x + (-1)$

**TOPIC 10 - REDUCTION TO LINEAR EQUATIONS**

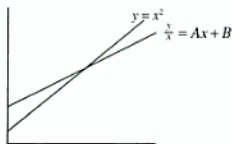
1. (a) (i)  $y = \frac{9x}{2} - 16x^2$   
 (ii)  $x = 1.30$   
 (b)  $c = 5, d = 2$
2. (a) (i)  $A = 2.0, k = 1.6$   
 (ii)  $x = 3.4$   
 (b) (i)  $p = 6, q = -8$   
 (ii)  $k = 5\frac{1}{2}$
3. (a) (i)  $\lg y = \lg A + x \lg k$   
 (ii)  $A = 3.1, k = 5.01$   
 (iii)  $x = 3.26$   
 (b)  $\frac{1}{2}$  or 2



4. (a)
- 
- (i)  $y = 2x + \frac{5}{2}$   
 (ii)  $x = \pm 3.54$
- (b) (i)  $y = 2x^2 + x - 2$   
 (ii) 3.24  
 (iii)  $x = 1.8; y = 6.28$
5. (a)
- 
- (i)  $a = 9.0; b = -1.5$   
 (ii)  $x = 1.3$   
 (b) (i)  $c = 1\frac{1}{2}, d = -\frac{1}{2}$   
 (ii)  $y = \frac{3}{7}$

6. (i), (iii)

$x$	50	100	150	200	250
$y$	3 700	11 000	21 600	36 000	53 500
$\frac{y}{x}$	74	110	144	180	214

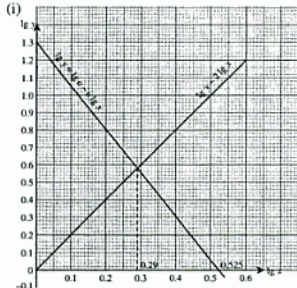


- (ii)  $A = 0.7$ ;  $B = 40$   
 (iii) The value of  $x$  given by the point of intersection of the two lines gives the dimension of the rectangle when it is a square.  
 (iv) The ratio approaches the constant 0.7.

7. (i), (ii), (iii)

	$Y$	$X$	$m$	$c$
$y = ab^x$	$\lg y$	$x$	$\lg b$	$\lg a$
$y = Ax^2$	$\lg y$	$\lg x$	$k$	$\lg A$
$px + qy = xy$	$y$	$\frac{y}{x}$	$q$	$p$

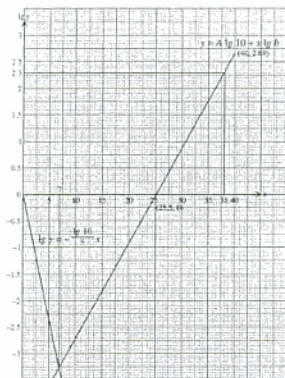
8. (i)



- (ii)  $a = 20$   
 $n = 2.5$   
 (iii)  $x = 1.9$

9. (i)

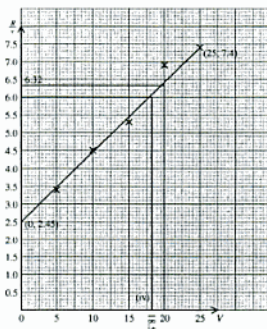
$x$	15	20	2.5	30	35	40
$v$	0.15	0.38	0.95	2.32	5.90	14.80
$\lg v$	-1.90	-0.97	-0.05	0.84	1.77	2.69



- (ii)  $b = 1.20$   
 $A = 2.06$   
 (iii)  $x = 38$   
 (iv)  $x = 7$

10. (i)  $\frac{R}{v} = av + b$

$v$	5	10	15	20	25
$R$	17	45	80	138	185
$\frac{R}{v}$	3.4	4.5	5.3	6.9	7.4



- (ii) 126  
 (iii)  $a = 0.20$ ,  $b = 2.45$   
 (iv)  $v = 18.5$

- (ii)  $x = 0.382, 2.62, 3$
10. (i)  $(x-1)(x-k)(x-k^2) = 0$   
 $(x-1)(x-k)(x-k^2) = Q(x-2) + 7$   
 Let  $x = 2$   
 $(2-1)(2-k)(2-k^2) = Q(2-2) + 7$   
 $(2-k)(2-k^2) = 7$   
 $4 - 2k^2 - 2k + k^3 = 7$   
 $k^3 - 2k^2 - 2k - 3 = 0$
- (ii) The only real value of  $k$  for this equation is 3.
11.  $a = 5, b = -2$

#### TOPIC 25 PERMUTATIONS AND COMBINATIONS

- (i) 126  
(ii) 56, 12
- (a) There are 60 ways in which the cast can be made.  
(b) 200 such number can be made.
- (i) 210  
(ii) 7  
(iii) 175
- (a) 322, 560  
(b) 40
- (a) (i) 120  
(ii) 36  
(b) The number of different digits = 96  
Number of these 5-digit numbers which are even = 60
- (a) 3024  
(b) 910 ways

#### TOPIC 26 BINOMIAL THEOREM: POSITIVE INTEGRAL INDEX

- 4
- (a) -270  
(b) First 4 terms =  $1 + 7p + 21p^2 + 35p^3$   
Coefficient of  $x^3 = 119$
- 672
- $-3\frac{9}{25}$
- (i)  $n = 4$   
(ii) -32
- $64 + 192x + 240x^2 + \dots; 48$
- (i)  $243x^5 - 405x^4 + 270x^3 - \dots$   
(ii) 135
- $a = 3, n = 7$   
 $b = 238$
- $n = 8, p = -1\frac{1}{2}, q = -189$
- $k = 3$

- (i)  $1 + 5px + 10p^2x^2$   
(ii)  $5p + q = -10; 10p^2 + 5pq = 15$   
 $p = -3, q = 5$

#### TOPIC 27 MATRICES

- \$27 600
- $\frac{1}{3} \begin{pmatrix} 6 & -3 \\ -7 & 4 \end{pmatrix}$   
 $x = 2, y = -5$
- The three matrices are (0.60 0.20 0.50),

$$\begin{pmatrix} 8 & 6 & 6 & 5 \\ 5 & 4 & 3 & 2 \\ 3 & 3 & 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 40 \\ 50 \\ 50 \\ 60 \end{pmatrix}$$

Total cost = \$1111

- $A^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$   
 $B^{-1} = \frac{1}{8} \begin{pmatrix} 3 & -1 \\ -3 & 7 \end{pmatrix}$   
 (i)  $\begin{pmatrix} 3 & -1 \\ -3 & 7 \end{pmatrix}$   
 (ii)  $\frac{1}{8} \begin{pmatrix} 9 & 5 \\ 14 & 6 \end{pmatrix}$   
 or  $\begin{pmatrix} 1\frac{1}{8} & \frac{5}{8} \\ 1\frac{7}{8} & \frac{3}{4} \end{pmatrix}$
- $A^{-1} = \frac{1}{23} \begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}; x = 1, y = -2$
- $p = 2, k = 5$
- (i)  $\begin{pmatrix} 5 & 8 & 4 & 10 \\ 300 & 60 & 40 \\ 150 & 50 & 20 \\ 120 & 40 & 0 \\ 100 & 0 & 0 \end{pmatrix}$   
 (ii)  $\begin{pmatrix} 4180 & 860 & 360 \end{pmatrix}$   
 (iii)  $\begin{pmatrix} 5\% \\ 10\% \\ 20\% \end{pmatrix}$   
 (iv) (367)
- (i)  $x = 3, y = 4$   
 (ii)  $B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

8. (i)  $S = \frac{1}{2}OA \times OB \times \sin AOB + \frac{1}{2}OC \times OD \times \sin COD$   
 $= \frac{1}{2} \times 2 \times 2 \times \sin(90^\circ - x) + \frac{1}{2} \times 4 \times 4 \times \sin x$   
 $S = 2 \cos x + 8 \sin x$
- (ii)  $S = \sqrt{68} \sin(x + 14.0^\circ)$   
 $R = \sqrt{68}$ ;  $\alpha = 14.0^\circ$
- (iii)  $32.7^\circ$
- (iv)  $S = \sqrt{68}$ ;  $x = 76.0^\circ$
- (v)  $26.6^\circ$
9. (a) (i)  $x = 60^\circ, 104.5^\circ, 255.5^\circ, 300^\circ$   
 (ii)  $y = 97.2^\circ$  or  $262.8^\circ$
- (b)  $z = 0.538$  or  $2.11$
10. (a) (i)  $x = 0^\circ, 30^\circ, 150^\circ, 180^\circ$   
 (ii)  $y = 108.4^\circ$  or  $161.6^\circ$
- (b) (i)  $5 \cos(\theta + 0.644)$   
 (ii)  $\theta = 0.516$
11.  $143.1^\circ, 323.1^\circ$
12. (a)  $x = 113.6^\circ, 246.4^\circ$   
 (b)  $y = 1.70, 3.28$
13. (a)  $x = 210^\circ, 330^\circ$   
 (b)  $y = 0.66, 3.80$
14. (i)  $\theta = 78.7^\circ$   
 (ii) The constant value is 13.
15. (a)  $x = 19.5^\circ$  or  $160.5^\circ$   
 (b)  $y = 3.93$  or  $5.50$
16. (b)  $x = 45^\circ, 116.6^\circ, 225^\circ, 296.6^\circ$   
 (b)  $y = 1.56$  or  $2.15$
17.  $x = 19.5^\circ$  or  $90^\circ$
18. (i)  $P = 2(AB + BC)$   
 $= 2(AB + 2AO)$   
 $= 2(OB \sin \theta + 2OB \cos \theta)$   
 $= 2(OB)(\sin \theta + 2 \cos \theta)$   
 $= 2 \times 4(\sin \theta + 2 \cos \theta)$   
 $P = 16 \cos \theta + 8 \sin \theta$
- (ii)  $P = 8\sqrt{5} \cos(\theta - 26.57^\circ)$
- (iii) Maximum value of  $P$  is  $8\sqrt{5}$ .  
 Maximum value of  $\theta = 26.6$
- (iv)  $\theta = 59.6$

### TOPIC 32 SOME GEOMETRY THEOREMS

1. (i) In  $\triangle ABC$ ,  $BC = AB \sin 60^\circ$   
 $= \frac{\sqrt{3}}{2}c$   
 $CD = \frac{1}{2}BC$   
 $= \frac{\sqrt{3}}{4}c$   
 $AC = AB \cos 60^\circ$   
 $= \frac{c}{2}$

In  $\triangle ADC$ ,  $AD^2 = CD^2 + AC^2$  Pythagoras'

$$= \left(\frac{\sqrt{3}}{4}c\right)^2 + \left(\frac{c}{2}\right)^2$$

$$= \frac{3}{16}c^2 + \frac{c^2}{4}$$

$$= \frac{7}{16}c^2$$

$$\therefore AD = \frac{\sqrt{7}c}{4}$$

- (ii)  $\tan CAD = \frac{CD}{AC}$   
 $= \frac{\frac{\sqrt{3}}{4}c}{\frac{c}{2}} = \frac{\sqrt{3}}{2}$   
 $= \frac{\sqrt{3}}{2}$   
 $\angle CAD = \angle BAC - \angle CAD$   
 $= 60^\circ - \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$
2. (i)  $\angle PRQ = \angle QPT$  ext.  $\angle = \angle$  in the alt. segment  
 (ii)  $\angle RNP = 90^\circ$   $RN \perp SPT$   
 $\angle PMR = 90^\circ$   $PM \perp QR$   
 $\therefore \angle RNP + \angle PMR = 180^\circ$   
 $\therefore NRMP$  is a cyclic quadrilateral  
 i.e.  $PR$  is a diameter of the circle  $NRMP$   
 (iii) Since a circle passes through  $NRMP$   
 $\angle MPT = \angle NRM$  ext.  $\angle$  of cyclic quad.  
 $\angle QPT = \angle PRQ$  proved  
 $\angle MPT - \angle QPT = \angle NRM - \angle PRQ$   
 $\angle MPQ = \angle NRP$   
 but  $\angle NRP = \angle NMP$   
 $\angle s$  in the same segment,  
 $NRMP$  is a cyclic quad.  
 $\therefore \angle MPQ = \angle NMP$   
 $\therefore MN \parallel QP$  alt.  $\angle s$  equal

### TOPIC 33 CIRCLES, SECTORS AND SEGMENTS

1. (i) 50.3 cm  
 (ii) 15.3 cm<sup>2</sup>
2. (i) 16.9 cm  
 (ii) 12.6 cm<sup>2</sup>
3. (i) 18.3 cm  
 (ii) 38.0 cm<sup>2</sup>  
 (iii) 39.9 cm
4. (i) 62.0 cm  
 (ii) 298 cm<sup>2</sup>
5.  $r = 2$ ,  $\theta = 5$  or  $r = 5$ ,  $\theta = \frac{4}{5}$
6. (i)  $\angle AOB = 1.2$  radians  
 (ii)  $DE = 7.46$  cm  
 (iii)  $\angle DOE \approx 0.485$   
 (iv) 9.28 cm<sup>2</sup>
7. 32 cm

8. (i) Sector  $COB = 38.4 \text{ cm}^2$   
 (ii) Sector  $CAD = 52.3 \text{ cm}^2$   
 (iii) Area of the shaded region =  $15.9 \text{ cm}^2$
9. (i)  $P = r(\tan \theta + \theta + \frac{1}{\cos \theta} - 1)$   
 (ii)  $A = \frac{1}{2}r^2(\tan \theta - \theta)$   
 (iii)  $r = 15.0$   
 (iv)  $A = 154$

### TOPIC 34 VECTOR GEOMETRY

1. (i)  $\vec{OB} = 7\mathbf{i} - \mathbf{j}$   
 (ii) The angle between  $AC$  and  $OB = 63.4^\circ$ .
2. (i)  $\vec{OD} = p\mathbf{b} + (1-p)\mathbf{a}$   
 (ii)  $\vec{OD} = q(\mathbf{a} + 2\mathbf{b})$   
 $p = \frac{2}{3}, q = \frac{1}{3}; k = 7$
3. (a) (i)  $-13$   
 (ii)  $\sqrt{290}$   
 The angle between  $\mathbf{a}$  and  $\mathbf{b}$   
 $= \cos^{-1} \frac{-13}{\sqrt{290}}$   
 $= 139.8^\circ$  (or  $40.2^\circ$ )  
 (iii)  $1\frac{1}{2}$
- (b) (i)  $\frac{2}{3}(2\mathbf{e} + 3\mathbf{d})$   
 (ii)  $\lambda = 2\frac{1}{2}, \mu = \frac{1}{2}$
4.  $\frac{4}{3}\mathbf{i} - \frac{3}{5}\mathbf{j}$
5. (i)  $k = 3$   
 (ii)  $63.4^\circ$
- (iii)  $\begin{pmatrix} 12\lambda - 8 \\ 6 - 4\lambda \end{pmatrix}$
6. (a)  $0.8$   
 (b)  $3$
7. (a) (i)  $\frac{8}{12} = \frac{2}{3} = \frac{4}{6}$   
 $\therefore \vec{AO} \parallel \vec{OB}$   
 But they meet at  $O$ .  
 $\therefore O, A$  and  $B$  lie on the same straight line.  
 (ii)  $d = 5$
- (b) (i)  $\vec{PQ} = \mathbf{q} - \frac{1}{4}\mathbf{p}$   
 $\vec{OS} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$   
 (ii)  $\vec{ST} = \frac{2}{3}\mathbf{q} - \frac{1}{4}\mathbf{p}$   
 (iii)  $\vec{ST} = (\frac{1}{2} + \mu)\mathbf{q} - \frac{1}{2}\mathbf{p}$   
 $\lambda = 2; \mu = \frac{1}{2}$
8. (a) (i)  $\frac{1}{3}(\mathbf{c} + 2\mathbf{b})$

(ii)  $\vec{OQ} = \mathbf{c} + 2\mathbf{b} - 2\mathbf{a}$   
 $\vec{CQ} = \vec{OQ} - \vec{OC}$   
 $= (\mathbf{c} + 2\mathbf{b} - 2\mathbf{a}) - \mathbf{c}$   
 $= 2(\mathbf{b} - \mathbf{a})$   
 $= 2(\vec{OB} - \vec{OA})$   
 $= 2\vec{AB}$

$\therefore \vec{CQ}$  is parallel to  $\vec{AB}$ .

- (b) (i)  $2\mathbf{i} - 2\mathbf{j}$  (ii)  $4\mathbf{i} + 4\mathbf{j}$   
 (iii)  $6\mathbf{i} + 5\mathbf{j}$
- (iv)  $\vec{OA} = 3\mathbf{i} + 4\mathbf{j}$   
 $\vec{OA}' = 3\mathbf{i} - 4\mathbf{j}$   
 $\vec{A}'C = \vec{OC} - \vec{OA}'$   
 $= (5\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - 4\mathbf{j})$   
 $= 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{i} + 4\mathbf{j}$   
 $= 2\mathbf{i} + 6\mathbf{j}$   
 $\vec{CD} = \vec{OD} - \vec{OC}$   
 $= (6\mathbf{i} + 5\mathbf{j}) - (5\mathbf{i} + 2\mathbf{j})$   
 $= 6\mathbf{i} + 5\mathbf{j} - 5\mathbf{i} - 2\mathbf{j}$   
 $= \mathbf{i} + 3\mathbf{j}$   
 $= \frac{1}{2}(2\mathbf{i} + 6\mathbf{j})$   
 $= \frac{1}{2}\vec{A}'C$   
 $\therefore \vec{A}'C \parallel \vec{CD}$   
 But they concur at  $C$ .  
 $\therefore A', C$  and  $D$  are collinear.  
 $A'C : CD = A'C : \frac{1}{2}A'C$   
 $= 2 : 1$

9. (i)  $\vec{AP} = \frac{1}{3}\mathbf{b} - \mathbf{a}, \vec{OM} = \frac{\mathbf{a} + \mathbf{b}}{2}$   
 (ii)  $\vec{OQ} = \frac{2}{3}(\mathbf{a} + \mathbf{b})$   
 (iii)  $\vec{OQ} = (1 - \mu)\mathbf{a} + \frac{\mu}{3}\mathbf{b}$   
 (iv)  $\lambda = \frac{1}{2}, \mu = \frac{1}{4}$
10.  $k = 2$
11.  $k = 2\frac{2}{3}$
12. (i)  $\vec{OX} = \frac{\pi}{3}(\mathbf{q} + 2\mathbf{p})$   
 (ii)  $\vec{PX} = \frac{\pi}{3}(2\mathbf{q} - 5\mathbf{p})$   
 (iii)  $n = \frac{5}{9}$   
 $m = \frac{2}{3}$
13. (i)  $\vec{OP} = 6\mathbf{i} - 8\mathbf{j}$   
 $\vec{OQ} = 12\mathbf{i} + 9\mathbf{j}$   
 (ii)  $\lambda = 5$

### TOPIC 35 VELOCITY VECTORS

- 64 km h<sup>-1</sup>
  - 252 km h<sup>-1</sup>
  - 014.7°; 037° and 26 minutes
- 71.0°
  - 350.9°
  - 173.2°
  - 09 24
- From the direction of 108.5°
  - 883 km
- $P = 20\mathbf{i} + (50 + 10t)\mathbf{j}$   
 $Q = (80 - 10t)\mathbf{i} + (20 + 30t)\mathbf{j}$
  - 22.4 units
  - Velocity of  $P$  relative to  $Q$   
 $= (20\mathbf{i} + 10\mathbf{j}) - (-10\mathbf{i} + 30\mathbf{j})$   
 $= (20 + 10)\mathbf{i} - (-10 + 30)\mathbf{j}$   
 $= (20 + 10)\mathbf{i} + (10 - 30)\mathbf{j}$   
 $= 30\mathbf{i} - 20\mathbf{j}$   
 Position vector of  $P$  relative to  $Q$  at 1200 hour  
 $= 50\mathbf{j} - (80\mathbf{i} + 20\mathbf{j})$   
 $= -80\mathbf{i} + (50 - 20)\mathbf{j}$   
 $= -80\mathbf{i} + 30\mathbf{j}$   
 But  $\frac{30}{-80} \neq \frac{-20}{30}$ .  
 The two vectors are not even parallel.  
 Therefore,  $P$  and  $Q$  will never meet.
- Bearing of  $Q$  from  $P$  is 108.4°
  - 47 minutes
- 34 s
- 2.5 ms<sup>-1</sup>
  - The ferry must be steered upstream at an angle of 143.1° to the bank it is leaving.

### TOPIC 36 DIFFERENTIATION OF SUM AND PRODUCT

- $y = x + 2$   
 $B = (-1, 1)$
  - $R = (2, 4)$   
 $K = (2, 1)$
  - 2 : 3
  - 4½ units
- $\frac{1}{12}$
- The square of a real number cannot be negative. Therefore the curve  $y = f(x)$  has no turning point. Since its gradient cannot be zero. Since it has no turning point, there is always a one-to-one correspondence between the values of  $x$  and  $f(x)$ . Therefore the function  $f$  has an inverse.

- $f^{-1}(9) = x = 2$
- $c = \pm 12$
- $a = -9, b = 17$
  - $(-2, 27)$

### TOPIC 37 QUOTIENT RULE

- $k = 13$

### TOPIC 38 CHAIN RULE

- $-20(3 - 2x)^9$

### TOPIC 39 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

- $x(1 + 2 \ln x)$
  - 70.5 or 70.6
  - 55.5
  - 0.513 (or 4)
  - 0.96 to 0.97
- $\frac{4x+5}{(x+1)(2x+3)}$
  - $\frac{dy}{dx} = ky$   
 $k = 2; A = -4$
- $\frac{d}{dx}xe^{-x} = x(e^{-x})(-\frac{1}{x}) + e^{-x}(1)$   
 $= e^{-x} - \frac{1}{x}e^{-x}$   
 $= \frac{1}{x}e^{-x}(2 - x)$   
 $= \frac{1}{2}(2 - x)e^{-x}$
  - $\frac{1}{4}(x - 4)e^{-x}$
  - $M = (2, 2e^{-1})$  or  $(2, 0.736)$
  - $M$  is a maximum point.

### TOPIC 40 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

- (0.442, 1)
  - (2.21, -5.00)
- $\sin 2x$
- $\frac{dy}{dx} = \frac{4 \sin x - 1}{(\sin x - 4)^2}$   
 $x = 0.253, 2.89$
  - 0.64
  - 10.4
- $x^2(2x \cos 2x + 3 \sin 2x)$



- (b) (i)  $-2$   
 (ii)  $a = -2, b = -1$
4. (i)  $A = (-7, -7); B = (1, 1); C = (9, 9)$   
 (ii)  $\frac{1}{2}t^2$   
 (iii)  $y = 3x + 14$   
 $y = 3x - 18$   
 (v)  $4$   
 (vi)  $64y = x^3 - 3x^2 + 3x + 63$
5. (i)  $\frac{dy}{dx} = \frac{t}{1+t}$   
 (ii)  $17\frac{1}{2}$  units<sup>2</sup>  
 (iii)  $C = (5, 4); D = (1, 8)$   
 $AC = AD = \sqrt{10}$   
 (iv)  $k = 16$

#### TOPIC 46 SMALL INCREMENTS AND APPROXIMATIONS

1. (i)  $-\frac{4c}{(1+2c)^2}$   
 (ii)  $c = 18$
2.  $\frac{p}{60}$
3.  $\frac{200p}{1} \%$
4.  $\frac{dy}{dx} = \ln x$   
 Approximate change in  $y = 2p$
5. (i)  $\frac{dy}{dx} = 2x - \frac{6}{x^2}$   
 (ii)  $0.125$
6. (i)  $\frac{dy}{dx} = \frac{2}{2x-3}$   
 (ii)  $1 + 2p$
7. (i)  $-6$   
 (ii)  $-6p$

#### TOPIC 47 CONNECTED RATES OF CHANGE

1.  $\frac{dy}{dx} = -5; 10$  units/sec
2. (i)  $13.2$   
 (ii)  $t = 34.7$   
 (iii)  $0.177$  g/day
3. (i) Volume of water =  $\frac{8x^3}{27}$  cm<sup>3</sup>  
 (ii)  $28.6$  cm s<sup>-1</sup>
4. =  $0.0002$   
 $3\frac{1}{2}$  or  $-6\frac{1}{2}$
5. (i)  $\frac{60}{247}$  cm s<sup>-1</sup> ( $0.243$  cm s<sup>-1</sup>)  
 (ii)  $20$  cm
6. (i)  $t = 3$   
 (ii) The distance is  $42$  m when its velocity is a minimum.
7.  $48$

8. (i)  $A = 5; k = 0.0511$   
 (ii)  $-0.153$
9. (i)  $\frac{2x+3-2x \ln x}{x(2x+3)^2}$   
 (ii)  $\Delta y > \frac{p}{2}$   
 (iii) Rate of change of  $x = 0.6$  unit per second
10. (i) The gradient is  $\frac{1}{2}$ .  
 (ii)  $8 > 0$  and  $(x+2)^2 > 0$   
 $\therefore \frac{dy}{dx} = \frac{x}{(x+2)^2} \neq 0$  whatever the value of  $x$ .  
 This means the curve has no stationary point.
- (iii) The corresponding rate of change is  $0.005$  units/second.

#### TOPIC 48 INDEFINITE INTEGRALS

1.  $\frac{1}{8}(4x+5)^{\frac{3}{2}} + c$
2.  $3y = 2x^3 - 9x^2 + 45$
3.  $y = x^3 + 2 \ln x + 2$
4.  $c = \frac{1}{7(0.1-2)}$
5.  $y = 2x^3 + 3x^2 - 5x - 6$
6.  $y = 2x - \frac{4x}{32} - 3\frac{1}{2}$
7.  $(1\frac{3}{4}, 0)$
8. (i)  $\frac{dy}{dx} = x^2 + 3x^2 \ln x$   
 (ii)  $\ln x = -\frac{1}{3}$   
 (iii)  $\Delta y = 29.6 p$   
 (iv)  $\frac{1}{2}(x^3 \ln x - \frac{1}{3}x^3) + d$ ,  
 where  $d$  is also a constant.
9. (i)  $y = x^3 - x^2 - 5x - 3$   
 (ii) Gradient of the curve  
 $= 3x^2 - 2x - 5$   
 $= 3(x^2 - \frac{2}{3}x) - 5$   
 $= 3[x^2 - \frac{2}{3}x + (\frac{1}{3})^2] - 5 - 3(\frac{1}{3})^2$   
 $= 3(x - \frac{1}{3})^2 - 5 - \frac{1}{3}$   
 $= 3(x^2 - \frac{2}{3}x + \frac{1}{9}) - \frac{16}{3}$   
 $\geq -\frac{16}{3}$  since  $3(x - \frac{1}{3})^2 \geq 0$

#### TOPIC 49 DEFINITE INTEGRALS

1.  $\frac{1}{4}\pi$  or  $0.785$
2.  $\frac{5}{24}$
3. (a) (i)  $-1\frac{1}{2}$   
 (ii)  $4\frac{1}{2}$   
 (b)  $\frac{d}{dx}(\frac{1}{9-4x^2}) = \frac{8x}{(9-4x^2)^2}$   
 $\int_0^1 \frac{x}{(9-4x^2)^2} dx = \frac{1}{90}$
- (c)  $0.272$  units<sup>2</sup>

4. (a) (i)  $\frac{3}{8}$   
(ii) 0.708  
(b)  $k = 8\frac{1}{2}$
5. (a) (i)  $x \cos x + \sin x$   
(ii)  $\frac{\pi}{2} - 1$  or 0.571
6. 3.41 unit<sup>2</sup>
7.  $\frac{dy}{dx} = \frac{12x}{\sqrt{4x-3}}$   
(i)  $k = 12$   
(ii)  $6\frac{2}{3}$
8. (i)  $y = (x+2)\sqrt{x-1}$   
 $\frac{dy}{dx} = (x+2)\frac{d}{dx}\sqrt{x-1} + \sqrt{x-1}\frac{d}{dx}(x+2)$   
 $= (x+2)\frac{1}{2\sqrt{x-1}} + \sqrt{x-1}$   
 $= \frac{(x+2)+2(x-1)}{2\sqrt{x-1}} + \sqrt{x-1}$   
 $= \frac{x+2+2x-2}{2\sqrt{x-1}} + \sqrt{x-1}$   
 $= \frac{3x}{2\sqrt{x-1}} + \sqrt{x-1}$   
 $= \frac{3x}{\sqrt{x-1}}$   
 $k = \frac{3}{2}$   
 $= 1\frac{1}{2}$   
(ii)  $6\frac{2}{3}$
9. (i)  $\frac{dy}{dx} = 2 \sin 2x - 2 \sin x$   
 $\frac{d^2y}{dx^2} = 4 \cos 2x - 2 \cos x$   
(ii) The stationary point is a maximum point.  
(iii)  $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} y dx = 0.701$
10.  $\frac{\sqrt{x}}{4}$  or 0.433
11. 3
12. (i)  $\frac{4}{x-3} - \frac{3}{x+3}$   
(ii) 2.37
6. (a) (i) 6 units<sup>2</sup>  
(ii) 12 units<sup>2</sup>  
(b) (i)  $1\frac{1}{2}$  units<sup>2</sup>  
(ii)  $2\frac{2}{3}$  units<sup>2</sup>
7.  $\cos^2 2x = \frac{\cos 4x + 1}{2}$   
Area of the region = 1.93 units<sup>2</sup>.
8. (i)  $e^{2x}(2x+1)$   
(ii)  $x = -\frac{1}{2}$   
(iii) 156 units
9.  $\frac{4}{7}$  units<sup>2</sup>
10. (i)  $y = e^{\frac{1}{2}x} + 3e^{-\frac{1}{2}x}$   
 $\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x} + 3(-\frac{1}{2})e^{-\frac{1}{2}x}$   
 $= \frac{1}{2}e^{\frac{1}{2}x} - \frac{3}{2}e^{-\frac{1}{2}x}$   
At the stationary point,  
 $\frac{1}{2}e^{\frac{1}{2}x} - \frac{3}{2}e^{-\frac{1}{2}x} = 0$   
 $e^{\frac{1}{2}x} - 3e^{-\frac{1}{2}x} = 0$   
 $E^{\frac{1}{2}x} = 3e^{-\frac{1}{2}x}$   
Multiply each side by  $e^{-\frac{1}{2}x}$ ,  
 $e^x = 3$   
 $e^{\frac{1}{2}x} = \sqrt{3}$   
 $e^{-\frac{1}{2}x} = \frac{1}{\sqrt{3}}$   
 $y = \sqrt{3} + 3\left(\frac{1}{\sqrt{3}}\right)$   
 $= \sqrt{3} + \sqrt{3}$   
 $= 2\sqrt{3}$   
(ii) The stationary point is a minimum.  
(iii) 3.66 units<sup>2</sup>
11. 7 units<sup>2</sup>
12. (i) 18.5 units<sup>2</sup>  
(ii)  $a = 1, b = 4$   
The stationary point is a minimum point.
13. (i)  $P = (3, 3)$   
(ii)  $R = (-9, 0)$   
(iii)  $19\frac{1}{3}$  units<sup>2</sup>
14. (i)  $A = (-0.963, 0)$   
 $B = (0, 3)$   
(ii)  $C = (6, 0)$   
(iii) Area of the shaded region  
 $= \text{Area of } AOB + \text{Area of } OBC$   
 $= \int_{-0.963}^0 (4 - e^{-2x}) dx + \frac{1}{2} OC \times CB$   
 $= \left[ 4x - \frac{e^{-2x}}{-2} \right]_{-0.963}^0 + \frac{1}{2}(6)(3)$   
 $= \left[ 4x + \frac{e^{-2x}}{2} \right]_{-0.963}^0 + 9$   
 $= \left[ 0 + \frac{1}{2} - 4\left(-\frac{1}{2}e^{-1.926}\right) + \frac{1}{2} \right] + 9$   
 $= \left[ \frac{1}{2} + 2 \ln 4 - 2 \right] + 9$   
 $= 10.3$  units<sup>2</sup>

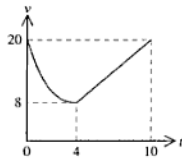
**TOPIC 50 PLANE AREAS BY INTEGRATION**

1. (i)  $A = (2, 12), B(6, 12)$   
(ii)  $21\frac{1}{3}$  units<sup>2</sup>
2. (ii)  $\frac{5}{8}$  units<sup>2</sup>
3. (i) 8.32 units<sup>2</sup>  
(ii) 3.68 units<sup>2</sup>  
 $k = 2$   
 $f(y) = 6 - \frac{12}{y}$
4. (a)  $3\frac{1}{6}$  units<sup>2</sup>  
(b) (i)  $\frac{a}{3}$   
(ii)  $a - \frac{a^2}{k}$
5. 0.272 units<sup>2</sup>

15. (i)  $5y = 21 - 6x$   
 (ii) At  $B$ ,  $y = 0$   
 $\therefore 5 \times 0 = 21 - 6x$   
 $6x = 21$   
 $x = 3.5$   
 $OA = 4.2$  units  
 (iii) Area of  $\triangle OAB = \frac{1}{2}(3.5)(4.2)$   
 $= 7.35$  units<sup>2</sup>  
 Area of shaded region  $= \frac{38}{15} + 3.75$   
 $= 6\frac{17}{60}$   
 Percentage of park area remaining  
 $= \frac{6\frac{17}{60}}{7.35}$   
 $= 0.8548752$   
 $\approx 85.5\%$

### TOPIC 51 KINEMATICS (WITHOUT CALCULUS)

1. (a) (i) 6 m/s  
 (ii) 4 s  
 (iii) 2 s  
 (b) (i)  $\frac{4}{5}$  m s<sup>-2</sup>  
 (ii) -4.8 m s<sup>-1</sup>  
 (iii)  $T = 40$   
 (iv) 32 m in the opposite direction
2. (i) Time = 4 s; Distance = 4.8 m  
 (ii)  $V = 15$ ;  $T = 6$   
 (iii) Time taken = 4.4  
 Magnitude of the deceleration  
 $3.41$  ms<sup>-2</sup> or  $3\frac{9}{22}$  ms<sup>-2</sup>
3. (i) 8 ms<sup>-1</sup>  
 (ii) 48 m  
 (iii) 6 s



$$t = \frac{120 - vt^2}{a}$$

### TOPIC 52 KINEMATICS (CALCULUS)

1. (i)  $10t + 1 - 3p$   
 (ii)  $p = 6$   
 (iii)  $t = \frac{2}{5}$  or 3
2. (i)  $t = 2\pi$  or 6.28  
 (ii) 26.9 m  
 (iii)  $2t - 4$   
 (iv)  $\frac{1}{3}t^3 - 2t^2 - 5t + 4$   
 (v)  $-29\frac{1}{3}$  m  
 (vi) 150 m  
 (vii)  $t = 2.77$   
 (viii) Acceleration  $= -2\frac{1}{2}$  ms<sup>-2</sup>  
 (ix) Displacement  $= 80 - 80e^{-2}$
5. (i)  $t = 3$   
 8 m  
 (ii) 4.65  
 $t^2 + t + 2$
6. (i)  $t = 3$   
 (ii) 146 m  
 (iii) 30 m/s<sup>2</sup>
7. (i)  $t = 3$   
 (ii) The distance is 42 m when its velocity is a minimum.
8. (c) (i)  $A = 5$ ;  $k = 0.0511$   
 (ii) -0.153
9.  $t = 2.30$   
 $A = 26.8$  m
10. (i)  $a = 1.4 - 0.6t$   
 $V = \int (1.4 - 0.6t) dt$   
 $= 1.4t - 0.3t^2 + c$  where  $c$  is a constant  
 If  $t = 0$   $V = 1.4(0) - 0.3(0)^2 + c = 0.5$   
 $c = 0.5$   
 If  $t = 5$   $V = 1.4(5) - 0.3(5)^2 + 0.5 = 7 - 7.5 + 0.5 = 0$   
 $\therefore$  the particle comes instantaneously to rest when  $t = 5$ .
- (ii) 40 m
11. (i) The velocity is 25 ms<sup>-1</sup>.  
 (ii)  $t = 8$   
 (iii)  $149\frac{1}{3}$  m  
 (iv)  $250\frac{2}{3}$  m

**SOLUTION TO O-LEVEL ADDITIONAL  
MATHEMATICS EXAMINATION  
NOVEMBER 2007**

**PAPER 1**

- $k > 18$      $k < 2$
- $6 - 2\sqrt{3}$
- (i)  $k = 0.0693$   
(ii)  $t = 15.8$
- (i)  $-3$   
(ii)  $2 \pm \sqrt{3}$
- (i)  $\cot x$   
(ii)  $0.693 \text{ units}^2$
- (i)  $A = \frac{1}{16} \begin{pmatrix} -1 & 3 \\ -6 & 2 \end{pmatrix}$  or  $\begin{pmatrix} -\frac{1}{16} & \frac{3}{16} \\ -\frac{3}{8} & \frac{1}{8} \end{pmatrix}$   
(ii)  $k = 3$
- (i)  $\frac{dy}{dx} = \frac{2x+3}{(x+2)^2} e^{2x}$   
(ii) At the turning point:  $\frac{2x+3}{(x+2)^2} e^{2x} = 0$   
 $(2x+3)e^{2x} = 0$

Since  $e^{2x} > 0$

$$2x + 3 = 0$$

$$2x = -3$$

$$x = -\frac{3}{2}$$

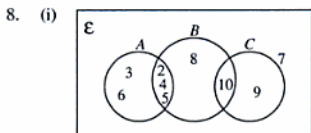
$$y = \frac{e^{-1}}{-\frac{1}{2} + 2} \times \frac{2}{2}$$

$$= \frac{2e^{-1}}{-\frac{1}{2} + 4}$$

$$= 2e^{-3}$$

$$= \frac{2}{e^3}$$

(iii) The turning point is a minimum.



(ii)  $A' \cap B' = \{7, 9\}$

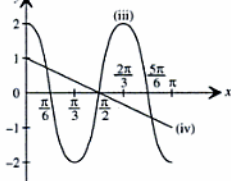
(iii)  $n(B \cup C) = 6$

$n\{(A \cup B \cup C)'\} = 1$

9. (i)  $\frac{2\pi}{3}$

(ii) 2

(iii), (iv)



(v) From the graphs of (iii) and (iv), this equation has three solutions.

10. (a)  $1 \leq x \leq 4$

(b) (i)  $c = -7$  or 3

(ii) If  $c = -7$                       If  $c = 3$

$y = |2x - 7|$                        $y = |2x - 3|$

$0 = |2x - 7|$                        $0 = |2x - 3|$

$2x = 7$                                $2x = 3$

If  $c = -7, x = 3.5$     If  $c = 3, x = 1.5$

11. (i)  $\vec{OY} = \frac{1}{2}(\mathbf{p} + 2\mathbf{q})$

$\vec{QR} = \mathbf{p} + \mathbf{q}$

(ii)  $\vec{PX} = \lambda(\mathbf{q} - \frac{1}{2}\mathbf{p})$

(iii)  $\vec{PX} = (\mu - 1)\mathbf{p} + (\mu + 1)\mathbf{q}$

(iv)  $\lambda = \frac{4}{3}, \mu = \frac{1}{3}$

12. EITHER

(i)  $\triangle BSC$  is an isosceles triangle

$\angle BSC = 2 \angle BSQ$

$= 2 \tan^{-1} \left( \frac{BQ}{QS} \right)$

$= 2 \tan^{-1} \left( \frac{4}{3} \right)$

$= 2 \times 0.46365$

$= 0.9273$

$= 0.927$

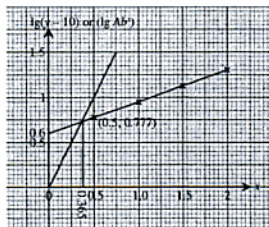
(ii) 33.2 cm

(iii) 86.8 cm<sup>2</sup>

OR

(i)

$x$	0.5	1.0	1.5	2.0
$y$	15.9	19.1	23.4	30.2
$y - 10$	5.9	9.1	13.4	20.2
$\lg(y - 10)$	0.771	0.959	1.13	1.31



$A = 3.98$

$b = 2.26$

(ii)  $= 0.365$

**PAPER 2**

$$\begin{aligned}
 1. \quad \text{R.H.S.} &= 1 - \sin^2 A \\
 &= \cos^2 A \\
 &= \frac{1}{\sec^2 A} \\
 &= \frac{1}{1 + \tan^2 A} \\
 &= \text{L.H.S.}
 \end{aligned}$$

$$2. \quad (5 \ 6 \ 8 \ 7) \begin{pmatrix} 200 & 500 & 200 \\ 300 & 600 & 0 \\ 0 & 400 & 300 \\ 0 & 700 & 0 \end{pmatrix} \begin{pmatrix} 145 \\ 75 \\ 20 \end{pmatrix}$$

Total amount of compost supplied = 1 539 000 litres

3.  $PQ = 18.0$  units

4. (i)  $-280$

(ii)  $-182$

5. (i)  $2 \sec^2(2x + 1)$

(ii) The secant function is never equal to zero.

$$\sec^2(2x + 1) > 0$$

$$2 \sec^2(2x + 1) > 0$$

The curve  $y = \tan(2x + 1)$  has no stationary point.

(iii)  $2.04 p$

6. (i)  $q = 50$

(ii)  $\frac{900 \text{ km}}{3 \text{ h}} = 300 \text{ km/h}$

$$p\mathbf{i} + 270\mathbf{i} = 300\mathbf{i}$$

$$p + 270 = 300$$

$$p = 30$$

(iii) Time taken =  $3\frac{3}{4}$  h or 3 h 45 min.

7. (i)  $x = 6$

(ii)  $y = 625$

8. (a) (i) 840

(ii) 60

(b) (i) 35

(ii) 30

9. (a)  $x = 210^\circ$  and  $330^\circ$

(b)  $y = 1.89, 5.03$

10. (a)  $c \geq -2$

(b) (i)  $(x + 1)^2 + 4$

(ii) The smallest possible value of  $k$  is  $-1$ .

(iii)  $g^{-1}: x \mapsto -1 + \sqrt{x - 4}$

11. (i)  $2x + 3y = 19$

(ii)  $C = (2, 5)$

(iii)  $D = (8, 1)$

(iv)  $AB^2 = (9 - 1)^2 + (9 + 3)^2$

$$= 64 + 144$$

$$= 208$$

$$= 4 \times 52$$

$$= 4 \times [36 + 16]$$

$$= 4 \times [6^2 + 4^2]$$

$$= 4[(2 - 8)^2 + (5 - 1)^2]$$

$$= 4CD^2$$

$$\therefore AB = 2CD$$

**12. EITHER**

(i)  $\frac{3}{5} \text{ ms}^{-2}$

(ii)  $t + \frac{1}{2}t^2 - \frac{1}{6}(4t + 9)^{\frac{3}{2}} + \frac{9}{2}$

**OR**

(i)  $P = (3, 3)$

(ii)  $\frac{7}{8} \text{ units}^2$



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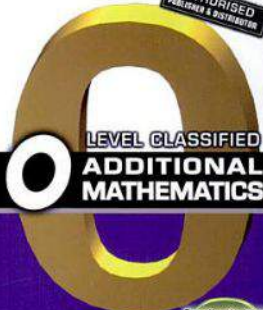
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