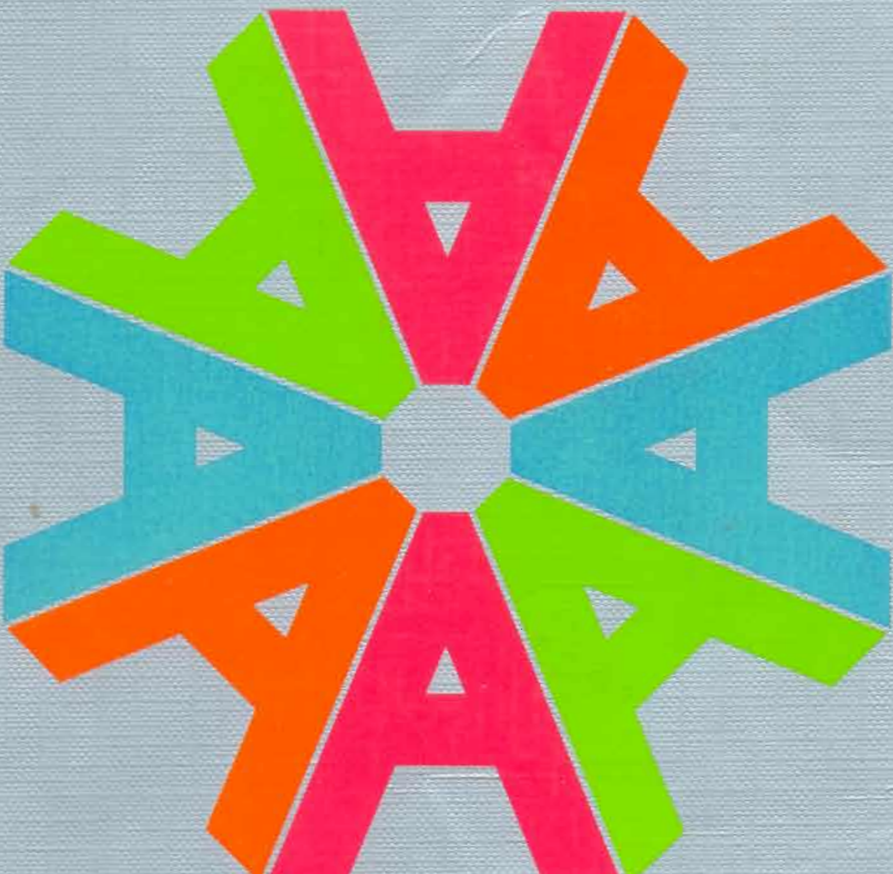
CORE MATHS for A-level

L. BOSTOCK S. CHANDLER



CORE MATHS for A-level

By the same authors

MODULE A: PURE MATHS 1
MODULE B: PURE MATHS 2
MATHEMATICS – THE CORE COURSE FOR A-LEVEL
FURTHER PURE MATHEMATICS – with C. Rourke
MATHEMATICS – MECHANICS AND PROBABILITY
FURTHER MECHANICS AND PROBABILITY
APPLIED MATHEMATICS I
APPLIED MATHEMATICS II
PURE MATHEMATICS I
PURE MATHEMATICS II

CORE MATHS for A-level

L. Bostock, B.Sc.

S. Chandler, B.Sc.

Stanley Thornes (Publishers) Ltd

First published in 1990 by Stanley Thornes (Publishers) Ltd,
Ellenborough House, Wellington Street, CHELTENHAM, GL50 1YD

Reprinted 1990
Reprinted 1992
Reprinted 1993

British Library Cataloguing in Publication Data

Bostock, L.
Core maths for A-level.
1. Mathematics
I. Title II. Chandler, S.
510

ISBN 0-7487-0067-6

Typeset by Tech-Set, Gateshead, Tyne & Wear, in 10/12pt Times
Printed and bound in Great Britain at The Bath Press, Avon

PREFACE

Why produce yet another A-level Maths textbook?

Now that GCSE courses have been introduced it can no longer be assumed that all students enter an A-level course with the algebraic skills and geometric knowledge that used to be expected. Many more students now move in to sixth-form colleges to do A-levels and hence come from a variety of backgrounds, including those who wish to embark on an A-level course from intermediate level GCSE.

In much the same way as multiplication tables are the tools needed to build a mathematics course from 11 to 16, skill in algebraic techniques are the tools necessary for building a body of mathematical knowledge beyond the 16+ level. This book starts with work designed to help those students acquire a facility in using algebra. To interest those students who already have these skills, new work is included in all chapters. Chapter 2 for example, includes an introduction to simple partial fractions.

All too many students regard A-level mathematics as being intrinsically difficult - an opinion with which we strongly disagree. Part of the reason for this myth may be that students, at an early stage in their course, tackle problems that are too sophisticated. The exercises in this book are designed to overcome this problem, all starting with straightforward questions. The more sophisticated A-level type questions are given in consolidation sections which appear at regular intervals throughout the book. These are intended for use at a later date to give practice in examination type questions when confidence and sophistication have been developed. The consolidation sections also include a summary of the work in preceeding chapters and a set of multiple choice questions, which are very useful for self-testing even if they do not form part of the examination to be taken.

There are many computer programs that aid in the understanding of mathematics. In particular, a good graph drawing package is invaluable for investigating graphical aspects of functions. In a few places we have indicated a program that we think is relevant. This is either *Super Graph* or a program from *132 Short Programs for the Mathematics Classroom*.

Super Graph by David Tall is a flexible graph drawing package and is available from Abco Design Ltd., Unit 11, Stirling Industrial Centre, Stirling, Boreham Wood, Herts WD6 2BT. Tel: 081 953 9292.
132 Short Programs for the Mathematics Classroom is published in book form by Stanley Thornes (Publishers) Ltd.

We are grateful to the following Examination Boards for permission to reproduce questions from their past examination papers (part questions are indicated by the suffix p):

University of London (U of L)
Joint Matriculation Board (JMB)
University of Cambridge Local Examinations Syndicate (C)
The Associated Examining Board (C)

L. Bostock
1990

S. Chandler

CONTENTS

NOTES ON USE OF THE BOOK

Notation

$=$	is equal to	$:$	is such that
$\equiv$	is identical to	$\mathbb{N}$	the natural numbers
$\approx$	is approximately equal to*	$\mathbb{Z}$	the integers
$>$	is greater than	$\mathbb{Q}$	the rational numbers
$\geqslant$	is greater than or equal to	$\mathbb{R}$	the real numbers
$<$	is less than	$\mathbb{R}^+$	the positive real numbers excluding zero
$\leqslant$	is less than or equal to		
∞	infinity; infinitely large	$\mathbb{C}$	the complex numbers
$\Rightarrow$	implies	$[a, b]$	the interval $\{x : a \leqslant x \leqslant b\}$
$\Leftarrow$	is implied by	$(a, b]$	the interval $\{x : a < x \leqslant b\}$
$\Leftrightarrow$	implies and is implied by	(a, b)	the interval $\{x : a < x < b\}$
$\in$	is a member of		

A stroke through a symbol negates it, e.g. $\neq$ means 'is not equal to'

Abbreviations

$\parallel$	is parallel to	w.r.t.	with respect to
+ve	positive	exp	exponential, e.g. $\exp x$ means e^x
−ve	negative		

Useful Formulae

For a cone with base radius r, height h and slant height l

volume $= \frac{1}{3}\pi r^2 h$ curved surface area $= \pi r l$

For a sphere of radius r

volume $= \frac{4}{3}\pi r^3$ surface area $= 4\pi r^2$

For any pyramid with height h and base area a

volume $= \frac{1}{3}ah$

*Practical problems rarely have exact answers. Where numerical answers are given they are correct to two or three decimal places depending on their context, e.g. π is 3.142 correct to 3 d.p. and although we write $\pi = 3.142$ it is understood that this is not an exact value. We reserve the symbol $\approx$ for those cases where the approximation being made is part of the method used.

Computer Program References

Marginal symbols indicate a computer program which is helpful, programs being identified in the following manner,

47 Program No. 47 from *132 Short Programs for the Mathematics Classroom*

SG *Super Graph*

Instructions for Answering Multiple Choice Exercises

These exercises are included in each consolidation section. The questions are set in groups, each group representing one of the variations that may arise in examination papers. The answering techniques are different for each group and are classified as follows:

TYPE I
These questions consist of a problem followed by several alternative answers, only *one* of which is correct.

Write down the letter corresponding to the correct answer.

TYPE II
In this type of question some information is given and is followed by a number of responses. *One or more* of these follow(s) directly and necessarily from the information given.

Write down the letter(s) corresponding to the correct response(s).
e.g. PQR is a triangle

A $\angle P + \angle Q + \angle R = 180°$
B PQ + QR is less than PR
C if $\angle P$ is obtuse, $\angle Q$ and $\angle R$ must both be acute.
D $\angle P = 90°$, $\angle Q = 45°$, $\angle R = 45°$

The correct responses are **A** and **C**.
B is definitely incorrect and **D** may or may not be true of triangle PQR, i.e. it does not follow directly and necessarily from the information given. Responses of this kind should not be regarded as correct.

TYPE III
A single statement is made. Write T if it is true and F if it is false.

CHAPTER 1

ALGEBRA 1

The ability to manipulate algebraic expressions is an essential base for any mathematics course beyond GCSE. Applying the processes involved needs to be almost as instinctive as the ability to manipulate simple numbers. This and the next two chapters present the facts and provide practice necessary for the development of these skills.

MULTIPLICATION OF ALGEBRAIC EXPRESSIONS

The multiplication sign is usually omitted, so that, for example,

$$2q \text{ means } 2 \times q$$

and $\quad x \times y$ can be simplified to xy

Remember also that if a string of numbers and letters are multiplied, the multiplication can be done in any order, for example

$$2p \times 3q = 2 \times p \times 3 \times q$$
$$= 6pq$$

Powers can be used to simplify expressions such as $x \times x$,

i.e. $\quad x \times x = x^2$

and $\quad x \times x^2 = x \times x \times x = x^3$

But remember that a power refers only to the number or letter it is written above, for example

$2x^2$ means that x is squared, but 2 is not.

Example 1a

Simplify (a) $(4pq)^2 \times 5$ (b) $\frac{ax^2}{y} \div \frac{x}{ay^2}$

(a)
$$(4pq)^2 \times 5 = 4pq \times 4pq \times 5$$
$$= 80p^2q^2$$

(b)
$$\frac{ax^2}{y} \div \frac{x}{ay^2} = \frac{ax^2}{x} \times \frac{ay^2}{y}$$
$$= a^2xy$$

EXERCISE 1a

Simplify

1. $3 \times 5x$
2. $x \times 2x$
3. $(2x)^2$
4. $5p \times 2q$
5. $4x \times 2x$
6. $2pq \times 5pr$
7. $(3a)^2$
8. $7a \times 9b$
9. $8t \times 3st$
10. $2a^2 \times 4a$
11. $25x^2 \div 15x$
12. $12m^2 \div 6m$
13. $b^2 \times 4ab$
14. $25x^2y \div 5x$
15. $(7pq)^2 \times (2p)^2$
16. $\frac{22ab}{11b}$
17. $\frac{18ax^2}{3x}$
18. $\frac{36xy}{18y}$
19. $\frac{72ab^2}{40a^2b}$
20. $\frac{2}{5} \div \frac{1}{x}$
21. $\frac{x^2}{y} \div \frac{y}{x}$

ADDITION AND SUBTRACTION OF EXPRESSIONS

The *terms* in an algebraic expression are the parts separated by a plus or minus sign.

Like terms contain the same combination of letters; like terms can be added or subtracted.

For example, $2ab$ and $5ab$ are like terms and can be added,

i.e. $2ab + 5ab = 7ab$

Unlike terms contain different algebraic expressions; they cannot be added or subtracted. For example, ab and ac are unlike terms and $ab + ac$ cannot be simplified.

Example 1b

Simplify $5x - 3(4 - x)$

$$5x - 3(4 - x) = 5x - 12 + 3x$$
$$= 8x - 12$$

Note that $-3(4 - x)$ means 'take away 3 times everything inside the bracket': remember that $(-3) \times (-x) = +3x$

EXERCISE 1b

Simplify

1. $2x^2 - 4x + x^2$
2. $5a - 4(a + 3)$
3. $2y - y(x - y)$
4. $8pq - 9p^2 - 3pq$
5. $4xy - y(x - y)$
6. $x^3 - 2x^2 + x^2 - 4x + 5x + 7$
7. $t^2 - 4t + 3 - 2t^2 + 5t + 2$
8. $2(a^2 - b) - a(a + b)$
9. $3 - (x - 4)$
   Note that $-(x - 4)$ means $-1(x - 4)$
10. $5x - 2 - (x + 7)$
11. $3x(x + 2) + 4(3x - 5)$
12. $a(b - c) - c(a - b)$
13. $2cT(3 - T) + 5T(c - 11T)$
14. $x^2(x + 7) - 3x^3 + x(x^2 - 7)$
15. $(3y^2 + 4y - 2) - (7y^2 - 20y + 8)$
16. $6RS + 5RF - R(R + S)$

COEFFICIENTS

We can identify a particular term in an expression by using the letter, or combination of letters, involved, for example

$2x^2$ is 'the term in x^2'

$3xy$ is 'the term in xy'

The number in front of the letters is called the *coefficient*, for example

in the term $2x^2$, 2 is the coefficient of x^2

in the term $3xy$, 3 is the coefficient of xy

If no number is written in front of a term, the coefficient is 1 or -1, depending on the sign of the term.

Consider the expression $x^3 + 5x^2y - y^3$

the coefficient of x^3 is 1

the coefficient of x^2y is 5

the coefficient of y^3 is -1

There is no term in x^2, so the coefficient of x^2 is zero.

EXERCISE 1c

1. Write down the coefficient of x in $x^2 - 7x + 4$

2. What is the coefficient of xy^2 in the expression $y^3 + 2xy^2 - 7xy$?

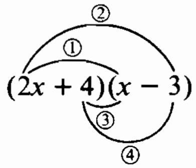
3. For the expression $x^2 - 5xy - y^2$ write down the coefficient of
   (a) x^2 (b) xy (c) y^2

4. For the expression $x^3 - 3x + 7$ write down the coefficient of
   (a) x^3 (b) x^2 (c) x

EXPANSION OF TWO BRACKETS

Expanding an expression means multiplying it out.

To expand $(2x + 4)(x - 3)$ each term in the first bracket is multiplied by each term in the second bracket. To make sure that nothing is missed out, it is sensible to follow the same order every time.
The order used in this book is:



$$(2x + 4)(x - 3) = 2x^2 - 6x + 4x - 12$$
$$= 2x^2 - 2x - 12$$

Use the next exercise to practice expanding and to develop the confidence to go straight to the simplified form.

EXERCISE 1d

Expand and simplify

1. $(x + 2)(x + 4)$
2. $(x + 5)(x + 3)$
3. $(a + 6)(a + 7)$
4. $(t + 8)(t + 7)$
5. $(s + 6)(s + 11)$
6. $(2x + 1)(x + 5)$

7. $(5y + 3)(y + 5)$
8. $(2a + 3)(3a + 4)$
9. $(7t + 6)(5t + 8)$
10. $(11s + 3)(9s + 2)$
11. $(x - 3)(x - 2)$
12. $(y - 4)(y - 1)$
13. $(a - 3)(a - 8)$
14. $(b - 8)(b - 9)$
15. $(p - 3)(p - 12)$
16. $(2y - 3)(y - 5)$
17. $(x - 4)(3x - 1)$
18. $(2r - 7)(3r - 2)$
19. $(4x - 3)(5x - 1)$
20. $(2a - b)(3a - 2b)$
21. $(x - 3)(x + 2)$
22. $(a - 7)(a + 8)$
23. $(y + 9)(y - 7)$
24. $(s - 5)(s + 6)$
25. $(q - 5)(q + 13)$
26. $(2t - 5)(t + 4)$
27. $(x + 3)(4x - 1)$
28. $(2q + 3)(3q - 5)$
29. $(x + y)(x - 2y)$
30. $(s + 2t)(2s - 3t)$

Difference of Two Squares

Consider the expansion of $(x - 4)(x + 4)$,

$$(x - 4)(x + 4) = x^2 - 4x + 4x - 16$$
$$= x^2 - 16$$

EXERCISE 1e

Expand and simplify

1. $(x - 2)(x + 2)$
2. $(5 - x)(5 + x)$
3. $(x + 3)(x - 3)$
4. $(2x - 1)(2x + 1)$
5. $(x + 8)(x - 8)$
6. $(x - a)(x + a)$

From Questions 1 to 6 it is clear that an expansion of the form $(ax + b)(ax - b)$ can be written down directly,

i.e. $$(ax + b)(ax - b) = a^2x^2 - b^2$$

Use this result to expand the following brackets.

7. $(x - 1)(x + 1)$
8. $(3b + 4)(3b - 4)$
9. $(2y - 3)(2y + 3)$
10. $(ab + 6)(ab - 6)$
11. $(5x + 1)(5x - 1)$
12. $(xy + 4)(xy - 4)$

Squares

$(2x + 3)^2$ means $(2x + 3)(2x + 3)$

$$\begin{aligned}\therefore \quad (2x + 3)^2 &= (2x + 3)(2x + 3) \\ &= (2x)^2 + (2)(2x)(3) + (3)^2 \\ &= 4x^2 + 12x + 9\end{aligned}$$

In general, $(ax + b)^2 = a^2x^2 + (2)(ax)(b) + b^2$

$= a^2x^2 + 2abx + b^2$

and $(ax - b)^2 = a^2x^2 - 2abx + b^2$

EXERCISE 1f

Use the results above to expand

1. $(x + 4)^2$
2. $(x + 2)^2$
3. $(2x + 1)^2$
4. $(3x + 5)^2$
5. $(2x + 7)^2$
6. $(x - 1)^2$
7. $(x - 3)^2$
8. $(2x - 1)^2$
9. $(4x - 3)^2$
10. $(5x - 2)^2$
11. $(3t - 7)^2$
12. $(x + y)^2$
13. $(2p + 9)^2$
14. $(3q - 11)^2$
15. $(2x - 5y)^2$

Important Expansions

The results from the last two sections should be memorised. They are summarised here.

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

$$(ax - b)^2 = a^2x^2 - 2abx + b^2$$

$$(ax + b)(ax - b) = a^2x^2 - b^2$$

The next exercise contains a variety of expansions including some of the forms given above.

Example 1g

Expand $(4p + 5)(3 - 2p)$

$$\begin{aligned}(4p + 5)(3 - 2p) &= (5 + 4p)(3 - 2p) \\ &= 15 + 2p - 8p^2\end{aligned}$$

EXERCISE 1g

Expand

1. $(2x - 3)(4 - x)$
2. $(x - 7)(x + 7)$
3. $(6 - x)(1 - 4x)$
4. $(7p + 2)(2p - 1)$
5. $(3p - 1)^2$
6. $(5t + 2)(3t - 1)$
7. $(4 - p)^2$
8. $(4t - 1)(3 - 2t)$
9. $(x + 2y)^2$
10. $(4x - 3)(4x + 3)$
11. $(3x + 7)^2$
12. $(R + 3)(5 - 2R)$
13. $(a - 3b)^2$
14. $(2x - 5)^2$
15. $(7a + 2b)(7a - 2b)$
16. $(3a + 5b)^2$
17. Write down the coefficients of x^2 and x in the expansion of
(a) $(2x - 4)(3x - 5)$ (b) $(5x + 2)(3x + 5)$
(c) $(2x - 3)(7x - 5)$ (d) $(9x + 1)^2$

FACTORISING QUADRATIC EXPRESSIONS

In the last four exercises, each bracket contained a *linear expression*, i.e. an expression that contained an x term and a number term.

An expression of the form $ax + b$, where a and b are numbers, is called a linear expression in x

When two linear expressions in x are multiplied, the result usually contains three terms: a term in x^2, a term in x and a number.

Expressions of this form, i.e. $ax^2 + bx + c$ where a, b and c are numbers and $a \neq 0$, are called *quadratic expressions in x*

Since the product of two linear brackets is quadratic, we might expect to be able to reverse this process. For instance, given a quadratic such as $x^2 - 5x + 6$, we could try to find two linear expressions in x whose product is $x^2 - 5x + 6$. To be able to do this we need to appreciate the relationship between what is inside the brackets and the resulting quadratic.

Consider the examples

$$(2x + 1)(x + 5) = 2x^2 + 11x + 5 \qquad [1]$$

$$(3x - 2)(x - 4) = 3x^2 - 14x + 8 \qquad [2]$$

$$(x - 5)(4x + 2) = 4x^2 - 18x - 10 \qquad [3]$$

The first thing to notice about the quadratic in each example is that

the coefficient of x^2 is the product of the coefficients of x in the two brackets,

the number is the product of the numbers in the two brackets,

the coefficient of x is the sum of the coefficients formed by multiplying the x term in one bracket by the number term in the other bracket.

The next thing to notice is the relationship between the signs.

Positive signs throughout the quadratic come from positive signs in both brackets, as in [1].

A positive number term and a negative coefficient of x in the quadratic come from a negative sign in each bracket, as in [2].

A negative number term in the quadratic comes from a negative sign in one bracket and a positive sign in the other, as in [3].

Examples 1h

1. Factorise $x^2 - 5x + 6$

The x term in each bracket is x as x^2 can only be $x \times x$
The sign in each bracket is $-$, so $x^2 - 5x + 6 = (x - \ \ )(x - \ \ )$
The numbers in the brackets could be 6 and 1 or 2 and 3
Checking the middle term tells us that the numbers must be 2 and 3

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Mentally expanding the brackets checks that they are correct.

2. Factorise $x^2 - 3x - 10$

The x term in each bracket is x $\Rightarrow$ $x^2 - 3x - 10 = (x - \ \ )(x + \ \ )$
The numbers could be 10 and 1 or 5 and 2
Checking the middle term shows that they are 5 and 2

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$

Mentally expanding the brackets confirms that they are correct.

EXERCISE 1h

Factorise

1. $x^2 + 8x + 15$	2. $x^2 + 11x + 28$	3. $x^2 + 7x + 6$
4. $x^2 + 7x + 12$	5. $x^2 - 10x + 9$	6. $x^2 - 6x + 9$
7. $x^2 + 8x + 12$	8. $x^2 - 9x + 8$	9. $x^2 + 5x - 14$
10. $x^2 + x - 12$	11. $x^2 - 4x - 5$	12. $x^2 - 10x - 24$
13. $x^2 + 9x + 14$	14. $x^2 - 2x + 1$	15. $x^2 - 9$
16. $x^2 + 5x - 24$	17. $x^2 + 4x + 4$	18. $x^2 - 1$
19. $x^2 - 3x - 18$	20. $x^2 + 10x + 25$	21. $x^2 - 16$
22. $4 + 5x + x^2$	23. $2x^2 - 3x + 1$	24. $3x^2 + 4x + 1$
25. $9x^2 - 6x + 1$	26. $6x^2 - x - 1$	27. $9 + 6x + x^2$
28. $4x^2 - 9$	29. $x^2 + 2ax + a^2$	30. $x^2y^2 - 2xy + 1$

Harder Factorising

When the number of possible combinations of terms for the brackets increases, common sense considerations can help to reduce the possibilities.

For example, if the coefficient of x in the quadratic is odd, then there must be an even number and an odd number in the brackets.

Example 1i

Factorise $12 - x - 6x^2$

The x terms in the brackets could be $6x$ and x, or $3x$ and $2x$, one positive and the other negative.

The number terms could be 12 and 1 or 3 and 4 (not 6 and 2 because the coefficient of x in the quadratic is odd).

Now we try various combinations until we find the correct one.

$$12 - x - 6x^2 = (3 + 2x)(4 - 3x)$$

EXERCISE 1I

Factorise

1. $6x^2 + x - 12$	**2.** $4x^2 - 11x + 6$	**3.** $4x^2 + 3x - 1$
4. $3x^2 - 17x + 10$	**5.** $4x^2 - 12x + 9$	**6.** $3 - 5x - 2x^2$
7. $25x^2 - 16$	**8.** $3 - 2x - x^2$	**9.** $5x^2 - 61x + 12$
10. $9x^2 + 30x + 25$	**11.** $3 + 2x - x^2$	**12.** $12 + 7x - 12x^2$
13. $1 - x^2$	**14.** $9x^2 + 12x + 4$	**15.** $x^2 + 2xy + y^2$
16. $1 - 4x^2$	**17.** $4x^2 - 4xy + y^2$	**18.** $9 - 4x^2$
19. $36 + 12x + x^2$	**20.** $40x^2 - 17x - 12$	**21.** $7x^2 - 5x - 150$
22. $36 - 25x^2$	**23.** $x^2 - y^2$	**24.** $81x^2 - 36xy + 4y^2$
25. $49 - 84x + 36x^2$	**26.** $25x^2 - 4y^2$	**27.** $36x^2 + 60xy + 25y^2$
28. $4x^2 - 4xy - 3y^2$	**29.** $6x^2 + 11xy + 4y^2$	**30.** $49p^2q^2 - 28pq + 4$

Common Factors

Consider $4x^2 + 8x + 4$

$$4x^2 + 8x + 4 = 4(x^2 + 2x + 1)$$

The quadratic inside the bracket now has smaller coefficients and can be factorised more easily:

$$4x^2 + 8x + 4 = 4(x + 1)(x + 1)$$
$$= 4(x + 1)^2$$

Not all quadratics factorise.

Consider $3x^2 - x + 5$

The options we can try are $(3x - 5)(x - 1)$ [1]

$(3x - 1)(x - 5)$ [2]

From [1], $(3x - 5)(x - 1) = 3x^2 - 8x + 5$

From [2], $(3x - 1)(x - 5) = 3x^2 - 16x + 5$

As neither of the possible pairs of brackets expand to give $3x^2 - x + 5$, we conclude that $3x^2 - x + 5$ has no factors of the form $ax + b$ where a and b are integers.

Example 1J

Factorise $2x^2 - 8x + 16$

$$2x^2 - 8x + 16 = 2(x^2 - 4x + 8)$$

The possible brackets are $(x - 1)(x - 8)$ and $(x - 2)(x - 4)$
Neither pair expands to $x^2 - 4x + 8$, so there are no further factors.

EXERCISE 1J

Factorise where possible

1. $x^2 + x + 1$	**2.** $2x^2 + 4x + 2$	**3.** $x^2 + 3x + 2$
4. $3x^2 + 12x - 15$	**5.** $x^2 + 4$	**6.** $x^2 - 4x - 6$
7. $x^2 + 3x + 1$	**8.** $2x^2 - 8x + 8$	**9.** $3x^2 - 3x - 6$
10. $2x^2 - 6x + 8$	**11.** $3x^2 - 6x - 24$	**12.** $x^2 - 4x - 12$
13. $x^2 + 1$	**14.** $4x^2 - 100$	**15.** $5x^2 - 25$
16. $7x^2 + x + 4$	**17.** $10x^2 - 39x - 36$	**18.** $x^2 + xy + y^2$

HARDER EXPANSIONS

Consider the product $(x - 2)(x^2 - x + 5)$

This expansion should be done in a systematic way.

First multiply each term of the quadratic by x, writing down the separate results as they are found. Then multiply each term of the quadratic by -2. Do not attempt to simplify at this stage.

$$(x - 2)(x^2 - x + 5)$$
$$= x^3 - x^2 + 5x - 2x^2 + 2x - 10$$

Now simplify

$$= x^3 - 3x^2 + 7x - 10$$

Example 1k

Expand $(x + 2)(2x - 1)(x + 4)$

First we expand the last two brackets.

$$\begin{aligned}(x + 2)(2x - 1)(x + 4) &= (x + 2)(2x^2 + 7x - 4)\\ &= 2x^3 + 7x^2 - 4x + 4x^2 + 14x - 8\\ &= 2x^3 + 11x^2 + 10x - 8\end{aligned}$$

EXERCISE 1k

Expand and simplify

1. $(x - 2)(x^2 + x + 1)$
2. $(3x - 2)(x^2 - x - 1)$
3. $(2x - 1)(2x^2 - 3x + 5)$
4. $(x - 1)(x^2 - x - 1)$
5. $(2x + 3)(x^2 - 6x - 3)$
6. $(x + 1)(x + 2)(x + 3)$
7. $(x + 4)(x - 1)(x + 1)$
8. $(x - 2)(x - 3)(x + 1)$
9. $(x + 1)(2x + 1)(x + 2)$
10. $(x + 2)(x + 1)^2$
11. $(2x - 1)^2(x + 2)$
12. $(3x - 1)^3$
13. $(4x + 3)(x + 1)(x - 4)$
14. $(x - 1)(2x - 1)(2x + 1)$
15. $(2x + 1)(x + 2)(3x - 1)$
16. $(x + 1)^3$
17. $(x - 2)(x + 2)(x + 1)$
18. $(x + 3)(2x + 3)(x - 1)$
19. $(3x - 2)(2x + 5)(4x - 1)$
20. $2(x - 7)(2x + 3)(x - 5)$
21. Expand and simplify $(x - 2)^2(3x - 4)$. Write down the coefficients of x^2 and x.
22. Find the coefficients of x^3 and x^2 in the expansion of $(x - 4)(2x + 3)(3x - 1)$
23. Expand and simplify $(x + y)^3$
24. Expand and simplify $(x + y)^4$

PASCAL'S TRIANGLE

58

It is sometimes necessary to expand expressions such as $(x + y)^4$ but the multiplication is tedious when the power is three or more.
We now describe a far quicker way of obtaining such expansions.

Consider the following expansions,

$$\begin{aligned}(x + y)^1 &= x + y\\ (x + y)^2 &= x^2 + 2xy + y^2\\ (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

The first thing to notice is that the powers of x and y in the terms of each expansion form a pattern. Looking at the expansion of $(x + y)^4$ we see that the first term is x^4 and then the power of x decreases by 1 in each succeeding term while the power of y increases by 1. For all the terms, the sum of the powers of x and y is 4 and the expansion ends with y^4. There is a similar pattern in the other expansions.

Now consider just the coefficients of the terms. Writing these as a triangular array gives

```
            1   1
          1   2   1
        1   3   3   1
      1   4   6   4   1
```

This array is called *Pascal's Triangle* and clearly it has a pattern.
Each row starts and ends with 1 and each other number is the sum of the two numbers in the row above it, as shown. When the pattern is known, Pascal's triangle can be written down to as many rows as needed. Using Pascal's triangle to expand $(x + y)^6$, for example, we go as far as row 6:

```
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
```

We then use our knowledge of the pattern of the powers, together with row 6 of the array, to fill in the coefficients,

i.e. $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

The following worked examples show how expansions of other brackets can be found.

Examples 1l

1. Expand $(x+5)^3$

From Pascal's triangle $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Replacing y by 5 gives $(x+5)^3 = x^3 + 3x^2(5) + 3x(5)^2 + (5)^3$

$= x^3 + 15x^2 + 75x + 125$

2. Expand $(2x-3)^4$

From Pascal's triangle, $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Replacing x by $2x$ and y by -3 gives

$$(2x-3)^4 = (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$$

$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

EXERCISE 1l

Expand

1. $(x+3)^3$	**2.** $(x-2)^4$	**3.** $(x+1)^4$
4. $(2x+1)^3$	**5.** $(x-3)^5$	**6.** $(p-q)^4$
7. $(2x+3)^3$	**8.** $(x-4)^5$	**9.** $(3x-1)^4$
10. $(1+5a)^4$	**11.** $(2a-b)^6$	**12.** $(2x-5)^3$

MIXED EXERCISE 1

1. Find the coefficient of x in the expansion of $(3x-7)(5x+4)$
2. Expand $5(3x-2)(3-7x)$
3. Write down the coefficient of y^2 in the expansion of $(2y+9)^3$
4. Factorise $3x^2 - 9x + 6$
5. Write down the coefficient of x^3 in the expansion of $(x-5)^5$
6. Factorise $4x^2 - 36$
7. Expand $x(2x-1)^2$
8. Find the factors of $25 + x^2 - 10x$
9. Find the coefficient of x in the expansion of $(x-4)(3-x)$

CHAPTER 2

FRACTIONS

SIMPLIFICATION OF FRACTIONS

The value of a fraction is unaltered if *numerator* and *denominator* are multiplied or divided by the same number,

e.g. $\frac{3}{6} = \frac{1}{2} = \frac{2}{4} = \frac{7}{14} = \ldots$

and $\frac{ax}{ay} = \frac{x}{y} = \frac{3x}{3y} = \frac{x(a+b)}{y(a+b)} = \ldots$

A fraction can be simplified by multiplying or dividing *top* and *bottom* by a factor which is common to both.

When simplifying fractions, it is sensible first to get rid of any fractions in the numerator and/or denominator. Then factorise the numerator and denominator and look for any common factors.

Examples 2a

1. Simplify $\frac{2a^2 - 2ab}{6ab - 6b^2}$

$$\frac{2a^2 - 2ab}{6ab - 6b^2} = \frac{\cancel{2}a\cancel{(a-b)}}{{}_3\cancel{6}b\cancel{(a-b)}}$$
$$= \frac{a}{3b}$$

2. Simplify $\frac{\frac{1}{2}x^2 - 2}{\frac{1}{4}y^2 + 3}$

$$\frac{\frac{1}{2}x^2 - 2}{\frac{1}{4}y^2 + 3} = \frac{2x^2 - 8}{y^2 + 12}$$ Multiplying top and bottom by 4
$$= \frac{2(x^2 - 4)}{y^2 + 12}$$
$$= \frac{2(x-2)(x+2)}{y^2 + 12}$$

EXERCISE 2a

Simplify where possible.

1. $\frac{x-2}{4x-8}$
2. $\frac{2x+4}{3x-6}$
3. $\frac{2a+8}{3a+12}$
4. $\frac{3p-3q}{5p-5q}$
5. $\frac{x^2+xy}{xy+y^2}$
6. $\frac{x-3p}{2x+p}$
7. $\frac{a-4}{a-2}$
8. $\frac{x^2y+xy^2}{y^2+\frac{2}{5}xy}$
9. $\frac{\frac{1}{3}a-b}{a+\frac{1}{6}b}$
10. $\frac{2x(b-4)}{6x^2(b+4)}$
11. $\frac{(x-4)(x-3)}{x^2-16}$
12. $\frac{4y^2+3}{y^2-9}$
13. $\frac{\frac{1}{3}(x-3)}{x^2-9}$
14. $\frac{x^2-x-6}{2x^2-5x-3}$
15. $\frac{(x-2)(x+2)}{x^2+x-2}$
16. $\frac{\frac{1}{2}(a+5)}{a^2-25}$
17. $\frac{3p+9q}{p^2+6pq+9q^2}$
18. $\frac{a^2+2a+4}{a^2+7a+10}$
19. $\frac{x^2+2x+1}{3x^2+12x+9}$
20. $\frac{4(x-3)^2}{(x+1)(x^2-2x-3)}$

MULTIPLICATION AND DIVISION

Fractions are multiplied by taking the product of the numerators and the product of the denominators,

e.g. $$\frac{x}{a}\times\frac{y}{b}=\frac{x\times y}{a\times b}=\frac{xy}{ab}$$

To divide by a fraction, we multiply by the reciprocal of that fraction, for example

$$\frac{x}{a}\div\frac{y}{b}=\frac{x}{a}\times\frac{b}{y}=\frac{xb}{ay}$$

Example 2b

Simplify $\dfrac{2\pi x^2}{7y^2}\div 4\pi x$

$$\frac{2\pi x^2}{7y^2}\div 4\pi x=\frac{\cancel{2}\cancel{\pi}x^{\cancel{2}}}{7y^2}\times\frac{1}{{}_2\cancel{4\pi x}}$$

$$=\frac{x}{14y^2}$$

EXERCISE 2b

Simplify

1. $\dfrac{4x}{y}\times\dfrac{x}{6y}$
2. $2st\times\dfrac{3t}{s^2}$
3. $\dfrac{4uv}{3}\div\dfrac{u}{2v}$
4. $\dfrac{4\pi r^2}{3}\div 2\pi r$
5. $x(2-x)\div\dfrac{2-x}{3}$
6. $\dfrac{3x^2}{2y}\Big/\dfrac{6xy}{9}$
7. $\dfrac{\pi x^3}{3}\div 8\pi x$
8. $\dfrac{1}{a^2+ab}\Big/\dfrac{1}{2}$
9. $\dfrac{1}{x^2-1}\div\dfrac{1}{x-1}$
10. $\left(\dfrac{1}{a}\right)^2\times\tfrac{1}{2}a$

11. $(x+1)\times\dfrac{1}{x^2-1}$
12. $\dfrac{x-4}{x+3}\div\dfrac{2(x-4)}{3}$
13. $\dfrac{a^2}{3}\times\left(\dfrac{a}{3}\right)^2$
14. $\dfrac{x^2}{6}\Big/\left(\dfrac{x}{2}\right)^2$
15. $\dfrac{2r^3}{3}\times\left(\dfrac{1}{rs}\right)^2$
16. $\dfrac{3x^2}{2y}\times\dfrac{y}{y-2}$
17. $\dfrac{ab}{c}\div\dfrac{ac}{b}$
18. $\dfrac{x^2+4x+3}{5}\times\dfrac{10}{x+1}$
19. $\dfrac{x^2+x-12}{3}\Big/(x+4)$
20. $\dfrac{4x^2-9}{(x-1)^2}\div\dfrac{2x+3}{x(x-1)}$

ADDITION AND SUBTRACTION OF FRACTIONS

Before fractions can be added or subtracted, they must be expressed with the same denominator, i.e. we have to find a common denominator. Then the numerators can be added or subtracted,

e.g. $$\frac{2}{p}+\frac{3}{q}=\frac{2q}{pq}+\frac{3p}{pq}=\frac{2q+3p}{pq}$$

Example 2c

Simplify $x-\dfrac{1}{x}$

$$x-\frac{1}{x}=\frac{x}{1}-\frac{1}{x}=\frac{x^2}{x}-\frac{1}{x}=\frac{x^2-1}{x}$$

$$=\frac{(x-1)(x+1)}{x}$$

EXERCISE 2c

Simplify

1. $\dfrac{1}{a}-\dfrac{1}{b}$
2. $\dfrac{1}{3x}+\dfrac{1}{5x}$

3. $\dfrac{1}{p} - \dfrac{1}{q}$

4. $\dfrac{1}{2x} + \dfrac{3}{5x}$

5. $x + \dfrac{1}{x}$

6. $\dfrac{x}{y} - \dfrac{y}{x}$

7. $2p - \dfrac{1}{p}$

8. $\dfrac{x}{3} + \dfrac{x+1}{4}$

9. $\frac{1}{2}(x-1) + \frac{1}{3}(x+1)$

10. $\dfrac{x+2}{5} - \dfrac{2x-1}{3}$

11. $\dfrac{1}{\sin A} + \dfrac{1}{\sin B}$

12. $\dfrac{1}{\cos A} + \dfrac{1}{\sin A}$

13. $3x + \dfrac{1}{4x}$

14. $x - \dfrac{2}{2x+1}$

15. $x + 1 + \dfrac{1}{x+1}$

16. $1 + \dfrac{1}{x} + \dfrac{1}{2x}$

17. $1 - x + \dfrac{1}{x}$

18. $\dfrac{1}{n} + \dfrac{1}{n^2}$

19. $\dfrac{x}{a^2} + \dfrac{x}{b^2}$

20. $1 + \dfrac{1}{a} + \dfrac{1}{a+1}$

Example 2d

Simplify $\dfrac{2}{x+2} - \dfrac{x-4}{2x^2+x-6}$

$$\begin{aligned}\frac{2}{x+2} - \frac{x-4}{2x^2+x-6} &= \frac{2}{x+2} - \frac{x-4}{(x+2)(2x-3)}\\ &= \frac{2(2x-3)}{(x+2)(2x-3)} - \frac{x-4}{(x+2)(2x-3)}\\ &= \frac{2(2x-3)-(x-4)}{(x+2)(2x-3)}\\ &= \frac{4x-6-x+4}{(x+2)(2x-3)}\\ &= \frac{3x-2}{(x+2)(2x-3)}\end{aligned}$$

EXERCISE 2d

Simplify

1. $\dfrac{1}{x+1} + \dfrac{1}{x-1}$

2. $\dfrac{1}{x+1} + \dfrac{1}{x-2}$

3. $\dfrac{4}{x+2} + \dfrac{3}{x+3}$

4. $\dfrac{1}{x^2-1} + \dfrac{1}{x+1}$

5. $\dfrac{2}{a^2-1} - \dfrac{3}{a-1}$

6. $\dfrac{1}{x^2+2x+1} + \dfrac{1}{x+1}$

7. $\dfrac{3}{4x^2+4x+1} - \dfrac{2}{2x+1}$

8. $\dfrac{2}{x^2+5x+4} - \dfrac{3}{x+1}$

9. $\dfrac{4}{(x+1)^2} + \dfrac{2}{x+1}$

10. $\dfrac{3}{(x+2)^2} - \dfrac{1}{x+4}$

11. $\dfrac{1}{2(x-1)} + \dfrac{2}{3(x+4)}$

12. $\dfrac{7}{5(x+2)} - \dfrac{2}{x+4}$

13. $\dfrac{4}{3(x+2)} - \dfrac{3}{2(3x-5)}$

14. $\dfrac{3}{x+1} - \dfrac{2}{x-2} + \dfrac{4}{x+3}$

15. $\dfrac{1}{x+1} - \dfrac{2}{x+2} + \dfrac{3}{x+3}$

16. $\dfrac{x+2}{(x+1)^2} - \dfrac{1}{x}$

17. $\dfrac{4t}{t^2+2t+1} + \dfrac{3}{t+1}$

18. $\dfrac{2t}{t^2+1} - \dfrac{t^2+1}{t^2-1}$

19. $\dfrac{1}{y^2-x^2} + \dfrac{3}{y+x}$

20. $1 + \dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2}$

PARTIAL FRACTIONS

In the last two exercises you were asked to express two separate fractions as a single fraction with a common denominator. Later on in the course it is necessary to take an expression such as $\dfrac{x-2}{(x+3)(x-4)}$ and express it as the sum of two separate fractions.

This process is called splitting up, or decomposing, into *partial fractions*.

Consider again $\dfrac{x-2}{(x+3)(x-4)}$

This fraction is a *proper fraction* because the highest power of x in the numerator (1 in this case) is less than the highest power of x in the denominator (2 in this case when the brackets are expanded).

Therefore its separate (or partial) fractions also will be proper,

i.e. $\dfrac{x-2}{(x+3)(x-4)}$ can be expressed as $\dfrac{A}{x+3}+\dfrac{B}{x-4}$

where A and B are numbers. The worked example which follows shows how the values of A and B can be found.

Example 2e

Express $\dfrac{x-2}{(x+3)(x-4)}$ in partial fractions.

$$\frac{x-2}{(x+3)(x-4)} = \frac{A}{x+3}+\frac{B}{x-4}$$

Expressing the separate fractions on the RHS as a single fraction over a common denominator gives

$$\frac{x-2}{(x+3)(x-4)} = \frac{A(x-4)+B(x+3)}{(x+3)(x-4)}$$

This is not an equation because the RHS is just another way of expressing the LHS. It follows that, as the denominators are identical the numerators also are identical.

i.e. $$x-2 = A(x-4)+B(x+3)$$

Remembering that this is *not* an equation but two ways of writing the same expression, it follows that LHS = RHS for any value that we choose to give to x.

Choosing to substitute 4 for x (to eliminate A) gives

$$2 = A(0)+B(7)$$

$$\Rightarrow \qquad B = \frac{2}{7}$$

Choosing to substitute -3 for x (to eliminate B) gives

$$-5 = A(-7)+B(0)$$

$$\Rightarrow \qquad A = \frac{5}{7}$$

Therefore
$$\frac{x-2}{(x+3)(x-4)} = \frac{5/7}{x+3}+\frac{2/7}{x-4}$$
$$= \frac{5}{7(x+3)}+\frac{2}{7(x-4)}$$

EXERCISE 2e

Express the following fractions in partial fractions.

1. $\dfrac{x-2}{(x+1)(x-1)}$
2. $\dfrac{x+4}{(x+3)(x-5)}$
3. $\dfrac{2x-1}{(x-1)(x-7)}$
4. $\dfrac{3x+1}{(2x-1)(x-1)}$
5. $\dfrac{4}{(x+3)(x-2)}$
6. $\dfrac{2x-3}{(x-2)(4x-3)}$
7. $\dfrac{7x}{(2x-1)(x+4)}$
8. $\dfrac{3-x}{(x+1)(2x-1)}$
9. $\dfrac{2}{x(x-2)}$
10. $\dfrac{3}{x(2x+1)}$
11. $\dfrac{2x-1}{x^2-3x+2}$
12. $\dfrac{4}{x^2-7x-8}$
13. $\dfrac{3}{x^2-9}$
14. $\dfrac{4x}{4x^2-9}$
15. $\dfrac{6x+7}{3x(x+1)}$
16. $\dfrac{4x-2}{x^2+2x}$
17. $\dfrac{9}{2x^2+x}$
18. $\dfrac{3x}{2x^2-2x-4}$
19. $\dfrac{x+1}{3x^2-x-2}$
20. $\dfrac{3x+2}{2x^2-4x}$

MIXED EXERCISE 2

1. Simplify (a) $\dfrac{x^2-9}{2x-6}$ (b) $\dfrac{1}{x^2-9}\div\dfrac{1}{x+3}$
2. Simplify (a) $\left(\dfrac{2p}{r}\right)^2\times\dfrac{ar}{p^3}$ (b) $\dfrac{2p}{r}-\dfrac{3}{p}$

3. Simplify (a) $\dfrac{2n-4}{3} \div (n^2-4)$ (b) $\dfrac{1}{x+1} + \dfrac{1}{2x-1} + \dfrac{1}{x}$

4. Express $\dfrac{3}{x^2-1}$ in partial fractions.

5. Express $\dfrac{4}{(2x+1)(x-3)}$ in partial fractions.

6. Express $\dfrac{5}{x^2-x}$ in partial fractions.

7. Simplify (a) $\dfrac{4x^2-25}{4x^2+20x+25}$ (b) $\dfrac{2t}{t^2+1} \div \dfrac{t^2-1}{t^2+1}$

8. Simplify (a) $\left(\dfrac{x-1}{x+1}\right)^2 \times (x^2-1)$ (b) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

9. Express $\dfrac{3x-2}{(x+1)(4x-3)}$ in partial fractions.

10. Express $\dfrac{2t}{t^2-1}$ in partial fractions.

CHAPTER 3

SURDS AND INDICES

SQUARE ROOTS

When we express a number as the product of two equal factors, that factor is called the *square root* of the number, for example

$$4 = 2 \times 2 \Rightarrow 2 \text{ is the square root of } 4$$

This is written $\qquad 2 = \sqrt{4}$

Now -2 is also a square root of 4, as $4 = -2 \times -2$ but we do *not* write $\sqrt{4} = -2$
The symbol $\sqrt{\ }$ is used *only for the positive square root.*
So, although $x^2 = 4 \Rightarrow x = \pm 2$, the only value of $\sqrt{4}$ is 2

The negative square root of 4 would be written as $-\sqrt{4}$ and, when both square roots are wanted, we write $\pm\sqrt{4}$

CUBE ROOTS

When a number can be expressed as the product of three equal factors, that factor is called the *cube root* of the number,

e.g. $\qquad 27 = 3 \times 3 \times 3 \qquad$ so 3 is the cube root of 27

This is written $\sqrt[3]{27} = 3$

OTHER ROOTS

The notation used for square and cube roots can be extended to represent fourth roots, fifth roots, etc,

e.g. $\qquad 16 = 2 \times 2 \times 2 \times 2 \Rightarrow \sqrt[4]{16} = 2$

and $\qquad 243 = 3 \times 3 \times 3 \times 3 \times 3 \Rightarrow \sqrt[5]{243} = 3$

In general, if a number, n, can be expressed as the product of p equal factors then each factor is called the pth root of n and is written $\sqrt[p]{n}$

RATIONAL NUMBERS

A number which is either an integer, or a fraction whose numerator and denominator are both integers, is called a *rational number*.

The square roots of certain numbers are rational,

e.g. $\sqrt{9} = 3, \quad \sqrt{25} = 5, \quad \sqrt{\tfrac{4}{49}} = \tfrac{2}{7}$

This is not true of all square roots however, e.g. $\sqrt{2}$, $\sqrt{5}$, $\sqrt{11}$ are not rational numbers. Such square roots can be given to as many decimal places as are required, for example

$\sqrt{3} = 1.73$ correct to 2 d.p.

$\sqrt{3} = 1.732\,05$ correct to 5 d.p.

but they can never be expressed exactly as a decimal. They are called *irrational numbers*.

The only way to give an exact answer when such irrational numbers are involved is to leave them in the form $\sqrt{2}$, $\sqrt{7}$ etc; in this form they are called *surds*. At this level of mathematics *answers should always be given exactly unless an approximate answer is asked for*, e.g. give your answer correct to 3 s.f.

Surds arise in many topics and the reader will find it necessary to be able to manipulate them.

Simplifying Surds

Consider $\sqrt{18}$

One of the factors of 18 is 9, and 9 has an exact square root,

i.e. $\sqrt{18} = \sqrt{(9 \times 2)} = \sqrt{9} \times \sqrt{2}$

But $\sqrt{9} = 3$, therefore $\sqrt{18} = 3\sqrt{2}$

$3\sqrt{2}$ is the simplest possible surd form for $\sqrt{18}$

Similarly $\sqrt{\dfrac{2}{25}} = \dfrac{\sqrt{2}}{\sqrt{25}} = \dfrac{\sqrt{2}}{5}$

EXERCISE 3a

Express in terms of the simplest possible surd.

1. $\sqrt{12}$	**2.** $\sqrt{32}$	**3.** $\sqrt{27}$	**4.** $\sqrt{50}$
5. $\sqrt{200}$	**6.** $\sqrt{72}$	**7.** $\sqrt{162}$	**8.** $\sqrt{288}$
9. $\sqrt{75}$	**10.** $\sqrt{48}$	**11.** $\sqrt{500}$	**12.** $\sqrt{20}$

Multiplying Surds

Consider $(4 - \sqrt{5})(3 + \sqrt{2})$

The multiplication is carried out in the same way and order as when multiplying two linear brackets,

i.e.
$$\begin{aligned}(4 - \sqrt{5})(3 + \sqrt{2}) &= (4)(3) + (4)(\sqrt{2}) - (3)(\sqrt{5}) - (\sqrt{5})(\sqrt{2})\\ &= 12 + 4\sqrt{2} - 3\sqrt{5} - \sqrt{5}\sqrt{2}\\ &= 12 + 4\sqrt{2} - 3\sqrt{5} - \sqrt{10}\end{aligned}$$

In this example there are no like terms to collect but if the same surd occurs in each bracket the expansion can be simplified.

Examples 3b

1. Expand and simplify $(2 + 2\sqrt{7})(5 - \sqrt{7})$

$$\begin{aligned}(2 + 2\sqrt{7})(5 - \sqrt{7}) &= (2)(5) - (2)(\sqrt{7}) + (5)(2\sqrt{7}) - (2\sqrt{7})(\sqrt{7})\\ &= 10 - 2\sqrt{7} + 10\sqrt{7} - 14\\ &= 8\sqrt{7} - 4\end{aligned}$$

2. Expand and simplify $(4 - \sqrt{3})(4 + \sqrt{3})$

$$\begin{aligned}(4 - \sqrt{3})(4 + \sqrt{3}) &= 16 + 4\sqrt{3} - 4\sqrt{3} - (\sqrt{3})(\sqrt{3})\\ &= 16 - 3\\ &= 13\end{aligned}$$

Example 3b number 2 is a special case because the result is a single rational number. The reader will notice that the two given brackets were of the form $(x - a)(x + a)$, i.e. the factors of $a^2 - x^2$.

The product of any two brackets of the type $(p - \sqrt{q})(p + \sqrt{q})$ is, similarly, $p^2 - (\sqrt{q})^2 = p^2 - q$, which is always rational.

This property has an important application in a later section of this chapter.

EXERCISE 3b

Expand and simplify where this is possible.

1. $\sqrt{3}(2-\sqrt{3})$
2. $\sqrt{2}(5+4\sqrt{2})$
3. $\sqrt{5}(2+\sqrt{75})$
4. $\sqrt{2}(\sqrt{32}-\sqrt{8})$
5. $(\sqrt{3}+1)(\sqrt{2}-1)$
6. $(\sqrt{3}+2)(\sqrt{3}+5)$
7. $(\sqrt{5}-1)(\sqrt{5}+1)$
8. $(2\sqrt{2}-1)(\sqrt{2}-1)$
9. $(\sqrt{5}-3)(2\sqrt{5}-4)$
10. $(4+\sqrt{7})(4-\sqrt{7})$
11. $(\sqrt{6}-2)^2$
12. $(2+3\sqrt{3})^2$

Multiply by a bracket which will make the product rational.

13. $(4-\sqrt{5})$
14. $(\sqrt{11}+3)$
15. $(2\sqrt{3}-4)$
16. $(\sqrt{6}-\sqrt{5})$
17. $(3-2\sqrt{3})$
18. $(2\sqrt{5}-\sqrt{2})$

Rationalising a Denominator

A fraction whose denominator contains a surd is more awkward to deal with than one where a surd occurs only in the numerator.

There is a technique for transferring the surd expression from the denominator to the numerator; it is called *rationalising the denominator* (i.e. making the denominator into a rational number).

Examples 3c

1. Rationalise the denominator of $\dfrac{2}{\sqrt{3}}$

The square root in the denominator can be removed if we multiply it by another $\sqrt{3}$. If this is done we must, of course, multiply the numerator also by $\sqrt{3}$, otherwise the value of the fraction is changed.

$$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})(\sqrt{3})} = \frac{2\sqrt{3}}{3}$$

2. Rationalise the denominator and simplify $\dfrac{3\sqrt{2}}{5-\sqrt{2}}$

We saw in Example 3b number 2, that a product of the type $(a-\sqrt{b})(a+\sqrt{b})$ is wholly rational so in this question we multiply numerator and denominator by $5+\sqrt{2}$

$$\frac{3\sqrt{2}}{5-\sqrt{2}} = \frac{3\sqrt{2}(5+\sqrt{2})}{(5-\sqrt{2})(5+\sqrt{2})}$$
$$= \frac{15\sqrt{2}+3(\sqrt{2})(\sqrt{2})}{25-(\sqrt{2})(\sqrt{2})}$$
$$= \frac{15\sqrt{2}+6}{23}$$

EXERCISE 3c

Rationalise the denominator, simplifying where possible.

1. $\dfrac{3}{\sqrt{2}}$
2. $\dfrac{1}{\sqrt{7}}$
3. $\dfrac{2}{\sqrt{11}}$
4. $\dfrac{3\sqrt{2}}{\sqrt{5}}$
5. $\dfrac{1}{\sqrt{27}}$
6. $\dfrac{\sqrt{5}}{\sqrt{10}}$
7. $\dfrac{1}{\sqrt{2}-1}$
8. $\dfrac{3\sqrt{2}}{5+\sqrt{2}}$
9. $\dfrac{2}{2\sqrt{3}-3}$
10. $\dfrac{5}{2-\sqrt{5}}$
11. $\dfrac{1}{\sqrt{7}-\sqrt{3}}$
12. $\dfrac{4\sqrt{3}}{2\sqrt{3}-3}$
13. $\dfrac{3-\sqrt{5}}{\sqrt{5}+1}$
14. $\dfrac{2\sqrt{3}-1}{4-\sqrt{3}}$
15. $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}$
16. $\dfrac{3}{\sqrt{3}-\sqrt{2}}$
17. $\dfrac{3\sqrt{5}}{2\sqrt{5}+1}$
18. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$
19. $\dfrac{2\sqrt{7}}{\sqrt{7}+2}$
20. $\dfrac{\sqrt{5}-1}{3-\sqrt{5}}$
21. $\dfrac{1}{\sqrt{11}-\sqrt{7}}$
22. $\dfrac{4-\sqrt{3}}{3-\sqrt{3}}$
23. $\dfrac{1-3\sqrt{2}}{3\sqrt{2}+2}$
24. $\dfrac{1}{3\sqrt{2}-2\sqrt{3}}$
25. $\dfrac{\sqrt{3}}{\sqrt{2}(\sqrt{6}-\sqrt{3})}$
26. $\dfrac{1}{\sqrt{3}(\sqrt{21}+\sqrt{7})}$
27. $\dfrac{\sqrt{2}}{\sqrt{3}(\sqrt{5}-\sqrt{2})}$

INDICES

Base and Index

In an expression such as 3^4, the *base* is 3 and the 4 is called the *power* or *index* (the plural is *indices*).

Working with indices involves using some properties which apply to any base, so we express these rules in terms of a general base a (i.e. a stands for any number).

Rule 1

Because a^3 means $a \times a \times a$ and a^2 means $a \times a$ it follows that

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5$$

i.e. $a^3 \times a^2 = a^{3+2}$

Similar examples with different powers all indicate the general rule that

$$a^p \times a^q = a^{p+q}$$

Rule 2

Now dealing with division we have

$$a^7 \div a^4 = \frac{\not a \times \not a \times \not a \times \not a \times a \times a \times a}{\not a \times \not a \times \not a \times \not a} = a^3$$

i.e. $a^7 \div a^4 = a^{7-4}$

Again this is just one example of the general rule

$$a^p \div a^q = a^{p-q}$$

When this rule is applied to certain fractions some interesting cases arise.

Consider $a^3 \div a^5$

$$\frac{a^3}{a^5} = \frac{\not a \times \not a \times \not a}{\not a \times \not a \times \not a \times a \times a} = \frac{1}{a^2}$$

But from Rule 2 we have

$$a^3 \div a^5 = a^{3-5} = a^{-2}$$

Therefore a^{-2} means $\dfrac{1}{a^2}$

In general $\qquad a^{-p} = \dfrac{1}{a^p}$

i.e. $\qquad a^{-p}$ means 'the reciprocal of a^p'

Now consider $a^4 \div a^4$

$$\frac{a^4}{a^4} = \frac{\not a \times \not a \times \not a \times \not a}{\not a \times \not a \times \not a \times \not a} = 1$$

From Rule 2, $\qquad \dfrac{a^4}{a^4} = a^{4-4} = a^0$

Therefore $\qquad a^0 = 1$

i.e. $\qquad$ any base to the power zero is equal to 1

Rule 3

$$(a^2)^3 = (a \times a)^3$$
$$= (a \times a) \times (a \times a) \times (a \times a)$$
$$= a^6$$

i.e. $(a^2)^3 = a^{2 \times 3}$

In general $\qquad (a^p)^q = a^{pq}$

Rule 4

This rule explains the meaning of a fractional index.

From the first rule we have

$$a^{1/2} \times a^{1/2} = a^{1/2 + 1/2} = a^1 = a$$

i.e. $\qquad a = a^{1/2} \times a^{1/2}$

But $\qquad a = \sqrt{a} \times \sqrt{a}$

Therefore $a^{1/2}$ means $\sqrt{a}$, i.e. the positive square root of a

Similarly $a^{1/3} \times a^{1/3} \times a^{1/3} = a^{1/3+1/3+1/3} = a^1 = a$

But $\qquad \sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

Therefore $a^{1/3}$ means $\sqrt[3]{a}$, i.e. the cube root of a

In general $\qquad a^{1/p} = \sqrt[p]{a}$, i.e. the pth root of a

For a more general fractional index, $\frac{p}{q}$, the third rule shows that

$$a^{p/q} = (a^p)^{1/q} \quad \text{or} \quad (a^{1/q})^p$$

For example

$$a^{3/4} = (a^3)^{1/4} \quad \text{or} \quad (a^{1/4})^3$$
$$= \sqrt[4]{a^3} \quad \text{or} \quad (\sqrt[4]{a})^3$$

i.e. $a^{3/4}$ represents either 'the fourth root of a^3'
or 'the cube of the fourth root of a'

All the general rules can be applied to simplify a wide range of expressions containing indices *provided that the terms all have the same base.*

Examples 3d

1. Simplify (a) $\dfrac{2^3 \times 2^7}{4^3}$ (b) $(x^2)^7 \times x^{-3}$ (c) $\sqrt[3]{(a^4b^5)} \times b^{1/3}/a$

(a) First we express all the terms to a base 2

$$\frac{2^3 \times 2^7}{4^3} = \frac{2^3 \times 2^7}{(2^2)^3}$$
$$= \frac{2^{3+7}}{2^{2\times 3}}$$
$$= \frac{2^{10}}{2^6} = 2^4$$

(b)
$$(x^2)^7 \times x^{-3} = x^{2\times 7} \times \frac{1}{x^3}$$
$$= x^{14} \times \frac{1}{x^3} = x^{11}$$

(c)
$$\sqrt[3]{(a^4b^5)} \times b^{1/3}/a = (a^{4/3})(b^{5/3})(b^{1/3})(a^{-1})$$
$$= (a^{4/3-1})(b^{5/3+1/3})$$
$$= a^{1/3}b^2$$

2. Evaluate (a) $(64)^{-1/3}$ (b) $\left(\dfrac{25}{9}\right)^{-3/2}$

(a)
$$(64)^{-1/3} = \frac{1}{(64)^{1/3}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

(b)
$$\left(\frac{25}{9}\right)^{-3/2} = \left(\frac{9}{25}\right)^{3/2} = \left(\sqrt{\frac{9}{25}}\right)^3 = \left(\frac{3}{5}\right)^3 = \frac{27}{125}$$

Note that $\left(\dfrac{9}{25}\right)^{3/2}$ could have been expressed as $\sqrt{\left(\dfrac{9}{25}\right)^3}$ but this form involves *much* bigger numbers.

EXERCISE 3d

Simplify

1. $\dfrac{2^4}{2^2 \times 4^3}$
2. $4^{1/2} \times 2^{-3}$
3. $(3^3)^{1/2} \times 9^{1/4}$
4. $\dfrac{x^{1/3} \times x^{4/3}}{x^{-1/3}}$
5. $\dfrac{p^{1/2} \times p^{-3/4}}{p^{-1/4}}$
6. $(\sqrt{t})^3 \times (\sqrt{t^5})$
7. $(y^2)^{3/2} \times y^{-3}$
8. $(16)^{5/4} \div 8^{4/3}$
9. $\dfrac{y^{1/2}}{y^{-3/4}} \times \sqrt{(y^{1/2})}$
10. $x^2 \times x^{5/2} \div x^{-1/2}$
11. $\dfrac{y^{1/6} \times y^{-2/3}}{y^{1/4}}$
12. $(p^{1/3})^2 \times (p^2)^{1/3} \div \sqrt[3]{p}$

Evaluate

13. $\left(\dfrac{1}{3}\right)^{-1}$
14. $\left(\dfrac{1}{4}\right)^{5/2}$
15. $(8)^{-1/3}$
16. $\dfrac{1}{(16)^{-1/4}}$
17. $\left(\dfrac{1}{9}\right)^{-3/2}$
18. $\left(\dfrac{27}{8}\right)^{2/3}$
19. $\left(\dfrac{100}{9}\right)^{0}$
20. $\dfrac{1}{4^{-2}}$
21. $(0.64)^{-1/2}$
22. $\left(-\dfrac{1}{5}\right)^{-1}$
23. $(121)^{3/2}$
24. $\left(\dfrac{125}{27}\right)^{-1/3}$

25. $18^{1/2} \times 2^{1/2}$

26. $3^{-3} \times 2^0 \times 4^2$

27. $\dfrac{8^{1/2} \times 32^{1/2}}{(16)^{1/4}}$

28. $5^{1/3} \times 25^0 \times 25^{1/3}$

29. $27^{1/4} \times 3^{1/4} \times (\sqrt{3})^{-2}$

30. $\dfrac{9^{1/3} \times 27^{-1/2}}{3^{-1/6} \times 3^{-2/3}}$

LOGARITHMS

Consider the statement $10^2 = 100$

If this is expressed in words we have

the base 10 raised to the power 2 gives 100

Now this relationship can be rearranged to give the same information, but with a different emphasis, i.e.

the power to which the base 10 must be raised to give 100 is 2

In this form the power is called a logarithm (log)

The whole relationship can then be abbreviated to read

the logarithm to the base 10 of 100 is 2

or $\log_{10} 100 = 2$

In the same way, $2^3 = 8 \Rightarrow \log_2 8 = 3$

and $3^4 = 81 \Rightarrow \log_3 81 = 4$

Similarly $\log_5 25 = 2 \Rightarrow 5^2 = 25$

and $\log_9 3 = \frac{1}{2} \Rightarrow 9^{1/2} = 3$

Although we have so far used only 10, 2 and 3, the base of a logarithm can be any positive number, or even an unspecified number represented by a letter, for example

$$b = a^c \quad \Leftrightarrow \quad \log_a b = c$$

Note that the symbol $\Leftrightarrow$ means that each of these facts implies the other.

Example 3e

(a) Write $\log_2 64 = 6$ in index form.
(b) Write $5^3 = 125$ in logarithmic form.
(c) Complete the statement $2^{-3} = ?$ and then write it in logarithmic form.

(a) If $\log_2 64 = 6$ then the base is 2, the number is 64 and the power (i.e. the log) is 6

$$\log_2 64 = 6 \quad \Rightarrow \quad 2^6 = 64$$

(b) If $5^3 = 125$ then the base is 5, the log (i.e. the power) is 3 and the number is 125

$$5^3 = 125 \quad \Rightarrow \quad \log_5 125 = 3$$

(c) $2^{-3} = \frac{1}{8}$

The base is 2, the power (log) is -3 and the number is $\frac{1}{8}$

$$2^{-3} = \tfrac{1}{8} \quad \Rightarrow \quad \log_2 (\tfrac{1}{8}) = -3$$

EXERCISE 3e

Convert each of the following facts to logarithmic form.

1. $10^3 = 1000$ **2.** $2^4 = 16$ **3.** $10^4 = 10\,000$
4. $3^2 = 9$ **5.** $4^2 = 16$ **6.** $5^2 = 25$
7. $10^{-2} = 0.01$ **8.** $9^{1/2} = 3$ **9.** $5^0 = 1$
10. $4^{1/2} = 2$ **11.** $12^0 = 1$ **12.** $8^{1/3} = 2$
13. $p = q^2$ **14.** $x^y = 2$ **15.** $p^q = r$

Convert each of the following facts to index form.

16. $\log_{10} 100\,000 = 5$ **17.** $\log_4 64 = 3$ **18.** $\log_{10} 10 = 1$
19. $\log_2 4 = 2$ **20.** $\log_2 32 = 5$ **21.** $\log_{10} 1000 = 3$
22. $\log_5 1 = 0$ **23.** $\log_3 9 = 2$ **24.** $\log_4 16 = 2$
25. $\log_3 27 = 3$ **26.** $\log_{36} 6 = \frac{1}{2}$ **27.** $\log_a 1 = 0$
28. $\log_x y = z$ **29.** $\log_a 5 = b$ **30.** $\log_p q = r$

Evaluating Logarithms

It is generally easier to solve a simple equation in index form than in log form so we often use an index equation in order to evaluate a logarithm. For example to evaluate $\log_{49} 7$ we can say

if $x = \log_{49} 7$ then $49^x = 7 \Rightarrow x = \frac{1}{2}$

therefore $\log_{49} 7 = \frac{1}{2}$

In particular, for any base b,

if $x = \log_b 1$ then $b^x = 1 \Rightarrow x = 0$

i.e. **the logarithm to any base of 1 is zero.**

EXERCISE 3f

Evaluate

1. $\log_2 4$	**2.** $\log_{10} 1\,000\,000$	**3.** $\log_2 64$	**4.** $\log_3 81$
5. $\log_8 64$	**6.** $\log_4 64$	**7.** $\log_9 3$	**8.** $\log_{1/2} 4$
9. $\log_{10} 0.1$	**10.** $\log_{121} 11$	**11.** $\log_5 1$	**12.** $\log_2 2$
13. $\log_{64} 4$	**14.** $\log_{99} 1$	**15.** $\log_{27} 3$	**16.** $\log_a a^3$

THE LAWS OF LOGARITHMS

When working with indices earlier in this chapter we found certain rules that powers obey in the multiplication and division of numbers. Because logarithm is just another word for index or power, it is to be expected that logarithms too obey certain laws and these we are now going to investigate.

Consider $x = \log_a b$ and $y = \log_a c$

$\Rightarrow$ $a^x = b$ and $a^y = c$

Now $bc = (a^x)(a^y)$

$\Rightarrow$ $bc = a^{x+y}$

Therefore $\log_a bc = x + y$

i.e. $\log_a bc = \log_a b + \log_a c$

This is the first law of logarithms and, as a can represent *any* base, this law applies to the log of *any* product *provided that the same base is used for all the logarithms in the formula.*

Using x and y again, a law for the log of a fraction can be found.

$$\frac{b}{c} = \frac{a^x}{a^y} \Rightarrow \frac{b}{c} = a^{x-y}$$

Therefore $\log_a (b/c) = x - y$

i.e. $\log_a (b/c) = \log_a b - \log_a c$

A third law allows us to deal with an expression of the type $\log_a b^n$

Using $x = \log_a b^n \Rightarrow a^x = b^n$

i.e. $a^{x/n} = b$

Therefore $x/n = \log_a b \Rightarrow x = n \log_a b$

i.e. $\log_a b^n = n \log_a b$

So we now have the three most important laws of logarithms. Because they are true for *any* base it is unnecessary to include a base in the formula but

in each of these laws every logarithm must be to the same base

$$\log bc = \log b + \log c$$
$$\log b/c = \log b - \log c$$
$$\log b^n = n \log b$$

Examples 3g

1. Express $\log pq^2\sqrt{r}$ in terms of $\log p$, $\log q$ and $\log r$

$$\log pq^2\sqrt{r} = \log p + \log q^2 + \log\sqrt{r}$$
$$= \log p + 2\log q + \tfrac{1}{2}\log r$$

2. Simplify $3\log p + n\log q - 4\log r$

$$3\log p + n\log q - 4\log r = \log p^3 + \log q^n - \log r^4$$
$$= \log\frac{p^3q^n}{r^4}$$

EXERCISE 3g

Express in terms of $\log p$, $\log q$, and $\log r$

1. $\log pq$
2. $\log pqr$
3. $\log p/q$
4. $\log pq/r$
5. $\log p/qr$
6. $\log p^2q$
7. $\log q/r^2$
8. $\log p\sqrt{q}$
9. $\log p^2q^3/r$
10. $\log \sqrt{(q/r)}$
11. $\log q^n$
12. $\log p^nq^m$

Simplify

13. $\log p + \log q$
14. $2 \log p + \log q$
15. $\log q - \log r$
16. $3 \log q + 4 \log p$
17. $n \log p - \log q$
18. $\log p + 2 \log q - 3 \log r$

MIXED EXERCISE 3

1. Simplify (a) $\sqrt{84}$ (b) $\sqrt{300}$ (c) $\sqrt{45}$
2. Expand and simplify (a) $(3 + \sqrt{2})(4 - 2\sqrt{2})$ (b) $(\sqrt{5} - \sqrt{2})^2$
3. Multiply by a bracket that will make the product rational
   (a) $(7 - \sqrt{3})$ (b) $(2\sqrt{2} + 1)$ (c) $(\sqrt{7} - \sqrt{5})$
4. Rationalise the denominator and simplify where possible
   (a) $\dfrac{5}{\sqrt{7}}$ (b) $\dfrac{3}{\sqrt{13} - 2}$ (c) $\dfrac{4}{\sqrt{3} - \sqrt{2}}$ (d) $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$
5. Simplify (a) $\dfrac{2^3 \times 4^{-2}}{2^{-1}}$ (b) $(x^3)^{-2} \times (x^2)^3$
6. Evaluate (a) $(64)^{-1/3}$ (b) $\left(\dfrac{49}{16}\right)^{-1/2}$ (c) $\left(\dfrac{8}{27}\right)^{3/2}$
7. Simplify (a) $8^{1/6} \times 2^0 \times 2^{-1/2}$ (b) $(\sqrt{5})^{-2} \times 75^{1/2} \times 25^{-1/4}$
8. Evaluate (a) $\log_2 128$ (b) $\log_{25} 5$ (c) $\log_{12} 1$
9. Express in terms of $\log a$, $\log b$ and $\log c$
   (a) $\log a^3/(bc^2)$ (b) $\log a^n/b$ (c) $\log ab/c$
10. Simplify (a) $3 \log a - \log b$ (b) $\log 1/a + \log 1$

CHAPTER 4

QUADRATIC EQUATIONS AND SIMULTANEOUS EQUATIONS

QUADRATIC EQUATIONS

When a quadratic expression has a particular value we have a quadratic equation, for example

$$2x^2 - 5x + 1 = 0$$

Using a, b and c to stand for any numbers, any quadratic equation can be written in the general form

$$ax^2 + bx + c = 0$$

Solution by Factorising

Consider the quadratic equation $x^2 - 3x + 2 = 0$

The quadratic expression on the left-hand side can be factorised,

i.e. $$x^2 - 3x + 2 = (x - 2)(x - 1)$$

Therefore the given equation becomes

$$(x - 2)(x - 1) = 0 \qquad [1]$$

Now if the product of two quantities is zero then one, or both, of those quantities must be zero.

Applying this fact to equation [1] gives

$$x - 2 = 0 \quad \text{or} \quad x - 1 = 0$$

i.e. $$x = 2 \quad \text{or} \quad x = 1$$

This is the solution of the given equation.
The values 2 and 1 are called the *roots* of that equation.

This method of solution can be used for any quadratic equation in which the quadratic expression factorises.

Example 4a

Find the roots of the equation $x^2 + 6x - 7 = 0$

$$x^2 + 6x - 7 = 0$$

$\Rightarrow$ $(x-1)(x+7) = 0$

$\therefore$ $x - 1 = 0$ or $x + 7 = 0$

$\therefore$ $x = 1$ or $x = -7$

The roots of the equation are 1 and -7

EXERCISE 4a

Solve the equations.

1. $x^2 + 5x + 6 = 0$

2. $x^2 + x - 6 = 0$

3. $x^2 - x - 6 = 0$

4. $x^2 + 6x + 8 = 0$

5. $x^2 - 4x + 3 = 0$

6. $x^2 + 2x - 3 = 0$

7. $2x^2 + 3x + 1 = 0$

8. $4x^2 - 9x + 2 = 0$

9. $x^2 + 4x - 5 = 0$

10. $x^2 + x - 72 = 0$

Find the roots of the equations.

11. $x^2 - 2x - 3 = 0$

12. $x^2 + 5x + 4 = 0$

13. $x^2 - 6x + 5 = 0$

14. $x^2 + 3x - 10 = 0$

15. $x^2 - 5x - 14 = 0$

16. $x^2 - 9x + 14 = 0$

Rearranging the Equation

The terms in a quadratic equation are not always given in the order $ax^2 + bx + c = 0$. When they are given in a different order they should be rearranged into the standard form.

For example

$x^2 - x = 4$ becomes $x^2 - x - 4 = 0$

$3x^2 - 1 = 2x$ becomes $3x^2 - 2x - 1 = 0$

$x(x-1) = 2$ becomes $x^2 - x = 2$ $\Rightarrow$ $x^2 - x - 2 = 0$

It is usually best to collect the terms on the side where the x^2 term is positive, for example

$2 - x^2 = 5x$ becomes $0 = x^2 + 5x - 2$

i.e. $x^2 + 5x - 2 = 0$

Losing a Solution

Quadratic equations sometimes have a common factor containing the unknown quantity. It is very tempting in such cases to divide by the common factor, but doing this results in the loss of part of the solution, as the following example shows.

First solution

$x^2 - 5x = 0$

$x(x-5) = 0$

$\therefore$ $x = 0$ or $x - 5 = 0$

$\Rightarrow$ $x = 0$ or 5

Second solution

$x^2 - 5x = 0$

$x - 5 = 0$ (Dividing by x)

$\therefore$ $x = 5$

The solution $x = 0$ has been lost.

Although dividing an equation by a numerical common factor is correct and sensible, dividing by a common factor containing the unknown quantity results in the loss of a solution.

Examples 4b

1. Solve the equation $4x - x^2 = 3$

$$4x - x^2 = 3$$

$\Rightarrow$ $0 = x^2 - 4x + 3$

$\Rightarrow$ $x^2 - 4x + 3 = 0$

$\Rightarrow$ $(x-3)(x-1) = 0$

$\Rightarrow$ $x - 3 = 0$ or $x - 1 = 0$

$\Rightarrow$ $x = 3$ or $x = 1$

2. Find the roots of the equation $x^2 = 3x$

$$x^2 = 3x$$
$$\Rightarrow \quad x^2 - 3x = 0$$
$$\Rightarrow \quad x(x-3) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = 3$$

Therefore the roots are 0 and 3

EXERCISE 4b

Solve the equations.

1. $x^2 + 10 - 7x = 0$

2. $15 - x^2 - 2x = 0$

3. $x^2 - 3x = 4$

4. $12 - 7x + x^2 = 0$

5. $2x - 1 + 3x^2 = 0$

6. $x(x+7) + 6 = 0$

7. $2x^2 - 4x = 0$

8. $x(4x+5) = -1$

9. $2 - x = 3x^2$

10. $6x^2 + 3x = 0$

11. $x^2 + 6x = 0$

12. $x^2 = 10x$

13. $x(4x+1) = 3x$

14. $20 + x(1-x) = 0$

15. $x(3x-2) = 8$

16. $x^2 - x(2x-1) + 2 = 0$

17. $x(x+1) = 2x$

18. $4 + x^2 = 2(x+2)$

19. $x(x-2) = 3$

20. $1 - x^2 = x(1+x)$

Solution by Completing the Square

When there are no obvious factors, another method is needed to solve the equation. One such method involves adding a constant to the x^2 term and the x term, to make a perfect square. This technique is called *completing the square.*

Consider $\quad x^2 - 2x$

Adding 1 gives $\quad x^2 - 2x + 1$

Now $x^2 - 2x + 1 = (x-1)^2$ which is a perfect square.

Adding the number 1 was not a guess, it was found by using the fact that

$$x^2 + 2ax + \boxed{a^2} = (x+a)^2$$

We see from this that the number to be added is always

$$(\text{half the coefficient of } x)^2$$

Hence $x^2 + 6x$ requires 3^2 to be added to make a perfect square,

i.e. $\quad x^2 + 6x + 9 = (x+3)^2$

To complete the square when the coefficient of x^2 is not 1, we first take out the coefficient of x^2 as a factor,

e.g. $\quad 2x^2 + x = 2(x^2 + \frac{1}{2}x)$

Now we add $(\frac{1}{2} \times \frac{1}{2})^2$ inside the bracket, giving

$$2(x^2 + \tfrac{1}{2}x + \tfrac{1}{16}) = 2(x + \tfrac{1}{4})^2$$

Take extra care when the coefficient of x^2 is negative

e.g. $\quad -x^2 + 4x = -(x^2 - 4x)$

Then $\quad -(x^2 - 4x + 4) = -(x-2)^2$

$\therefore \quad -x^2 + 4x - 4 = -(x-2)^2$

Examples 4c

1. Solve the equation $x^2 - 4x - 2 = 0$, giving the solution in surd form.

$$x^2 - 4x - 2 = 0$$

No factors can be found so we isolate the two terms with x in,

i.e. $\quad x^2 - 4x = 2$

Add $\{\frac{1}{2} \times (-4)\}^2$ to *both* sides

i.e. $\quad x^2 - 4x + 4 = 2 + 4$

$\Rightarrow \quad (x-2)^2 = 6$

$\therefore \quad x - 2 = \pm\sqrt{6}$

$\therefore \quad x = 2 + \sqrt{6} \text{ or } x = 2 - \sqrt{6}$

---

2. Find in surd form the roots of the equation $2x^2 - 3x - 3 = 0$

$$2x^2 - 3x - 3 = 0$$
$$2x^2 - 3x = 3$$
$$2(x^2 - \tfrac{3}{2}x) = 3$$
$$x^2 - \tfrac{3}{2}x = \tfrac{3}{2}$$
$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = \tfrac{3}{2} + \tfrac{9}{16}$$
$$(x - \tfrac{3}{4})^2 = \tfrac{33}{16}$$
$$\therefore \quad x - \tfrac{3}{4} = \pm\sqrt{\tfrac{33}{16}} = \pm\tfrac{1}{4}\sqrt{33}$$
$$\therefore \quad x = \tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{33}$$

The roots of the equation are $\frac{1}{4}(3 + \sqrt{33})$ and $\frac{1}{4}(3 - \sqrt{33})$

EXERCISE 4c

Add a number to each expression so that the result contains a perfect square as a factor.

1. $x^2 - 4x$
2. $x^2 + 2x$
3. $x^2 - 6x$
4. $x^2 + 10x$
5. $2x^2 - 4x$
6. $x^2 + 5x$
7. $3x^2 - 48x$
8. $x^2 + 18x$
9. $2x^2 - 40x$
10. $x^2 + x$
11. $3x^2 - 2x$
12. $2x^2 + 3x$

Solve the equations by completing the square, giving the solutions in surd form.

13. $x^2 + 8x = 1$
14. $x^2 - 2x - 2 = 0$
15. $x^2 + x - 1 = 0$
16. $2x^2 + 2x = 1$
17. $x^2 + 3x + 1 = 0$
18. $2x^2 - x - 2 = 0$
19. $x^2 + 4x = 2$
20. $3x^2 + x - 1 = 0$
21. $2x^2 + 4x = 7$
22. $x^2 - x = 3$
23. $4x^2 + x - 1 = 0$
24. $2x^2 - 3x - 4 = 0$

The Formula for Solving a Quadratic Equation

Solving a quadratic equation by completing the square is rather tedious. If the method is applied to a general quadratic equation, a formula can be derived which can then be used to solve any particular equation.

Using a, b and c to represent any numbers we have the general quadratic equation

$$ax^2 + bx + c = 0$$

Using the method of completing the square for this equation gives

$$ax^2 + bx = -c$$

i.e. $$a\left(x^2 + \frac{b}{a}x\right) = -c$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{(b^2 - 4ac)}{4a^2}}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

i.e. $$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Example 4d

Find, by using the formula, the roots of the equation $2x^2 - 7x - 1 = 0$ giving them correct to 3 decimal places.

$$2x^2 - 7x - 1 = 0$$

Comparing with $ax^2 + bx + c = 0$ gives $a = 2$, $b = -7$, $c = -1$

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$
$$= \frac{7 \pm \sqrt{\{49 - 4(2)(-1)\}}}{4}$$

Therefore, in surd form, $$x = \frac{7 \pm \sqrt{57}}{4}$$

Correct to 3 d.p. the roots are 3.637 and -0.137

EXERCISE 4d

Solve the equations by using the formula. Give the solutions in surd form.

1. $x^2 + 4x + 2 = 0$ **2.** $2x^2 + x - 2 = 0$

3. $x^2 + 5x + 1 = 0$ **4.** $2x^2 - x - 4 = 0$

5. $x^2 + 1 = 4x$ **6.** $2x^2 - x = 5$

7. $1 + x - 3x^2 = 0$ **8.** $3x^2 = 1 - x$

Find, correct to 3 d.p., the roots of the equations.

9. $5x^2 + 9x + 2 = 0$ **10.** $2x^2 - 7x + 4 = 0$

11. $4x^2 - 7x - 1 = 0$ **12.** $3x = 5 - 4x^2$

13. $4x^2 + 3x = 5$ **14.** $1 = 5x - 5x^2$

15. $8x - x^2 = 1$ **16.** $x^2 - 3x = 1$

SIMULTANEOUS EQUATIONS

When only one unknown quantity has to be found, only one equation is needed to provide a solution.

If two unknown quantities are involved in a problem we need two equations connecting them. Then, between the two equations we can eliminate one of the unknowns, producing just one equation containing just one unknown. This is then ready for solution.

SOLUTION OF THREE LINEAR EQUATIONS

63 For three unknown quantities we need three connections, i.e. three equations. Then one unknown at a time can be eliminated. One way to eliminate an unknown quantity is to add or subtract two of the equations and then go on to eliminate the second unknown in a similar way.

Examples 4e

1. Solve the equations $\begin{cases} x + y - z = 4 \\ 2x + z = 7 \\ 3x - 2y = 5 \end{cases}$

$$x + y - z = 4 \quad [1]$$
$$2x + z = 7 \quad [2]$$
$$3x - 2y = 5 \quad [3]$$

As z appears only in equations [1] and [2] we can eliminate z from these two equations – in this case by adding.

[1] + [2] gives $3x + y = 11$ [4]

Now bring in [3] $3x - 2y = 5$ [3]

[4] − [3] gives $3y = 6$

$\Rightarrow$ $y = 2$

Using $y = 2$ in [3] gives $3x - 4 = 5$

$\Rightarrow$ $3x = 9$

$\Rightarrow$ $x = 3$

Now using $x = 3$ in [2] gives $6 + z = 7$

Therefore the solution of the three simultaneous equations is

$$x = 3, \quad y = 2, \quad z = 1$$

It is not always as easy to eliminate the first of the unknown quantities. If all three unknowns occur in all three equations it is necessary to eliminate the same unknown from each of two different pairs of equations.

---

2. Solve the equations $\begin{cases} x - y + 2z = 6 \\ 2x + y + z = 3 \\ 3x - y + z = 6 \end{cases}$

$$x - y + 2z = 6 \quad [1]$$
$$2x + y + z = 3 \quad [2]$$
$$3x - y + z = 6 \quad [3]$$

The easiest letter to eliminate from two pairs of equations is y

$[1] + [2]$ gives $\quad 3x + 3z = 9$

Dividing by 3 gives $\quad x + z = 3 \qquad [4]$

$[2] + [3]$ gives $\quad 5x + 2z = 9 \qquad [5]$

Now we eliminate either x or z from [4] and [5]

$5 \times [4] - [5]$ gives $\quad 3z = 6$

$\Rightarrow \quad z = 2$

Using $z = 2$ in [4] gives $\quad x + 2 = 3$

$\Rightarrow \quad x = 1$

Then using $x = 1$ and $z = 2$ in [2] gives

$$2 + y + 2 = 3$$

$\Rightarrow \quad y = -1$

Therefore the solution is $x = 1, \; y = -1, \; z = 2$

This topic is needed later on in the course and will be revised and developed at that stage. For the present, a short exercise of simple sets of equations is provided.

EXERCISE 4e

Solve the following sets of equations.

Remember first to look for a letter which occurs in only two equations because it can be eliminated completely in one step.

1. $x + 2y = 4$
$x + 3z = 5$
$2y - z = 1$

2. $y - z = 3$
$x - 2y + z = -4$
$x + 2y = 11$

3. $x + y + 3z = 6$
$2x - y = 3$
$4x - z = 2$

4. $2x - y - z = 5$
$4y + 3z = 5$
$x + 2y = 7$

5. $x + y + 4z = 15$
$x - y + z = 2$
$x + 2y - 3z = -4$

6. $2x - 3y + z = 13$
$x + y - 2z = -1$
$3x - y + 2z = 17$

SOLUTION OF ONE LINEAR AND ONE QUADRATIC EQUATION

Another way to eliminate an unknown quantity from two equations is by substitution. From the linear equation we can express one unknown in terms of the other, and then substitute in the quadratic equation.

Example 4f

Solve the equations $\quad x - y = 2$

$$2x^2 - 3y^2 = 15$$

$x - y = 2 \qquad [1]$

$2x^2 - 3y^2 = 15 \qquad [2]$

Equation 1 is linear so we use it for the substitution, i.e. $x = y + 2$

Substituting $y + 2$ for x in [2] gives

$$2(y+2)^2 - 3y^2 = 15$$

$\Rightarrow \quad 2(y^2 + 4y + 4) - 3y^2 = 15$

$\Rightarrow \quad 2y^2 + 8y + 8 - 3y^2 = 15$

Collecting terms on the side where y^2 is positive gives

$$0 = y^2 - 8y + 7$$

$\Rightarrow \quad 0 = (y-7)(y-1)$

$\therefore \quad y = 7$ or 1

Now we use $x = y + 2$ to find corresponding values of x

y	7	1
x	9	3

$\therefore$ either $\quad x = 9$ and $y = 7$

or $\quad x = 3$ and $y = 1$

Note that the values of x and y must be given in *corresponding pairs*.

It is incorrect to write the answer as $y = 7$ or 1 and $x = 9$ or 3

because $\begin{cases} y = 7 \text{ with } x = 3 \\ y = 1 \text{ with } x = 9 \end{cases}$ are *not* solutions

---

EXERCISE 4f

Solve the following pairs of equations.

1. $x^2 + y^2 = 5$
   $y - x = 1$

2. $y^2 - x^2 = 8$
   $x + y = 2$

3. $3x^2 - y^2 = 3$
   $2x - y = 1$

4. $y = 4x^2$
   $y + 2x = 2$

5. $y^2 + xy = 3$
   $2x + y = 1$

6. $x^2 - xy = 14$
   $y = 3 - x$

7. $xy = 2$
   $x + y - 3 = 0$

8. $2x - y = 2$
   $x^2 - y = 5$

9. $y - x = 4$
   $y^2 - 5x^2 = 20$

10. $x + y^2 = 10$
    $x - 2y = 2$

11. $4x + y = 1$
    $4x^2 + y = 0$

12. $3xy - x = 0$
    $x + 3y = 2$

13. $x^2 + 4y^2 = 2$
    $2y + x + 2 = 0$

14. $x + 3y = 0$
    $2x + 3xy = 1$

15. $3x - 4y = 1$
    $6xy = 1$

16. $x^2 + 4y^2 = 2$
    $x + 2y = 2$

17. $xy = 9$
    $x - 2y = 3$

18. $4x + y = 2$
    $4x + y^2 = 8$

19. $1 + 3xy = 0$
    $x + 6y = 1$

20. $x^2 - xy = 0$
    $x + y = 1$

21. $xy + y^2 = 2$
    $2x + y = 3$

22. $xy + x = -3$
    $2x + 5y = 8$

PROPERTIES OF THE ROOTS OF A QUADRATIC EQUATION

A number of interesting facts can be observed by examining the formula used for solving a quadratic equation, especially when it is written in the form

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

The Sum of the Roots

The separate roots are

$$-\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

When the roots are added, the terms containing the square root disappear giving

$$\text{sum of roots} = -\frac{b}{a}$$

This fact is very useful as a check on the accuracy of roots that have been calculated.

The Nature of the Roots

In the formula there are two terms. The first of these, $-\dfrac{b}{2a}$, can always be found for any values of a and b.

The second term however, i.e. $\dfrac{\sqrt{(b^2 - 4ac)}}{2a}$, is not so straightforward as there are three different cases to consider.

1) If $b^2 - 4ac$ is positive, its square root can be found and, whether it is a whole number, a fraction or a decimal, it is a number of the type we are familiar with - it is called a *real* number.

The two square roots, i.e. $\pm\sqrt{(b^2 - 4ac)}$ have different (or distinct) values giving two different real values of x
So the equation has *two different real roots.*

2) If $b^2 - 4ac$ is zero then its square root also is zero and $x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{(b^2-4ac)}}{2a}$ gives

$$x = -\frac{b}{2a} + 0 \quad \text{and} \quad x = -\frac{b}{2a} - 0$$

i.e. there is just one value of x that satisfies the equation.

An example of this case is $x^2 - 2x + 1 = 0$

From the formula we get $x = -\dfrac{(-2)}{2} \pm 0$

i.e. $x = 1$ or 1

By factorising we can see that there are two equal roots,

i.e. $(x-1)(x-1) = 0$

$\Rightarrow$ $x = 1$ or $x = 1$

This type of equation can be said to have a *repeated root*.

3) If $b^2 - 4ac$ is negative we cannot find its square root because there is no real number whose square is negative. In this case the equation has *no real roots*.

From these three considerations we see that the roots of a quadratic equation can be

either	real and different
or	real and equal
or	not real

and that it is the value of $b^2 - 4ac$ which determines the nature of the roots, i.e.

Condition	Nature of Roots
$b^2 - 4ac > 0$	Real and different
$b^2 - 4ac = 0$	Real and equal
$b^2 - 4ac < 0$	Not real

Sometimes it matters only that the roots are real, in which case the first two conditions can be combined to give

if $b^2 - 4ac \geqslant 0$, the roots are real.

Examples 4g

1. Determine the nature of the roots of the equation $x^2 - 6x + 1 = 0$

$$x^2 - 6x + 1 = 0$$

$a = 1$, $b = -6$, $c = 1$

$$b^2 - 4ac = (-6)^2 - 4(1)(1) = 32$$

$b^2 - 4ac > 0$ so the roots are real and different.

---

2. If the roots of the equation $2x^2 - px + 8 = 0$ are equal, find the value of p.

$$2x^2 - px + 8 = 0$$

$a = 2$, $b = -p$, $c = 8$

The roots are equal so $b^2 - 4ac = 0$,

i.e. $(-p)^2 - 4(2)(8) = 0$

$\Rightarrow$ $p^2 - 64 = 0$

$\Rightarrow$ $p^2 = 64$

$\therefore$ $p = \pm 8$

---

3. Prove that the equation $(k-2)x^2 + 2x - k = 0$ has real roots whatever the value of k

$$(k-2)x^2 + 2x - k = 0$$

$a = k-2$, $b = 2$, $c = -k$

$$\begin{aligned} b^2 - 4ac &= 4 - 4(k-2)(-k) \\ &= 4 + 4k^2 - 8k \\ &= 4k^2 - 8k + 4 \\ &= 4(k^2 - 2k + 1) = 4(k-1)^2 \end{aligned}$$

Now $(k-1)^2$ cannot be negative whatever the value of k, so $b^2 - 4ac$ cannot be negative. Therefore the roots are always real.

EXERCISE 4g

Without solving the equation, write down the sum of its roots.

1. $x^2 - 4x - 7 = 0$ **2.** $3x^2 + 5x + 1 = 0$

3. $2 + x - x^2 = 0$ **4.** $3x^2 - 4x - 2 = 0$

5. $x^2 + 3x + 1 = 0$ **6.** $7 + 2x - 5x^2 = 0$

Without solving the equation, determine the nature of its roots.

7. $x^2 - 6x + 4 = 0$ **8.** $3x^2 + 4x + 2 = 0$

9. $2x^2 - 5x + 3 = 0$ **10.** $x^2 - 6x + 9 = 0$

11. $4x^2 - 12x - 9 = 0$ **12.** $4x^2 + 12x + 9 = 0$

13. $x^2 + 4x - 8 = 0$ **14.** $x^2 + ax + a^2 = 0$

15. $x^2 - ax - a^2 = 0$ **16.** $x^2 + 2ax + a^2 = 0$

17. If the roots of $3x^2 + kx + 12 = 0$ are equal, find k

18. If $x^2 - 3x + a = 0$ has equal roots, find a

19. The roots of $x^2 + px + (p - 1) = 0$ are equal. Find p

20. Prove that the roots of the equation $kx^2 + (2k + 4)x + 8 = 0$ are real for all values of k

21. Show that the equation $ax^2 + (a + b)x + b = 0$ has real roots for all values of a and b

22. Find the relationship between p and q if the roots of the equation $px^2 + qx + 1 = 0$ are equal.

Summary

Methods for solving quadratic equations.

1) Collect the terms in the order $ax^2 + bx + c = 0$, then factorise the left-hand side.

2) Arrange in the form $ax^2 + bx = -c$, then complete the square on the left-hand side, adding the appropriate number to *both* sides.

3) Use the formula $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

Note. Roots that are not rational should be given in surd form (i.e. the exact form) unless an approximate form (such as correct to 3 s.f.) is specifically asked for.

Properties of Roots

$b^2 - 4ac > 0 \Rightarrow$ real different roots

$b^2 - 4ac = 0 \Rightarrow$ real equal roots

$b^2 - 4ac \geqslant 0 \Rightarrow$ real roots

$b^2 - 4ac < 0 \Rightarrow$ no real roots

Sum of roots $= -\dfrac{b}{a}$

LOSING SOLUTIONS

It has already been shown that a solution is lost if an equation is divided by a *common factor containing the unknown quantity.*

There is another situation where a valid solution *may* be overlooked.

The Infinite Solution

Consider the equation $t(t - 1) = t^2 + 2$
This gives $t^2 - t = t^2 + 2$
and $t = -2$ appears to be the only solution.

Suppose, however, that the value of t can be very large indeed, so large that t approaches infinity (we write, $t \to \infty$). Then, in the equation $t(t - 1) = t^2 + 2$, we see that

1 is so small compared with the value of t that $t(t - 1)$ is very nearly equal to t^2

also 2 is so small compared with the value of t^2 that $t^2 + 2$ is very nearly equal to t^2

That is, for very large values of t, $t(t - 1)$ is nearly equal to $t^2 + 2$ and, the larger t becomes the more nearly is the equation satisfied. Therefore, in an equation where the squared terms cancel, we must always *consider* a solution of the type $t \to \infty$

In real problems the unknown quantity usually represents something specific and in most cases it could not possibly have an infinitely large value.

There are cases, however, when the infinite solution is meaningful. The reader will meet some of these later on and will probably have met one already, i.e. if the unknown quantity, t, represents the tangent of an angle, then $t \to \infty$ gives an angle of 90°

MIXED EXERCISE 4

In each Question from 1 to 10

(a) write down the value of $-b/a$

(b) use any suitable method to find the roots of the equation, giving any irrational roots in surd form.

(c) find the sum of the roots and check that it is equal to the answer to (a).

1. $x^2 - 5x - 6 = 0$ **2.** $x^2 - 6x - 5 = 0$

3. $2x^2 + 3x = 1$ **4.** $5 - 3x^2 = 4x$

5. $x(2 - x) = 1$ **6.** $4x^2 - 3 = 11x$

7. $(x - 1)(x + 2) = 1$ **8.** $x^2 + 4x + 4 = 16$

9. $x^2 + 2x = 2$ **10.** $2(x^2 + 2) = x(x - 4)$

In Questions 11 to 16, solve the equations giving *all possible* solutions.

11. $x(x - 2) = 0$ **12.** $x(x + 3) = 4$

13. $x^2 + x + 8 = x(x + 5)$ **14.** $x^2 + 5x + 2 = 2(2x + 1)$

15. $x(x - 5) = 2(x + 5)$ **16.** $2x(x + 3) = x(2x - 1) + 7$

17. Determine the nature of the roots of the equations

(a) $x^2 + 3x + 7 = 0$ (b) $3x^2 - x - 5 = 0$

(c) $ax^2 + 2ax + a = 0$ (d) $2 + 9x - x^2 = 0$

18. For what values of p does the equation $px^2 + 4x + (p - 3) = 0$ have equal roots?

19. Show that the equation $2x^2 + 2(p + 1)x + p = 0$ always has real roots.

20. The equation $x^2 + kx + k = 1$ has equal roots. Find k

In Questions 21 and 22 solve the set of equations. (Choose your substitution carefully, to keep the amount of squaring to a minimum.)

21. $2x^2 - y^2 = 7$

$x + y = 9$

22. $2x = y - 1$

$x^2 - 3y + 11 = 0$

23. Use the formula to solve the equation $3x^2 - 17x + 10 = 0$

(a) Are the roots of the equation rational or irrational?

(b) What does your answer to (a) tell you about the LHS of the equation?

CONSOLIDATION A

SUMMARY

TERMS AND COEFFICIENTS

In an algebraic expression, terms are separated by plus or minus signs. An individual term is identified by the combination of letters involved. The coefficient of a term is the number in the term, e.g. ②x^2y

EXPANSION OF BRACKETS

Important results are

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$
$$(ax - b)^2 = a^2x^2 - 2abx + b^2$$
$$(ax + b)(ax - b) = a^2x^2 - b^2$$

PASCAL'S TRIANGLE

```
            1       1
        1       2       1
    1       3       3       1
1       4       6       4       1
```

The 1st, 2nd, 3rd, ... rows in this array give the coefficients in the expansion of $(1 + x)^1$, $(1 + x)^2$, $(1 + x)^3$, ...

PARTIAL FRACTIONS

A fraction like $\dfrac{3}{(x + 3)(2x - 1)}$ can be expressed as $\dfrac{A}{x + 3} + \dfrac{B}{2x - 1}$

INDICES

$$a^n \times a^m = a^{n+m}$$
$$a^n \div a^m = a^{n-m}$$
$$(a^n)^m = a^{nm}$$
$$\sqrt[n]{a} = a^{1/n}$$
$$a^0 = 1$$

LOGARITHMS

$$\log_a b = c \iff a^c = b$$

$$\log_a b + \log_a c = \log_a bc$$

$$\log_a b - \log_a c = \log_a b/c$$

$$\log_a b^n = n \log_a b$$

QUADRATIC EQUATIONS

The general quadratic equation is $ax^2 + bx + c = 0$

The roots of this equation can be found by
factorising when this is possible,

or completing the square,

or by using the formula $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

When $b^2 - 4ac > 0$, the roots are real and different.

When $b^2 - 4ac = 0$, the roots are real and equal.

When $b^2 - 4ac < 0$, the roots are not real.

MULTIPLE CHOICE EXERCISE A

TYPE I

1. The roots of the equation $x^2 - 3x + 2 = 0$ are

A 2, 1 **B** −2, −1 **C** −3, 2 **D** 0, $\frac{2}{3}$ **E** not real

2. The coefficient of xy in the expansion of $(x - 3y)(2x + y)$ is

A 1 **B** 6 **C** 5 **D** 0 **E** −5

3. The value of $\log_5 0.04$ is

A 4 **B** 5 **C** $\frac{1}{2}$ **D** −2 **E** 0.25

4. $\dfrac{1 - \sqrt{2}}{1 + \sqrt{2}}$ is equal to

A 1 **B** −1 **C** $3 - \sqrt{2}$ **D** $1 - \frac{2}{3}\sqrt{2}$ **E** $2\sqrt{2} - 3$

5. Expanding $(1 + \sqrt{2})^3$ gives

A $3 + 3\sqrt{2}$ **B** $7 + 5\sqrt{2}$ **C** $1 + 3\sqrt{2}$ **D** $3 + \sqrt{6}$ **E** $1 + 2\sqrt{2}$

6. The fraction $\dfrac{1}{(x+1)(x-1)}$ can be expressed as

A $\dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$ **B** $\dfrac{1}{x+1} + \dfrac{1}{x-1}$ **C** $\dfrac{2}{x+1} - \dfrac{2}{x-1}$ **D** $\dfrac{1}{x^2} - \dfrac{1}{1}$ **E** $\dfrac{1}{x+1} - \dfrac{1}{x-1}$

7. If $\dfrac{x+p}{(x-1)(x-3)} \equiv \dfrac{q}{x-1} + \dfrac{2}{x-3}$, the values of p and q are

A $p = -2, q = 1$ **B** $p = 2, q = 1$ **C** $p = 1, q = -2$ **D** $p = 1, q = 1$ **E** $p = 1, q = -1$

8. If $x^2 + px + 6 = 0$ has equal roots and $p > 0$, p is

A $\sqrt{48}$ **B** 0 **C** $\sqrt{6}$ **D** 3 **E** $\sqrt{24}$

9. If $x^2 + 4x + p \equiv (x + q)^2 + 1$, the values of p and q are

A $p = 5, q = 2$ **B** $p = 1, q = 2$ **C** $p = 2, q = 5$ **D** $p = -1, q = 5$ **E** $p = 0, q = -1$

10. $\dfrac{p^{-1/2} \times p^{3/4}}{p^{-1/4}}$ simplifies to

A 1 **B** $p^{-1/2}$ **C** $p^{3/4}$ **D** p **E** $p^{1/2}$

11. In the expansion of $(a - 2b)^3$ the coefficient of b^2 is

A $-2a^2$ **B** $-8a$ **C** $12a$ **D** $-4a$ **E** −12

12. If $\log_x y = 2$ then

A $x = 2y$ **B** $x = y^2$ **C** $x^2 = y$ **D** $y = 2x$ **E** $y = \sqrt{x}$

13. $\dfrac{2}{(x+1)(x-1)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-1}$ corresponds to

A $A = 1, B = 1$
B $A = -1, B = 1$
C $A = x, B = 1$
D $A = 0, B = 2$
E $A = x - 1, B = x + 1$

14. $\log 5 - 2\log 2 + \frac{3}{2}\log 16$ is equal to

A $\log 80$
B 10
C 0
D $2\log 12$
E 1

TYPE II

15. When $(3 - 5x)^4$ is expanded

A the coefficient of x^4 is 1
B the coefficient of x is -540
C there are four terms after all simplification.

16. $f(x) = \dfrac{2}{(x+1)(x-1)}$

A $f(x) = 0$ has two real roots
B $f(x) = \dfrac{1}{x-1} - \dfrac{1}{x+1}$
C $f(x) = \dfrac{1}{x^2}$

17. $f(x) = x^2 - 2x + 2$

A $f(x) = (x-1)^2 + 1$
B $f(1) = 0$
C $f(x) = 0$ has equal roots.

18. $\frac{1}{2}\log 16 - 1$

A can be expressed as a single logarithm
B has an exact decimal value
C is equal to $\log 7$

19. $f(x) = 2x^2 + 3x - 2$

A $f(x)$ can be expressed as the sum of two partial fractions
B the equation $f(x) = 0$ has two real distinct roots
C $x + 2$ is a factor of $f(x)$

20. $\dfrac{2\sqrt{3}-2}{2\sqrt{3}+2}$

A can be expressed as a fraction with a rational denominator
B is an irrational number
C is equal to -1

TYPE III

21. If $x - a$ is a factor of $x^2 + px + q$, the equation $x^2 + px + q = 0$ has a root equal to a

22. $3\log x + 1 = \log 10x^3$ is an equation.

23. In the expansion of $(1 + x)^6$ the coefficient of x is 6

24. Values of A and B can be found such that
$$\frac{x}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$$

MISCELLANEOUS EXERCISE A

1. Express $3x^2 + 6x - 4$ in the form $a[(x+b)^2 + c]$

2. Find the coefficient of x^3 in the expansion of $(3 - 2x)^4$

3. Find the values of A and B for which
$$\frac{x-2}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

4. Find the value of a) $\log_3 3\sqrt{3}$ b) $\log_{25} 125\sqrt{5}$

5. Find the value of a for which $x - 1$ is a factor of $2x^2 - 3x + a$

6. Find the value of k for which the equation $x^2 - 9x + k$ has equal roots.

7. Find the values of p and q for which
$$x^2 - 4x + p \equiv (x - q)^2 + 4$$

8. Given that k is a real constant such that $0 < k < 1$, show that the roots of the equation
$$kx^2 + 2x + (1 - k) = 0$$
are (a) always real
(b) always negative. (U of L 88)

CHAPTER 10

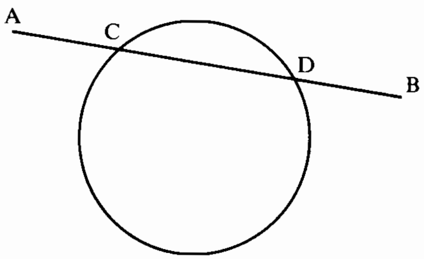
CIRCLE GEOMETRY

PARTS OF A CIRCLE

We start this chapter with a reminder of the language used to describe parts of a circle.

Part of the circumference is called an *arc*.
If the arc is less than half the circumference it is called a *minor arc*; if it is greater than half the circumference it is called a *major arc*.

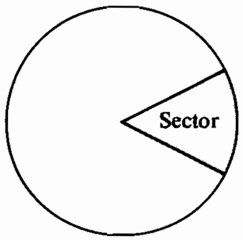
A straight line which cuts a circle in two distinct points is called a *secant*. The part of the line inside the circle is called a *chord*.

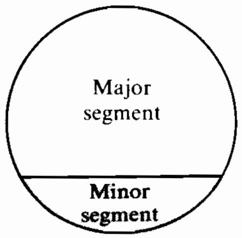


AB is a secant,
CD is a chord.

The area enclosed by two radii and an arc is called a *sector*.

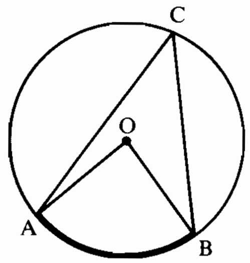
The area enclosed by a chord and an arc is called a *segment*. If the segment is less than half a circle it is called a *minor segment*; if it is greater than half a circle it is called a *major segment*.





THE ANGLE SUBTENDED BY AN ARC

Consider the points A, B and C on the circumference of a circle whose centre is O.



We say that $\angle ACB$ stands on the minor arc AB.

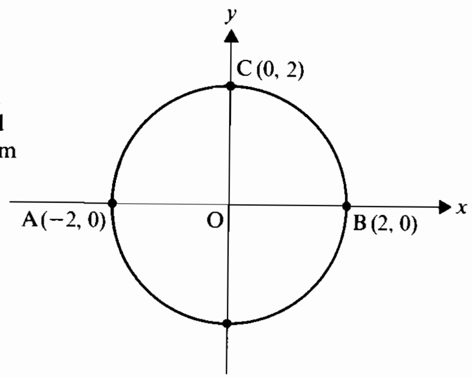
The minor arc AB is said to *subtend* the angle ACB at the circumference (and the angle is *subtended* by the arc).

In the same way, the arc AB is said to subtend the angle AOB at the centre of the circle.

Example 10a

A circle of radius 2 units which has its centre at the origin, cuts the x-axis at the points A and B and cuts the y-axis at the point C. Prove that $\angle ACB = 90°$

All the information given in the question, and gleaned from the known properties of the figure, can be marked in the diagram as shown. The diagram can then be referred to as justification for steps taken in the solution.



From the diagram, the gradient of AC is $\dfrac{2-0}{0-(-2)} = 1$

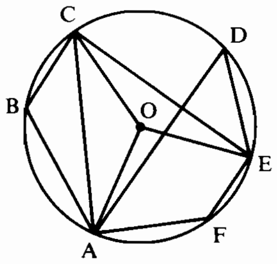
and the gradient of BC is $\dfrac{2-0}{0-2} = -1$

$\therefore$ (gradient of AC) × (gradient of BC) $= -1$

i.e. AC is perpendicular to BC $\Rightarrow$ $\angle ACB = 90°$

EXERCISE 10a

1\.



Name the angles subtended

(a) at the circumference by the minor arc AE

(b) at the circumference by the major arc AE

(c) at the centre by the minor arc AC

(d) at the circumference by the major arc AC

(e) at the centre by the minor arc CE

(f) at the circumference by the minor arc CD

(g) at the circumference by the minor arc BC.

2\. AB is a chord of a circle, centre O, and M is its midpoint. The radius from O is drawn through M. Prove that OM is perpendicular to AB.

3\. C(5, 3) is the centre of a circle of radius 5 units.

(a) Show that this circle cuts the x-axis at A(1, 0) and B(9, 0)

(b) Prove that the radius that is perpendicular to AB goes through the midpoint of AB.

(c) Find the angle subtended at C by the minor arc AB.

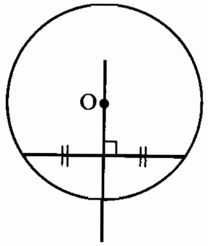
(d) The point D is on the major arc AB and DC is perpendicular to AB. Find the coordinates of D and hence find the angle subtended at D by the minor arc AB.

4\. A and B are two points on the circumference of a circle centre O. C is a point on the major arc AB. Draw the lines AC, BC, AO, BO and CO, extending the last line to a point D inside the sector AOB. Prove that $\angle$AOD is twice $\angle$ACO and that $\angle$BOD is twice angle $\angle$BCO. Hence show that the angle subtended by the minor arc AB at the centre of the circle is twice the angle that it subtends at the circumference of the circle.

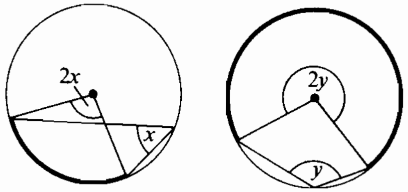
ANGLES IN A CIRCLE

The solutions to questions in the last exercise illustrate two important results.

1) The perpendicular bisector of a chord of a circle goes through the centre of the circle.

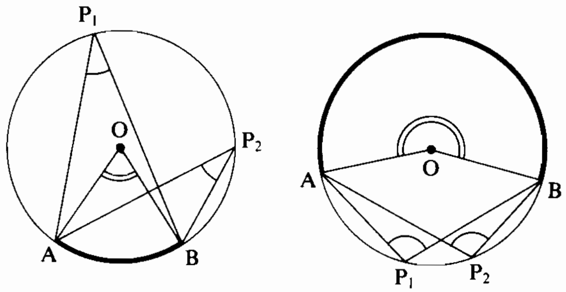


2) The angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference by the same arc.



Further important results follow from the last fact.

3)



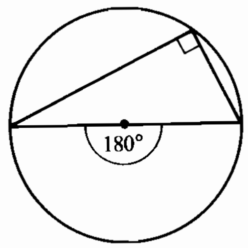
In both diagrams, $\angle AOB = 2\angle P_1 = 2\angle P_2$

So it follows that $\angle P_1 = \angle P_2$

i.e.

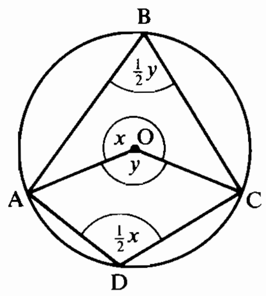
all angles subtended at the circumference by the same arc are equal.

4) A semicircle subtends an angle of 180° at the centre of the circle; therefore it subtends an angle half that size, i.e. 90°, at any point on the circumference. This angle is called the angle in a semicircle.



Hence the angle in a semicircle is 90°

5) If all four vertices of a quadrilateral ABCD lie on the circumference of a circle, ABCD is called a *cyclic quadrilateral*.



In the diagram, O is the centre of the circle.

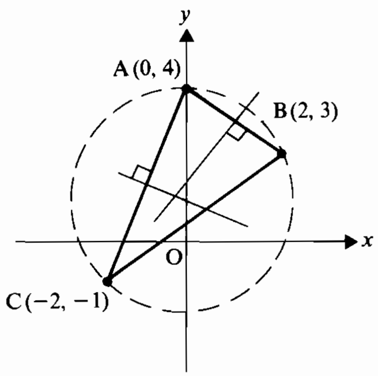
$\therefore$ $\angle ADC = \frac{1}{2}x$ and $\angle ABC = \frac{1}{2}y$

But $x + y = 360°$, therefore $\angle ADC + \angle ABC = 180°$

i.e. the opposite angles of a cyclic quadrilateral are supplementary

Example 10b

A circle circumscribes a triangle whose vertices are at the points A(0, 4), B(2, 3) and C(−2, −1). Find the centre of the circle.



When a circle circumscribes a figure, the vertices of the figure lie on the circumference of the circle. The centre of the circle can be found by locating the point of intersection of the perpendicular bisectors of two chords.

The gradient of AC is $\dfrac{4-(-1)}{0-(-2)} = \dfrac{5}{2}$

and its midpoint is $\left[\dfrac{0-2}{2}, \dfrac{4-1}{2}\right] \Rightarrow (-1, \frac{3}{2})$,

$\therefore$ the gradient of the perpendicular bisector of AC is $-\frac{2}{5}$ and its equation is

$$y = -\tfrac{2}{5}x + \tfrac{11}{10} \Rightarrow 4x + 10y - 11 = 0 \qquad [1]$$

Similarly the gradient of AB is $-\frac{1}{2}$ and its midpoint is $(1, \frac{7}{2})$

$\therefore$ the gradient of the perpendicular bisector of AB is 2 and its equation is

$$y = 2x + \tfrac{3}{2} \Rightarrow 4x - 2y + 3 = 0 \qquad [2]$$

Solving equations [1] and [2] simultaneously gives

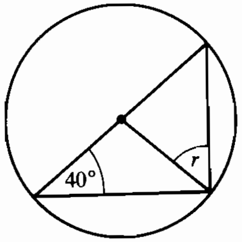
$$12y - 14 = 0 \Rightarrow y = \tfrac{7}{6} \text{ and } x = -\tfrac{1}{6}$$

Therefore the centre of the circle is the point $(-\frac{1}{6}, \frac{7}{6})$.

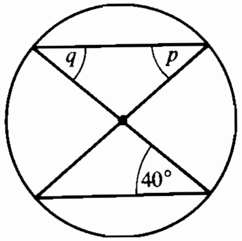
EXERCISE 10b

1. Find the size of each marked angle.

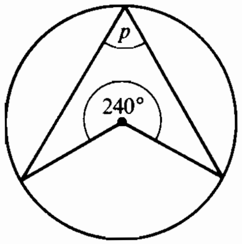
(a)



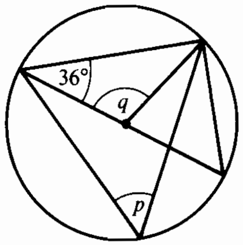
(b)



(c)



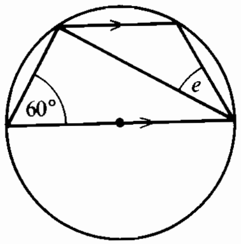
(d)



2. AB is a diameter of a circle centre O. C is a point on the circumference. D is a point on AC such that OD bisects ∠AOC. Prove that OD is parallel to BC.

3. A triangle has its vertices at the points A(1, 3), B(5, 1) and C(7, 5). Prove that △ABC is right-angled and hence find the coordinates of the centre of the circumcircle of △ABC.

4. Find the size of the angle marked e in the diagram.



5. AB and CD are two chords of a circle that cut at E. (E is not the centre of the circle.) Show that △s ACE and BDE are similar.

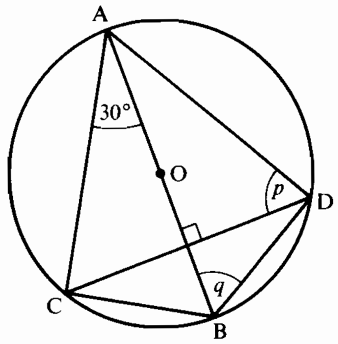
6. A circle with centre O circumscribes an equilateral triangle ABC. The radius drawn through O and the midpoint of AB meets the circumference at D. Prove that △ADO is equilateral.

7. The line joining A(5, 3) and B(4, −2) is a diameter of a circle. If P(a, b) is a point on the circumference find a relationship between a and b

8. ABCD is a cyclic quadrilateral. The side CD is produced to a point E outside the circle. Show that ∠ABC = ∠ADE.

9. A triangle has its vertices at the points A(1, 3), B(−2, 5) and C(4, −2). Find the coordinates of the centre, and correct to 3 s.f., the radius of the circle that circumscribes △ABC.

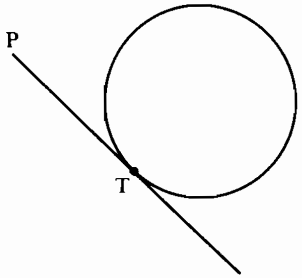
10. In the diagram, O is the centre of the circle and CD is perpendicular to AB. If ∠CAB = 30° find the size of each marked angle.



TANGENTS TO CIRCLES

If a line and a circle are drawn on a plane then there are three possibilities for the position of the line in relation to the circle. The line can miss the circle, or it can cut the circle in two distinct points, or it can touch the circle at one point. In the last case the line is called a *tangent* and the point at which it touches the circle is called the *point of contact*.

The length of a tangent drawn from a point to a circle is the distance from that point to the point of contact.



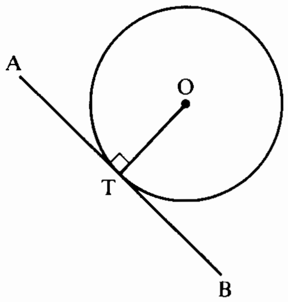
T is the point of contact.

PT is the length of the tangent from P.

Properties of Tangents to Circles

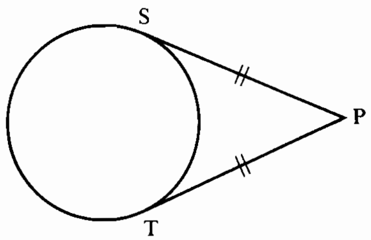
There are two important and useful properties of tangents to circles.

A tangent to a circle is perpendicular to the radius drawn through the point of contact,
i.e. AB is perpendicular to OT.



The two tangents drawn from an external point to a circle are equal in length,

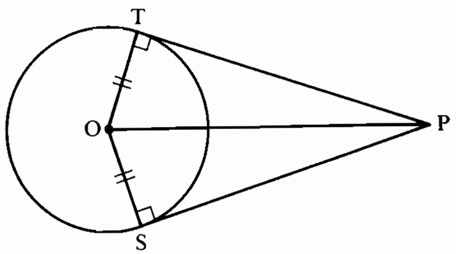
i.e. PS = PT



The second property is proved in the following worked example.

Examples 10c

1. PS and PT are two tangents drawn from a point P to a circle whose centre is O. Prove that PT = PS.



In Δs OTP, OSP $\quad \angle T = \angle S = 90°$

OS = OT (radii)

and OP is common

$\therefore$ Δs OTP, OSP are congruent.

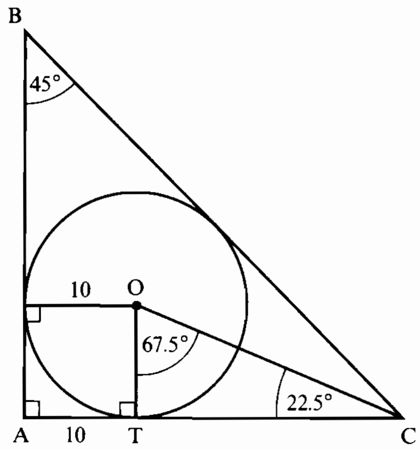
Hence PT = PS.

Another useful property follows from the last example, namely

when two tangents are drawn from a point to a circle, the line joining that point to the centre of the circle bisects the angle between the tangents.

---

2. A circle of radius 10 units is circumscribed by a right-angled isosceles triangle. Find the lengths of the sides of the triangle.



A circle is *circumscribed* by a figure when all the sides of the figure touch the circle. Note also that the circle is inscribed in the figure.

From the diagram

in ΔOTC, $\quad TC = 10 \tan 67.5°$

$= 24.14$

$AT = 10$

$\therefore \quad AC = 34.14 = AB$

in ΔABC, $\quad BC = \sqrt{(34.14^2 + 34.14^2)}$ (Pythagoras)

$= 48.28$

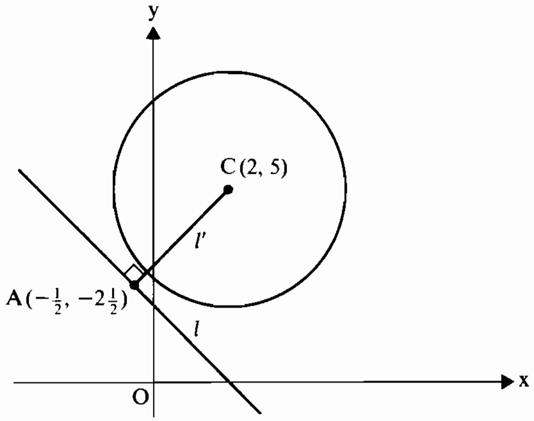
$\therefore$ correct to 3 s.f. the lengths of the sides of the triangle are 34.1 units, 34.1 units and 48.3 units.

---

3. The centre of a circle of radius 3 units is the point C(2, 5). The equation of a line, l, is $x + y - 2 = 0$

   (a) Find the equation of the line through C, perpendicular to l

   (b) Find the distance of C from l and hence determine whether l is a tangent to the circle.



(a) The line l' is perpendicular to $x + y - 2 = 0$

so its equation is $x - y + k = 0$

The point (2, 5) lies on l'

$\therefore \quad 2 - 5 + k = 0 \quad \Rightarrow \quad k = 3$

$\therefore$ the equation of l' is $x - y + 3 = 0$

(b) To find the distance of C from l we need the coordinates of A, the point of intersection of l and l'.

Adding the equations of l and l' gives $2x + 1 = 0$

$\Rightarrow \quad x = -\frac{1}{2}$ and $y = \frac{5}{2}$

so A is the point $(-\frac{1}{2}, \frac{5}{2})$

$\therefore$ CA $= \sqrt{[\{2 - (-\frac{1}{2})\}^2 + \{5 - \frac{5}{2}\}^2]} = 3.54$ to 3 s.f.

For the line to be a tangent, CA would have to be 3 units exactly (i.e. equal to the radius).

CA > 3, therefore l is not a tangent.

EXERCISE 10c

1. The two tangents from a point to a circle of radius 12 units are each of length 20 units. Find the angle between the tangents.

2. Two circles with centres C and O have radii 6 units and 3 units respectively and the distance between O and C is less than 9 units. AB is a tangent to both circles, touching the larger circle at A and the smaller circle at B, where AB is of length 4 units. Find the length of OC.

3. The two tangents from a point A to a circle touch the circle at S and at T. Find the angle between one of the tangents and the chord ST given that the radius of the circle is 5 units and that A is 13 units from the centre of the circle.

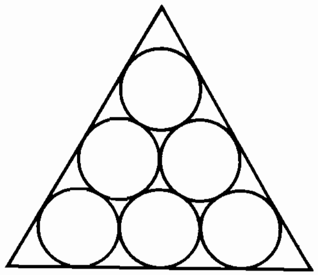
4. An equilateral triangle of side 25 cm circumscribes a circle. Find the radius of the circle.

5. AB is a diameter of the circle and C is a point on the circumference. The tangent to the circle at A makes an angle of 30° with the chord AC. Find the angles in $\triangle$ABC.

6. The centre of a circle is at the point C(4, 8) and its radius is 3 units. Find the length of the tangents from the origin to the circle.

7. A circle touches the y-axis at the origin and goes through the point A(8, 0). The point C is on the circumference. Find the greatest possible area of $\triangle$OAC.

8. A triangular frame is made to enclose six identical spheres as shown. Each sphere has a radius of 2 cm. Find the lengths of the sides of the frame.



9. Given that $\log(2x - 4) + \log 3 = 3\log y$ find an expression for x in terms of y

10. Express $\dfrac{3}{x(x+1)}$ in partial fractions.

11. Find the values of p and q for which

$$2x^2 + px + 3 = 2[(x-1)^2 + q]$$

Hence show that there are no real values of x for which $2[(x-1)^2 + q] = 0$

12. Find the values of a and b for which

$$(2x - a)^3 = 8x^3 + bx^2 + 6a^2x - 27$$

13. Find the values of x and y which satisfy the equations

$$2\log x = \log y + \log 3$$
$$x + y = 1$$

14. Find the value of c for which $x = 2$ is a root of the equation $3x^2 - 4x + c = 0$

CHAPTER 5

STRAIGHT LINE GEOMETRY

PROOF

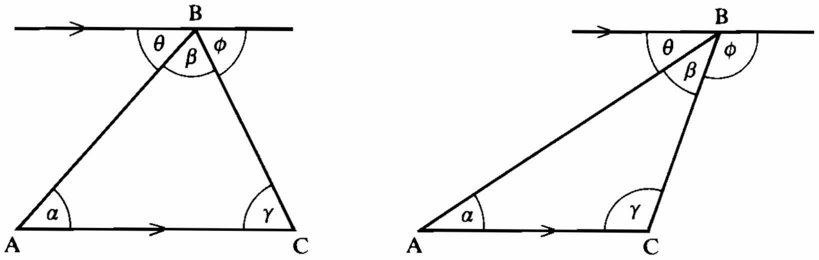
This chapter contains some geometric facts and definitions that will be needed later in this course.

Up to now, many rules have been based on investigating a few particular cases. For example the reader may have accepted that the sum of the interior angles in any triangle is 180°, only because the measured angles in some specific triangles had this property.
This fact may be reinforced by our not being able to find a triangle whose angles have a different sum but that does not rule out the possibility that such a triangle exists.

It is no longer satisfactory to assume that a fact is *always* true without *proving that it is*, because as a mathematics course progresses, one fact is often used to produce another. Hence it is very important to distinguish between a 'fact' that is assumed from a few particular cases and one that has been *proved* to be true, as results deduced from an assumption cannot be reliable.

A proof deals with a general case, e.g. a triangle in which the sides and angles are not specified. The formal statement of a proved result is called a *theorem*.

PROOF THAT THE ANGLES OF A TRIANGLE ADD UP TO 180°



Let ABC be any triangle.

Draw a straight line through B parallel to AC.

Using the notation on the diagram, and the fact that alternate angles are equal, we have

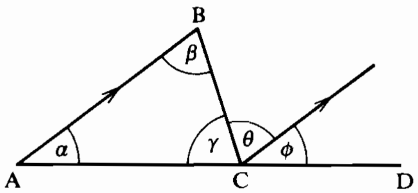
$$\alpha = \theta \quad \text{and} \quad \gamma = \phi$$

At B, $\theta + \beta + \phi = 180°$ (angles on a straight line)

$\therefore$ $\alpha + \beta + \gamma = 180°$

We have proved the general case and hence can now be certain that, for *all* triangles *the angles of a triangle add up to 180°*

Proof that in any triangle, an exterior angle is equal to the sum of the two interior opposite angles



Let ABC be any triangle, with AC produced to D.

Draw a line through C parallel to AB.

Using the notation on the diagram,

$\theta = \beta$ (alternate $\angle$s) and $\phi = \alpha$ (corresponding $\angle$s)

$\therefore$ $\theta + \phi = \beta + \alpha$

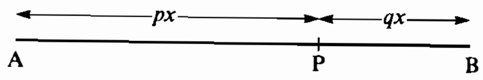
i.e. $\angle BCD = \angle BAC + \angle CBA$

DEFINITIONS

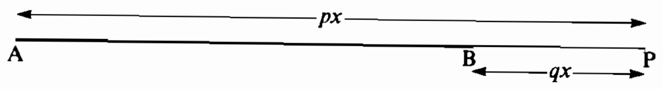
Division of a Line in a Given Ratio

A point P is said to divide a line AB *internally* if P is between A and B.
Further, if P divides AB internally in the ratio $p:q$, then

$$AP:PB = p:q$$



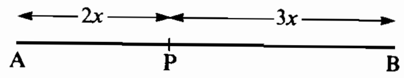
If a point P is on AB produced, or on BA produced, then P is said to divide a line AB *externally*.



Further, if P divides AB externally in the ratio $p:q$ then $AP:PB = p:q$

Examples 5a

1. A line AB of length 15 cm is divided internally by P in the ratio 2:3. Find the length of PB.



As P divides AB internally in the ratio 2:3, P is between A and B and is nearer to A than to B.

AB is divided into 5 portions of which PB is 3 portions. If one portion is x cm, then

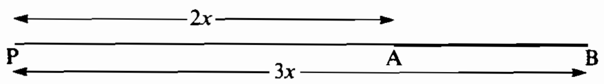
$$AB = 5x = 15$$

$\Rightarrow$ $x = 3$

$\therefore$ $PB = 3x = 9$

i.e. PB is 9 cm long.

2. A line AB of length 15 cm is divided externally by P in the ratio 2:3. Find the length of PB.



As AP : PB = 2:3, P is nearer to A than to B, so P is on BA produced.

$$AB = 3x - 2x = 15$$
$$\Rightarrow \quad x = 15$$
$$\therefore \quad PB = 3x = 45$$

i.e. PB is 45 cm long.

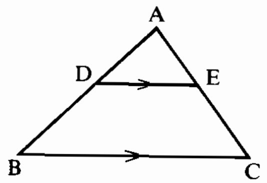
EXERCISE 5a

1. A line AB of length 12 cm is divided internally by P in the ratio 1:5. Find the length of AP.
2. A line AB of length 18 cm is divided externally by P in the ratio 5:6. Find the length of AP.
3. The point D divides a line PQ internally in the ratio 3:4. If PQ is of length 35 cm, find the length of DQ.
4. A line LM is divided externally by a point T in the ratio 5:7 Find the length of MT if (a) LM = 24 cm (b) LM = $2x$ cm.
5. P is a point on a line AB of length 12 cm and AP = 5 cm. Find the ratio in which P divides AB.
6. AB is a line of length 16 cm and M is a point on AB produced such that AM = 24 cm. Find the ratio in which M divides AB.
7. AB is a line of length x units and P is a point on AB such that AP is of length y units. Find, in terms of x and y, the ratio in which P divides AB.
8. PQ is a line of length a units. It is divided externally in the ratio $n:m$ by a point L, where $n > m$. Find, in terms of a, n and m, the length of QL.
9. ST is a line of length a units and L is a point on ST produced such that TL is of length b units. Find, in terms of a and b, the ratio in which L divides ST.

THE INTERCEPT THEOREM

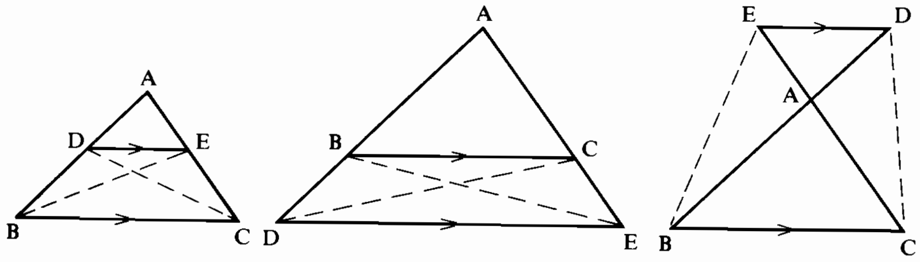
A straight line drawn parallel to one side of a triangle divides the other two sides in the same ratio.

i.e.



if DE is parallel to BC then AD/DB = AE/EC

Proof



Triangle ABC is any triangle and DE is parallel to BC.

Three different positions for DE must be considered, as shown in the diagrams. The proof that follows applies to all three cases. This proof uses the fact that the area of a triangle can be found by multiplying half the base by the height.

Area $\triangle DEC$ = area $\triangle DEB$

($\triangle$s have same base DE and equal heights)

$\therefore$ area $\triangle AED$: area $\triangle DEB$ = area $\triangle AED$: area $\triangle DEC$

Now area $\triangle AED$: area $\triangle DEB$ = AD : DB

($\triangle$s have the same heights)

and area $\triangle AED$: area $\triangle DEC$ = AE : EC

($\triangle$s have the same heights)

Therefore AD : DB = AE : EC

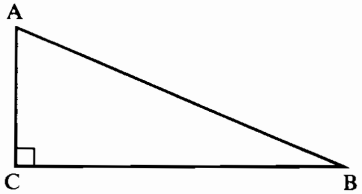
The converse of this theorem is also true, i.e. if a line divides two sides of a triangle in the same ratio, then it is parallel to the third side of the triangle.

PYTHAGORAS' THEOREM AND ITS CONVERSE

Pythagoras' theorem is familiar and very useful. Here is a reminder.

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

i.e.



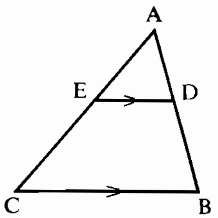
$$AB^2 = AC^2 + BC^2$$

The converse of this theorem is less well known, but equally useful. It states that

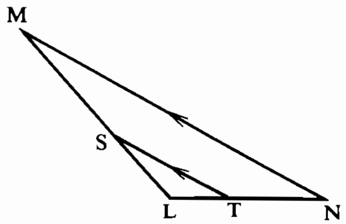
if the square on one side of a triangle is equal to the sum of the squares on the other two sides, then the angle opposite the first side is a right-angle.

EXERCISE 5b

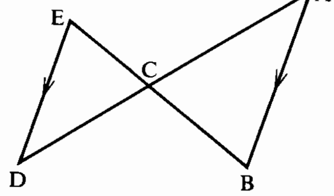
1. In triangle ABC, DE is parallel to BC.
If AD = 2 cm, DB = 3 cm and AE = 2.5 cm, find EC.



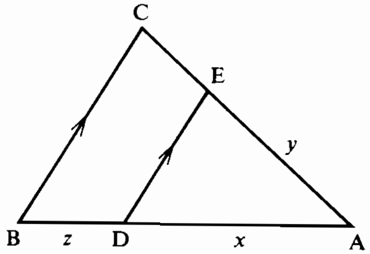
2. In triangle LMN, ST is parallel to MN.
If LM = 9 cm, LS = 4 cm and LT = 2 cm, find
(a) TN (b) LN.



3. In the diagram, ED is parallel to AB.
AC = 3 cm, CE = 1.5 cm and CD = 2 cm. Find BC.



4. Using the measurements given in the diagram find the length of CE in terms of x, y and z



5. ABC is a triangle and a line PQ is drawn parallel to BC, but on the opposite side of A from BC. PQ cuts BA produced at P and cuts CA produced at Q.
AC = 2.5 cm, AQ = 1 cm and AP = 0.5 cm. Find
(a) the length of AB (b) the ratio in which P divides AB.

6. Determine whether or not a triangle is right-angled if the lengths of its sides are
(a) 3, 5, 4 (b) 2, 1, $\sqrt{3}$ (c) 2, 2, $\sqrt{8}$
(d) $5x$, $12x$, $13x$ (e) 7, 5, 12 (f) $\sqrt{2}$, $\sqrt{3}$, 1

SIMILAR TRIANGLES

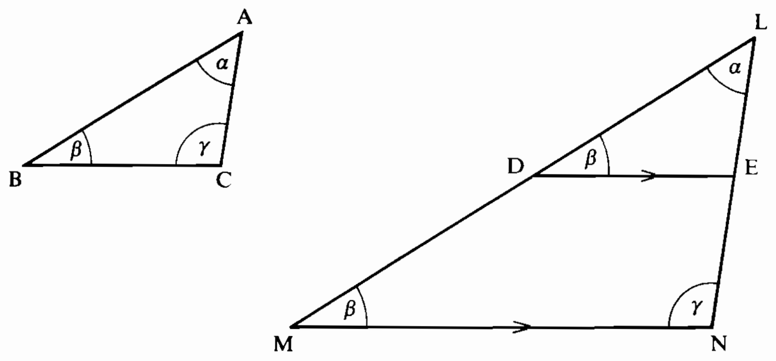
If one triangle is an enlargement of another triangle, then the two triangles are *similar*.

This means that the three angles of one triangle are equal to the three angles of the other triangle *and* that the corresponding sides of the two triangles are in the same ratio.

However, to prove that triangles are similar it is necessary only to show that *one* of these conditions is satisfied because the other one follows, i.e.

if two triangles contain the same angles then their corresponding sides are in the same ratio.

Proof



The angles of triangle ABC are equal to the angles of triangle LMN.

It follows that we can find a point D on LM such that LD = AB and a point E on LN such that LE = AC.

Joining DE, it is clear that $\triangle LDE$ and $\triangle ABC$ are identical, i.e. congruent.

$\therefore \quad \angle ABC = \angle LDE$

These are corresponding angles with respect to DE and MN.

$\therefore$ DE is parallel to MN

The intercept theorem then tells us that DE divides LM and LN in the same ratio, i.e.

$$LD:LM = LE:LN$$

$$\Rightarrow \qquad AB:LM = AC:LN$$

The converse of this theorem is also true, i.e.

if two triangles are such that their corresponding sides are in the same ratio, then corresponding pairs of angles are equal.

It is left to the reader to prove this in Question 11 in the next exercise.

SIMILAR FIGURES

Two figures are similar if one figure is an enlargement of the other. This means that their corresponding sides are in the same ratio and that the angles in one figure are equal to the corresponding angles in the other figure.

To show that figures other than triangles are similar, both the side and the angle property have to be proved. In the case of triangles, we have seen that it is necessary only to show that one of these conditions is satisfied to prove the triangles similar, because the other condition follows, i.e.

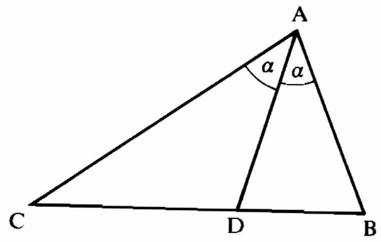
two triangles are similar if we can show
either that the angles of the triangles are equal
or that the corresponding sides of the triangles are in the same ratio

THE ANGLE BISECTOR THEOREM

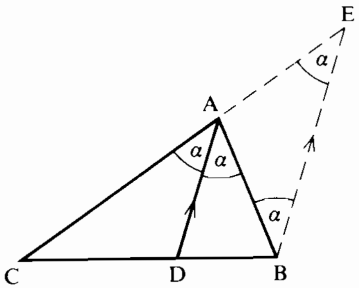
Another useful fact concerning triangles and ratios is

the line bisecting an angle of a triangle divides the side opposite to that angle in the ratio of the sides containing the angle.

e.g. if AD bisects $\angle A$, then $BD:DC = AB:AC$



Proof



In $\triangle ABC$, AD bisects the angle at A.

Drawing BE parallel to DA to cut CA produced at E, we have

$$\angle BEA = \angle DAC \quad \text{(corresponding angles)}$$

and $$\angle EBA = \angle BAD \quad \text{(alternate angles)}$$

$\therefore$ $\triangle BEA$ is isosceles $\Rightarrow$ $EA = AB$

In $\triangle BCE$ the intercept theorem gives

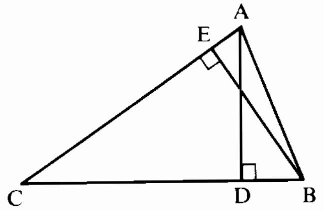
$$BD:DC = EA:AC$$

$\Rightarrow$ $$BD:DC = AB:AC$$

ALTITUDES AND MEDIANS

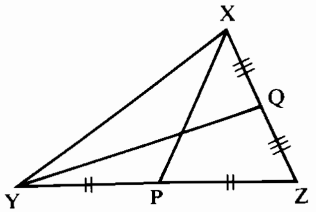
We end this chapter with a couple of definitions.
The line drawn from a vertex of a triangle, perpendicular to the opposite side, is called an *altitude*, for example

AD is the altitude through A,
and BE is the altitude through B.



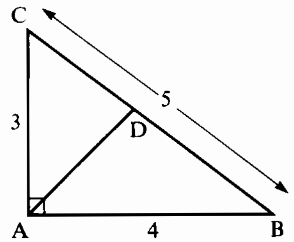
A *median* of a triangle is the line joining a vertex to the midpoint of the opposite side, for example

XP is the median through X,
and YQ is the median through Y.



Example 5c

In $\triangle ABC$, $AB = 4$ cm, $AC = 3$ cm and $\angle A = 90°$. The bisector of $\angle A$ cuts BC at D. Find the length of BD.



From Pythagoras' theorem, $BC = 5$ cm.

From the angle bisector theorem, $BD:DC = AB:AC = 4:3$

$\therefore$ $$BD:BC = 4:7$$

$\Rightarrow$ $$BD = \tfrac{4}{7} \times 5$$
$$= 2\tfrac{6}{7}$$

$\therefore$ BD is $2\frac{6}{7}$ cm long.

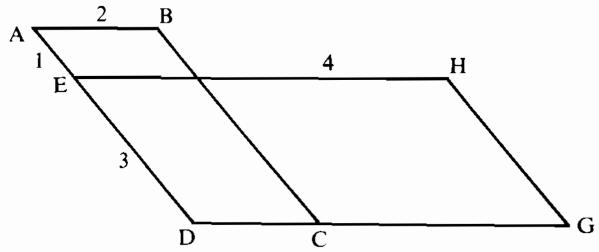
EXERCISE 5c

1. XYZ is a triangle with a right angle at X, XW is the altitude from X. Show that triangles XYZ, XWZ and XYW are all similar.

2. In $\triangle ABC$, $AB = 6$ cm, $AC = 5$ cm and $BC = 7$ cm. BD is the median from B to AC and BE is the bisector of $\angle B$ to AC. Find the length of DE.

3. Triangles PQR and XYZ are such that $\angle P = \angle X$ and $\angle Q = \angle Z$. $XY = 3$ cm, $YZ = 4$ cm, $PQ = 7$ cm and $PR = 12$ cm. Find the lengths of XZ and QR.

4.



ABCD and EDGH are parallelograms. Prove that they are similar.

5. PQR is a triangle in which PQ = 4 cm, PR = 3 cm and QR = 6 cm. T is a point outside the triangle on the side of QR, and ∠RQT = ∠PRQ and ∠QRT = ∠QPR. Find the lengths of QT and RT.

6. ABC is any triangle and equilateral triangles ABD and ACE are drawn on the sides AB and AC respectively. The bisector of ∠BAC meets BC at F such that BF : FC = 3 : 2. Find the ratio of the areas of the two equilateral triangles.

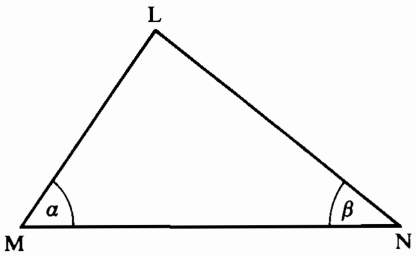
7. D and E are two points on the side BC of △ABC such that AD is the bisector of ∠BAC and AE is an altitude of the triangle. If ∠ACB = 40° and ∠ABC = 60° find ∠DAE.

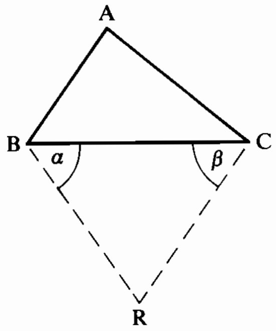
8. In triangle ABC, AB = 24 cm, BC = 7 cm and AC = 25 cm. Show that △ABC is right-angled. The bisector of ∠BAC meets BC at D. Find the lengths of BD, DC and AD.

9. AB and CD are two lines that intersect at E. AC and DB are parallel. Show that triangles ACE and EDB are similar.

10. Triangle ABC has a right-angle at B and BE is an altitude of the triangle. BC = 5 cm and BE = 4 cm. Calculate the length of EC and of AC.

11. (Proof that if the corresponding sides of two triangles are in the same ratio, then the triangles contain the same angles.)





Triangles LMN and ABC are such that
LM : AB = MN : BC = LN : AC.
R is a point such that ∠CBR = ∠LMN and ∠BCR = ∠LNM.
Prove that LM : BR = LM : AB and hence that BR = AB.
Similarly show that CR = AC. *Hence* show that △LMN and △ABC have equal angles.

CHAPTER 6

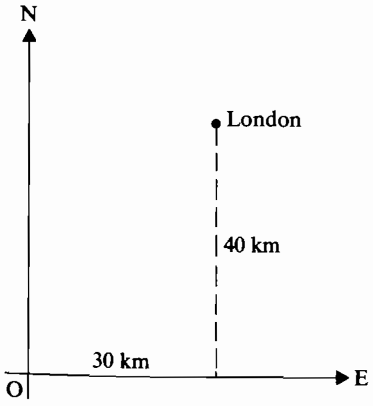
COORDINATE GEOMETRY

LOCATION OF A POINT IN A PLANE

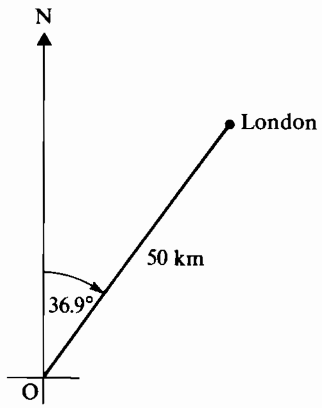
Graphical methods lend themselves particularly well to the investigation of the geometrical properties of many curves and surfaces. At this stage we will restrict ourselves to rectilinear plane figures (i.e. two dimensional figures bounded by straight lines). To begin, we need a simple and unambiguous way of describing the position of a point on a graph.

Consider the problem of describing the location of a city, London say.

There are many ways in which this can be done, but they all require reference to at least one known place and known directions. This is called a *system* or *frame of reference*. Within this frame of reference, two measurements are needed to locate the city precisely. These measurements are called coordinates.



[i]



[ii]

The position of London is described in two alternative ways in the diagrams above.

In [i] the frame of reference comprises a fixed point O and the directions due east and due north from O. The coordinates of London are 30 km east of O and 40 km north of O.

In [ii] the frame of reference comprises a fixed point O and the direction due north from O. The coordinates of London are 50 km from O and a bearing of 036.9°

The system used at this level for graphical work is based on the first of the two practical systems described above.

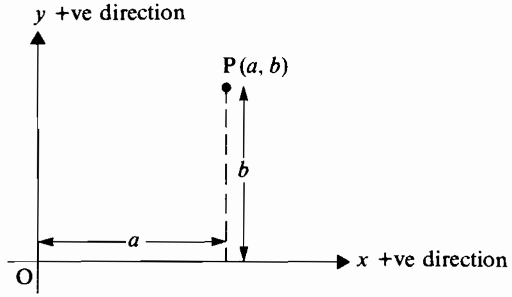
CARTESIAN COORDINATES

This system of reference uses a fixed point O, called *the origin*, and a pair of perpendicular lines through O. One of these lines is drawn horizontally and is called the x-axis. The other line is drawn vertically and is called the y-axis.

The coordinates of a point P are the directed distances of P from O parallel to the axes.

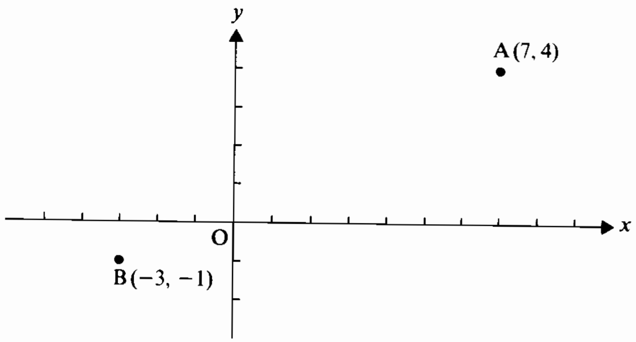
A positive coordinate is a distance measured in the positive direction of the axis and a negative coordinate is a distance in the opposite direction.

The coordinates are given as an ordered pair (a, b) with the x-*coordinate* or *abscissa* first and the y-*coordinate* or *ordinate* second.



The diagram opposite represents the points whose Cartesian coordinates are $(7, 4)$ and $(-3, -1)$

These points are referred to in future simply as the points $(7, 4)$ and $(-3, -1)$



EXERCISE 6a

1. Represent on a diagram the points whose coordinates are
   (a) $(1, 6)$ (b) $(0, 5)$ (c) $(-4, 0)$ (d) $(-3, -2)$ (e) $(3, -4)$

2. Two adjacent corners of a square are the points $(3, 5)$ and $(3, -1)$. What could the coordinates of the other two corners be?

3. The two opposite corners of a square are $(-2, -3)$ and $(3, 2)$. Write down the coordinates of the other two corners.

COORDINATE GEOMETRY

Coordinate geometry is the name given to the graphical analysis of geometric properties. For this analysis we need to refer to three types of points:

1) fixed points whose coordinates are known, e.g. the point $(1, 2)$

2) fixed points whose coordinates are not known numerically. These points are referred to as (x_1, y_1), (x_2, y_2), ... etc. or (a, b), etc.

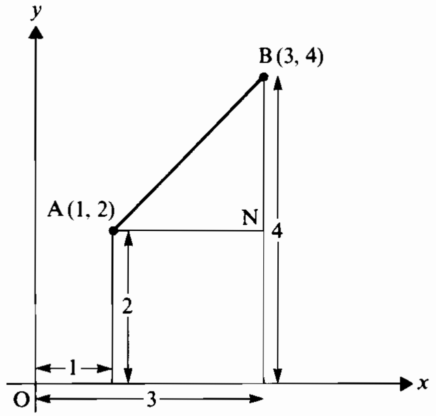
3) points which are not fixed. We call these general points and we refer to them as (x, y), (X, Y), etc.

It is conventional to use the letters A, B, C, ... for fixed points and the letters P, Q, R, ... for general points.

It is also conventional to graduate the axes using identical scales. This avoids distorting the shape of figures.

THE LENGTH OF A LINE JOINING TWO POINTS

Consider the line joining the points A(1, 2) and B(3, 4)

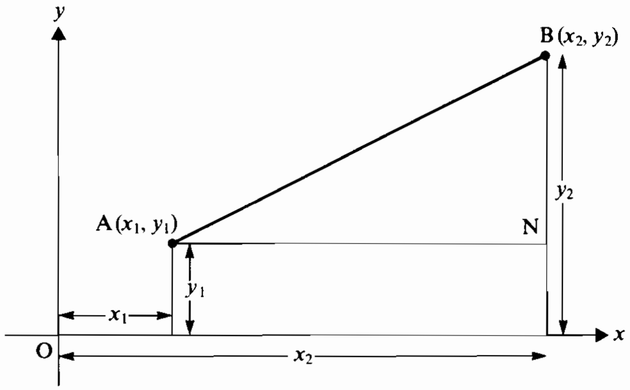


The length of the line joining A and B can be found by using Pythagoras' theorem, i.e.

$$AB^2 = AN^2 + BN^2$$
$$= (3-1)^2 + (4-2)^2$$
$$= 8$$

Therefore $AB = \sqrt{8} = 2\sqrt{2}$

In the same way the length of the line joining any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found.



From the diagram, $AB^2 = AN^2 + BN^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$\Rightarrow$ $$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$$

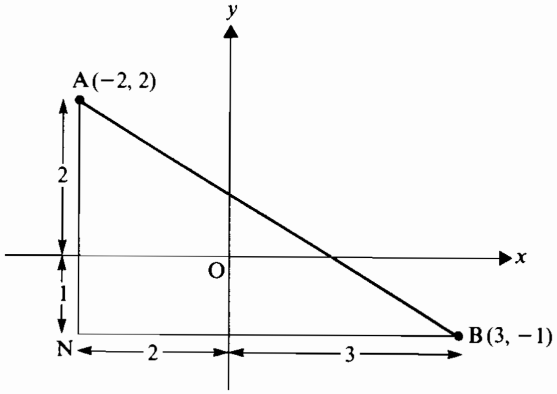
i.e. the length of the line joining $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$$

This formula still holds when some, or all, of the coordinates are negative. This is illustrated in the next worked example.

Examples 6b

1. Find the length of the line joining A(−2, 2) to B(3, −1)



$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$$
$$= \sqrt{[(3 - \{-2\})^2 + (-1-2)]}$$
$$= \sqrt{(5^2 + \{-3\}^2)}$$
$$= \sqrt{34}$$

From the diagram, $BN = 3 + 2 = 5$

and $AN = 2 + 1 = 3$

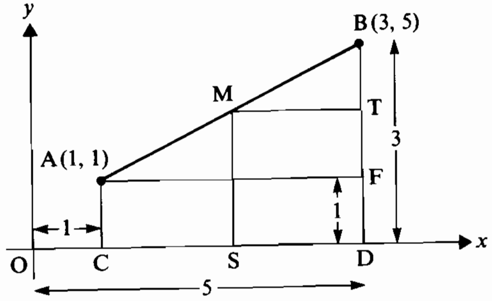
$\Rightarrow$ $AB^2 = 5^2 + 3^2 = 34$

$\Rightarrow$ $AB = \sqrt{34}$

This confirms that the formula used above is valid when some of the coordinates are negative.

THE MIDPOINT OF THE LINE JOINING TWO GIVEN POINTS

Consider the line joining the points A(1, 1) and B(3, 5)



Using the intercept theorem, we see that if M is the midpoint of AB then S is the midpoint of CD.

Therefore the x-coordinate of M is given by OS, where

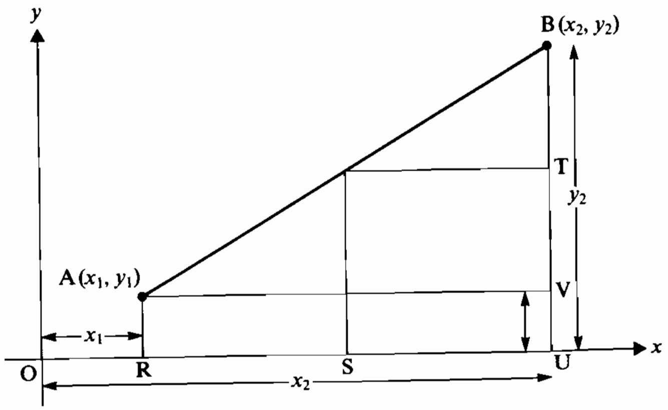
$$OS = OC + \tfrac{1}{2}CD = 1 + \tfrac{1}{2}(5 - 1) = 3$$

Similarly, T is the midpoint of BF, so the y-coordinate of M is given by SM (= DT), where

$$DT = DF + \tfrac{1}{2}FB = 1 + \tfrac{1}{2}(3 - 1) = 2$$

Therefore M is the point (3, 2)

In general, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the coordinates of M, the midpoint of AB, can be found in the same way.



At M, $\quad x = OS = OR + \tfrac{1}{2}RU$

$$= x_1 + \tfrac{1}{2}(x_2 - x_1) = \tfrac{1}{2}(x_1 + x_2)$$

and $\quad y = SM = UT = UV + \tfrac{1}{2}BV$

$$= y_1 + \tfrac{1}{2}(y_2 - y_1) = \tfrac{1}{2}(y_1 + y_2)$$

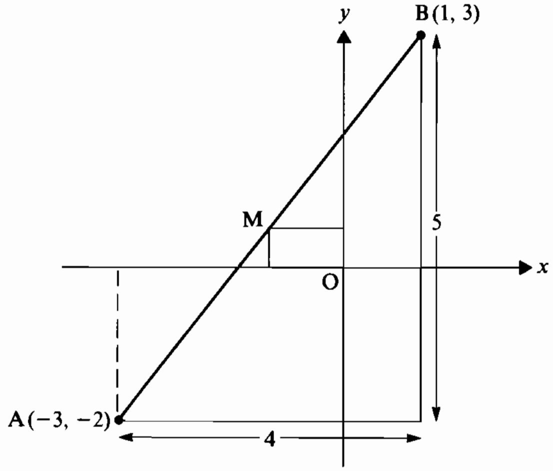
Note that the coordinates of M are the average of the coordinates of A and B. Hence

> the coordinates of the midpoint of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $[\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)]$

The next worked example shows that this formula holds when some of the coordinates are negative.

Examples 6b (continued)

2. Find the coordinates of the midpoint of the line joining A(−3, −2) and B(1, 3).



The coordinates of M are $[\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)]$

$$= [\tfrac{1}{2}(-3 + 1), \tfrac{1}{2}(-2 + 3)] = (-1, \tfrac{1}{2})$$

Alternatively, from the diagram, M is half-way from A to B horizontally and vertically, i.e.

at M $\quad x = -3 + \tfrac{1}{2}(4) = -1 \quad$ and $\quad y = -2 + \tfrac{1}{2}(5) = \tfrac{1}{2}$

This confirms that the formula works when some of the coordinates are negative.

EXERCISE 6b

1. Find the length of the line joining
(a) A(1, 2) and B(4, 6) (b) C(3, 1) and D(2, 0)
(c) J(4, 2) and K(2, 5)

2. Find the coordinates of the midpoints of the lines joining the points in Question 1.

3. Find (i) the length, (ii) the coordinates of the midpoint of the line, joining
(a) A(−1, −4), B(2, 6) (b) S(0, 0), T(−1, −2)
(c) E(−1, −4), F(−3, −2)

4. Find the distance from the origin to the point (7, 4)

5. Find the length of the line joining the point (−3, 2) to the origin.

6. Find the coordinates of the midpoint of the line from the point (4, −8) to the origin.

7. Show, by using Pythagoras' Theorem, that the lines joining A(1, 6), B(−1, 4) and C(2, 1) form a right-angled triangle.

8. A, B and C are the points (7, 3), (−4, 1) and (−3, −2) respectively.
(a) Show that △ABC is isosceles.
(b) Find the midpoint of BC.
(c) Find the area of △ABC.

9. The vertices of a triangle are A(0, 2), B(1, 5) and C(−1, 4)
Find
(a) the perimeter of the triangle
(b) the coordinates of D such that AD is a median of △ABC
(c) the length of AD.

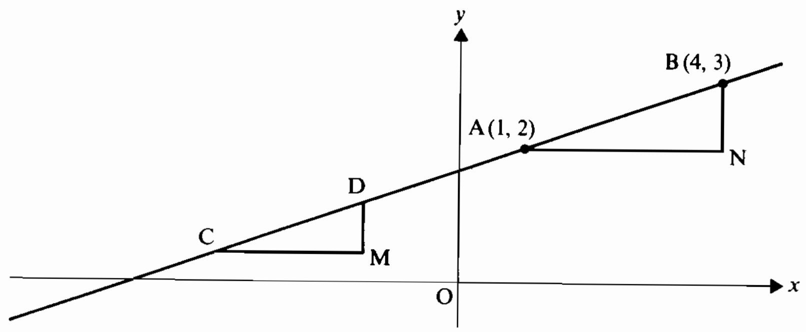
10. Show that the lines OA and OB are perpendicular where A and B are the points (4, 3) and (3, −4) respectively.

11. M is the midpoint of the line joining A to B. The coordinates of A and M are (5, 7) and (0, 2) respectively. Find the coordinates of B.

GRADIENT

The gradient of a straight line is a measure of its slope with respect to the x-axis. Gradient is defined as

the increase in y divided by the increase in x between one point and another point on the line.



Consider the line passing through A(1, 2) and B(4, 3).

From A to B, the increase in y is 1
the increase in x is 3

Therefore the gradient of AB is $\frac{1}{3}$.

Now NB measures the increase in the y-coordinate and AN measures the increase in the x-coordinate, so the gradient can be written as $\frac{NB}{AN}$.

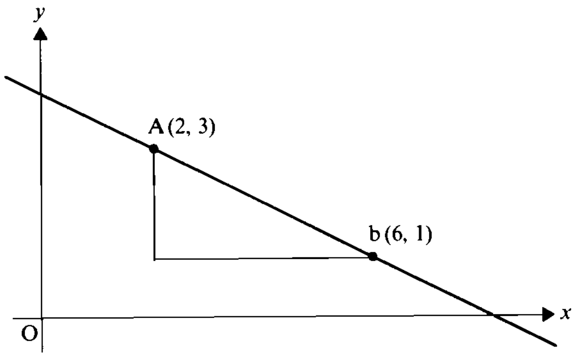
If C and D are any other two points on the line then △ABN and △CDM are similar, so

$$\frac{NB}{AN} = \frac{MD}{CM} = \frac{1}{3}$$

i.e.

the gradient of a line may be found from *any* two points on the line.

Now consider the line through the points A(2, 3) and B(6, 1)



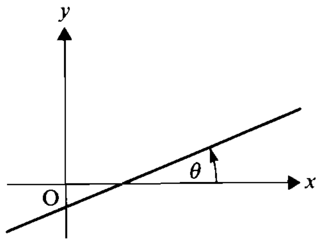
Moving from A to B $\dfrac{\text{increase in } y}{\text{increase in } x} = \dfrac{-2}{4} = -\dfrac{1}{2}$

Alternatively, moving from B to A $\dfrac{\text{increase in } y}{\text{increase in } x} = \dfrac{2}{-4} = -\dfrac{1}{2}$

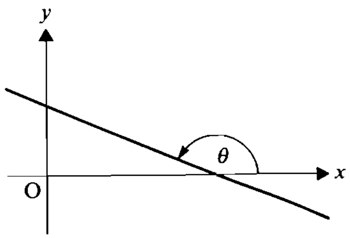
This shows that it does not matter in which order the two points are considered, provided that they are considered in the *same* order when calculating the increases in x and in y.

From these two examples we see that the gradient of a line may be positive or negative.

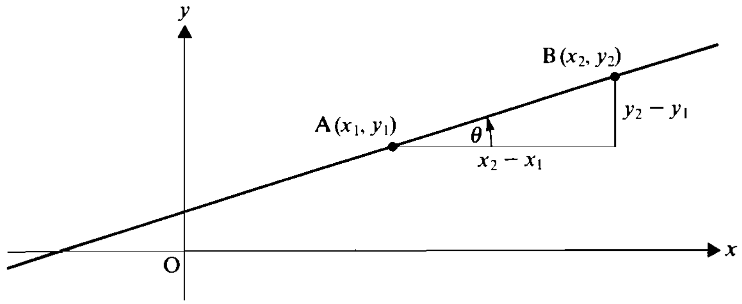
A positive gradient indicates an 'uphill' slope with respect to the positive direction of the x-axis, i.e. the line makes an acute angle with the positive sense of the x-axis.



A negative gradient indicates a 'downhill' slope with respect to the positive direction of the x-axis, i.e. the line makes an obtuse angle with the positive sense of the x-axis.



In general,



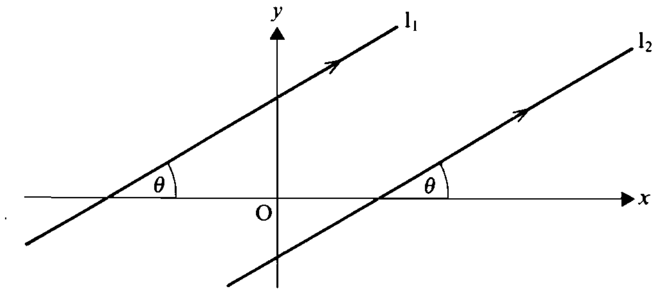
the gradient of the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{\text{the increase in } y}{\text{the increase in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As the gradient of a straight line is the increase in y divided by the increase in x from one point on the line to another,

gradient measures the increase in y per unit increase in x, i.e. the rate of increase of y with respect to x.

PARALLEL LINES

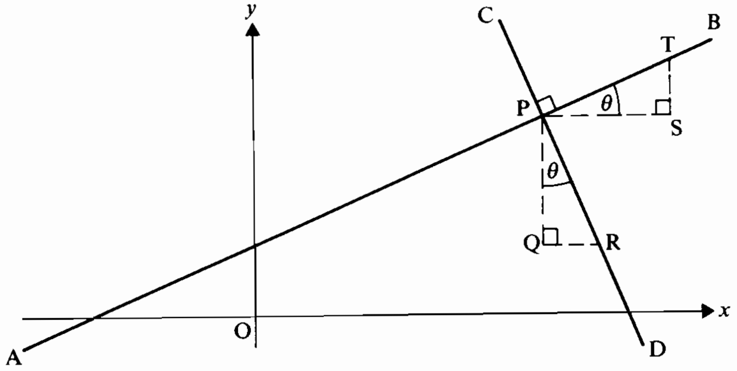


If l_1 and l_2 are parallel lines, they are equally inclined to the positive direction of the x-axis, i.e.

parallel lines have equal gradients.

PERPENDICULAR LINES

Consider the perpendicular lines AB and CD whose gradients are m_1 and m_2 respectively.



If AB makes an angle θ with the x-axis then CD makes an angle θ with the y-axis. Therefore triangles PQR and PST are similar.

Now the gradient of AB is $\dfrac{ST}{PS} = m_1$

and the gradient of CD is $\dfrac{-PQ}{QR} = m_2$, i.e. $\dfrac{PQ}{QR} = -m_2$

But $\dfrac{ST}{PS} = \dfrac{QR}{PQ}$ (△s PQR and PST are similar)

therefore $m_1 = -\dfrac{1}{m_2}$ or $m_1 m_2 = -1$

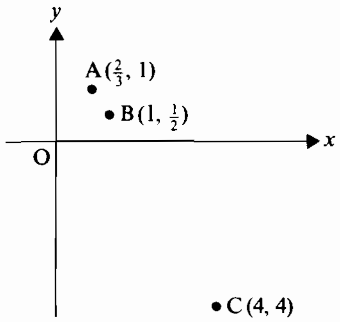
i.e.

the product of the gradients of perpendicular lines is -1, or, if one line has gradient m, any line perpendicular to it has gradient $-\dfrac{1}{m}$

Example 6c

Determine, by comparing gradients, whether the following three points are collinear (i.e. lie on the same straight line).

$$A(\tfrac{2}{3}, 1),\ B(1, \tfrac{1}{2}),\ C(4, -4)$$



The gradient of AB is $\dfrac{1-\frac{1}{2}}{\frac{2}{3}-1} = -\dfrac{3}{2}$

The gradient of BC is $\dfrac{-4-\frac{1}{2}}{4-1} = -\dfrac{3}{2}$

As the gradients of AB and BC are the same, A, B and C are collinear.

The diagram, although not strictly necessary, gives a check that the answer is reasonable.

EXERCISE 6c

1. Find the gradient of the line through the pair of points.
   (a) $(0, 0), (1, 3)$ (b) $(1, 4), (3, 7)$ (c) $(5, 4), (2, 3)$
   (d) $(-1, 4), (3, 7)$ (e) $(-1, -3), (-2, 1)$ (f) $(-1, -6), (0, 0)$
   (g) $(-2, 5), (1, -2)$ (h) $(3, -2), (-1, 4)$ (i) $(h, k), (0, 0)$

2. Determine whether the given points are collinear.
   (a) $(0, -1), (1, 1), (2, 3)$ (b) $(0, 2), (2, 5), (3, 7)$
   (c) $(-1, 4), (2, 1), (-2, 5)$ (d) $(0, -3), (1, -4), (-\frac{1}{2}, -\frac{5}{2})$

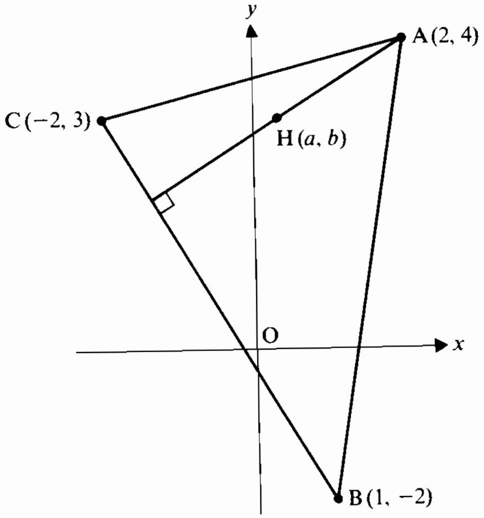
3. Determine whether AB and CD are parallel, perpendicular or neither.
   (a) $A(0, -1), B(1, 1), C(1, 5), D(-1, 1)$
   (b) $A(1, 1), B(3, 2), C(-1, 1), D(0, -1)$
   (c) $A(3, 3), B(-3, 1), C(-1, -1), D(1, -7)$
   (d) $A(2, -5), B(0, 1), C(-2, 2), D(3, -7)$
   (e) $A(2, 6), B(-1, -9), C(2, 11), D(0, 1)$

PROBLEMS IN COORDINATE GEOMETRY

This chapter ends with a miscellaneous selection of problems on coordinate geometry. A clear and reasonably accurate diagram showing all the given information will often suggest the most direct method for solving a particular problem.

Example 6d

The vertices of a triangle are the points A(2, 4), B(1, −2) and C(−2, 3) respectively. The point H(a, b) lies on the altitude through A. Find a relationship between a and b



As H is on the altitude through A, AH is perpendicular to BC.

The gradient of AH is $\dfrac{4-b}{2-a}$,

the gradient of BC is $\dfrac{3-(-2)}{-2-1} = -\dfrac{5}{3}$

The product of the gradients of perpendicular lines is −1

Therefore $\left[\dfrac{4-b}{2-a}\right]\left[-\dfrac{5}{3}\right] = -1$

$\Rightarrow \quad \dfrac{-20+5b}{6-3a} = -1$

$\Rightarrow \quad 5b = 3a + 14$

EXERCISE 6d

1. A(1, 3), B(5, 7), C(4, 8), D(a, b) form a rectangle ABCD. Find a and b

2. The triangle ABC has its vertices at the points A(1, 5), B(4, −1) and C(−2, −4)
   (a) Show that △ABC is right-angled.
   (b) Find the area of △ABC.

3. Show that the point $(-\frac{32}{3}, 0)$ is on the altitude through A of the triangle whose vertices are A(1, 5), B(1, −2) and C(−2, 5)

4. Show that the triangle whose vertices are (1, 1), (3, 2), (2, −1) is isosceles.

5. Find, in terms of a and b, the length of the line joining (a, b) and $(2a, 3b)$

6. The point (1, 1) is the centre of a circle whose radius is 2. Show that the point (1, 3) is on the circumference of this circle.

7. A circle, radius 2 and centre the origin, cuts the x-axis at A and B and cuts the positive y-axis at C. *Prove* that ∠ACB = 90°

8. Find in terms of p and q, the coordinates of the midpoint of the line joining C(p, q) and D(q, p). Hence show that the origin is on the perpendicular bisector of the line CD.

9. The point (a, b) is on the circumference of the circle of radius 3 whose centre is at the point (2, 1). Find a relationship between a and b

10. ABCD is a quadrilateral where A, B, C and D are the points (3, −1), (6, 0), (7, 3) and (4, 2). Prove that the diagonals bisect each other at right angles and hence find the area of ABCD.

11. The vertices of a triangle are at the points A(a, 0), B(0, b) and C(c, d) and ∠B = 90°. Find a relationship between a, b, c and d

12. A point P(a, b) is equidistant from the y-axis and from the point (4, 0). Find a relationship between a and b

CHAPTER 7

OBTUSE ANGLES, SINE AND COSINE FORMULAE

TRIGONOMETRIC RATIOS OF ACUTE ANGLES

The sine, cosine and tangent of an acute angle in a right-angled triangle are defined in terms of the sides of the triangle as follows

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

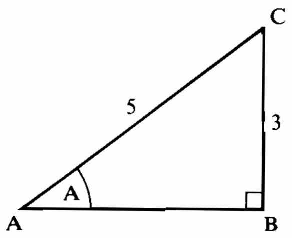
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

If any of these trig ratios is given as a fraction, the lengths of two of the sides of the right-angled triangle can be marked. Then the third side can be calculated by using Pythagoras' theorem.

Example 7a

Given that $\sin A = \frac{3}{5}$ find $\cos A$ and $\tan A$

Because $\sin A = \dfrac{\text{opp}}{\text{hyp}}$, we can draw a right-angled triangle with the side opposite to angle A of length 3 units and a hypotenuse of length 5 units.



Applying Pythagoras' theorem to $\triangle ABC$ gives

$$(AB)^2 + (BC)^2 = (AC)^2$$

i.e. $$(AB)^2 + 3^2 = 5^2$$

$\Rightarrow$ $$(AB)^2 = 25 - 9 = 16$$

$\Rightarrow$ $$AB = 4$$

Then $$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}$$

and $$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$

EXERCISE 7a

If any of the square roots in this exercise are not integers, leave them in surd form.

1. If $\tan A = \frac{12}{5}$ find $\sin A$ and $\cos A$
2. Given that $\cos X = \frac{4}{5}$ find $\tan X$ and $\sin X$
3. If $\sin P = \frac{40}{41}$ find $\cos P$ and $\tan P$
4. Tan $A = 1$ Find $\sin A$ and $\cos A$
5. If $\cos Y = \frac{2}{3}$ find $\sin Y$ and $\tan Y$
6. Given that $\sin A = \frac{1}{2}$ what is $\cos A$? Use your calculator to find the size of angle A
7. If $\sin X = \frac{?}{25}$ and $\tan X = \frac{7}{?}$ find $\cos X$

In each question from 8 to 12, use $\sin X = \frac{3}{5}$.

8. Find $\cos X$ and hence calculate $\cos^2 X - \sin^2 X$. Use a calculator to determine the value of angle X and hence find $\cos 2X$ correct to 2 s.f. What conclusion can you draw?
9. Find $\cos^2 X + \sin^2 X$
10. Evaluate $2 \sin X \cos X$ as a decimal. Find correct to 2 s.f. the value of $\sin 2X$ and draw any conclusion that you can.

11. Work out the value of $\frac{2\tan X}{1-\tan^2 X}$. Compare this with the value you found in Question 10 for $\sin 2X$

12. Work out the value of $\frac{1-\tan^2 X}{1+\tan^2 X}$. How does this quantity compare with the value of $\cos 2X$ found in Question 8?

TRIGONOMETRIC RATIOS OF OBTUSE ANGLES

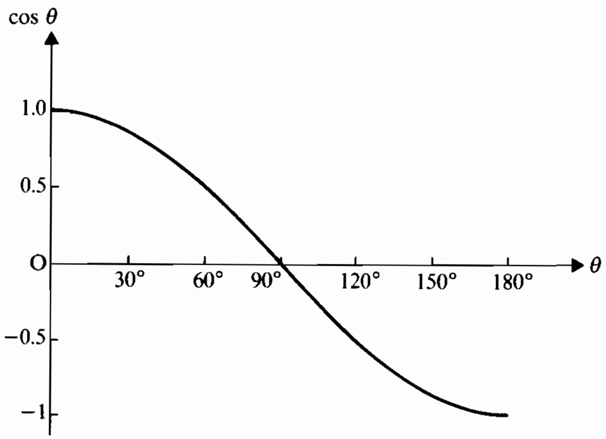
The Cosine of an Obtuse Angle

Clearly the definition in the preceeding paragraph is restricted to acute angles, so if we want to work with the cosine of an obtuse angle we need a broader definition.

First however we will examine the values given by a calculator for the cosines of angles from 0 to 180°

θ	0	30°	45°	60°	90°	120°	135°	150°	180°
$\cos\theta$ (to 2 d.p.)	1	0.87	0.71	0.50	0	−0.50	−0.71	−0.87	−1

Plotting these figures on graph paper gives a shape which is called a cosine curve. Note that θ, the symbol used for the angle, is the most commonly used symbol for a varying angle.



From the graph or the table it can be seen that

$$\cos 60° = 0.5$$

and $\cos 120° = -0.5$

i.e. $\cos 120° = -\cos 60°$ $(120° + 60° = 180°)$

also $\cos 45° = 0.71$

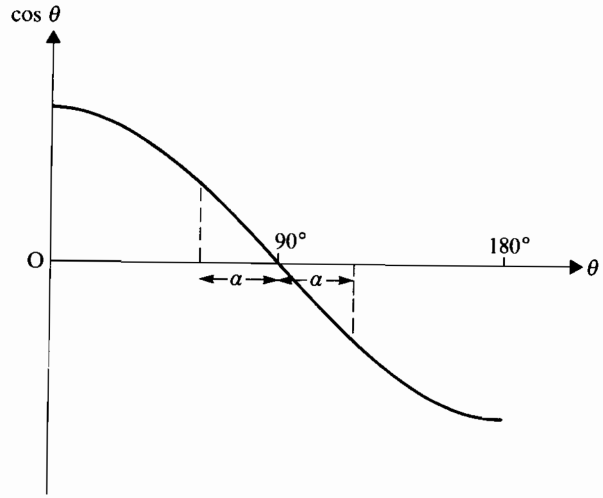
and $\cos 135° = -0.71$

i.e. $\cos 135° = -\cos 45°$ $(135° + 45° = 180°)$

The reader can find many more pairs of angles where the relationship is

$$\cos\theta = -\cos(180° - \theta)$$

This property is confirmed by looking again at the graph.



It appears that the curve has rotational symmetry about the point (90°, 0), suggesting that

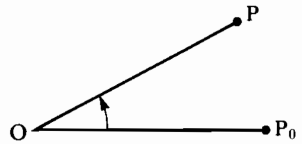
$$\cos(90° - \alpha) = -\cos(90° + \alpha)$$

But $\theta = 90° - \alpha$, so $\cos\theta = -\cos(180° - \theta)$

So far, it just *looks as though* this relationship is true and, at this level of study, we should now look for a more general explanation of this relationship. To do this we first consider a broader concept of what an angle is.

GENERAL DEFINITION OF AN ANGLE

Consider a line which can rotate from its initial position OP_0 about the point O to any other position OP.



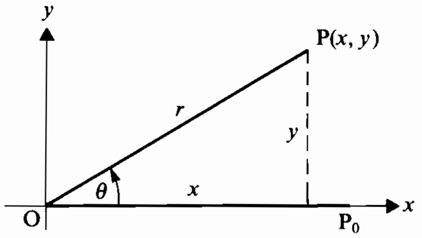
The amount of rotation is indicated by the angle between OP_0 and OP, i.e.

an angle is a measure of the rotation of a line about a fixed point.

The anticlockwise sense of rotation is taken as positive and clockwise rotation is negative. It follows that an angle formed by the anticlockwise rotation of OP is a positive angle.

When we define an angle in this way no triangle is involved, so we need a more general way of looking at the cosine of an angle.

GENERAL DEFINITION OF THE COSINE RATIO

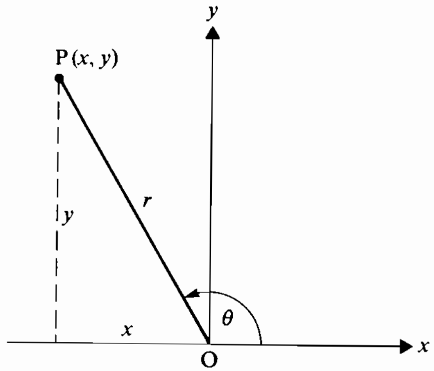


Using standard x and y axes, let the line OP_0 be drawn on the x-axis as shown and let OP be the position reached after this line has rotated through an angle θ. The coordinates of P are (x, y) and we shall refer to the length of OP as r. Using these symbols, the cosine of θ is defined by

$$\cos\theta = \frac{x}{r}$$

Note that this is consistent with the earlier definition, i.e.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

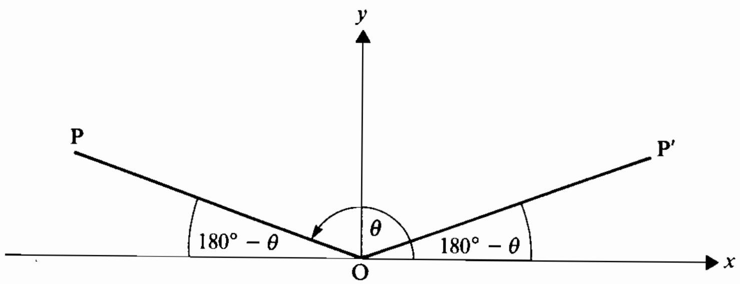


The definition $\cos\theta = \frac{x}{r}$ continues to be used when OP has rotated through an obtuse angle, but now it can be seen that the x-coordinate of P is negative. The value of r is always positive, because $r = \sqrt{(x^2 + y^2)}$, so it follows that $\frac{x}{r}$ is negative, i.e.

the cosine of an obtuse angle is negative.

This explains the *signs* of the values given in a calculator for the cosines of obtuse angles.

Further, if OP′ is the reflection in the y-axis of OP, then OP′ represents a rotation of $(180° - \theta)$



Both r and x have the same numerical values for OP′ and OP therefore the *numerical values* of $\cos\theta$ and $\cos(180° - \theta)$ are equal.

We have now proved, from the general definition, that

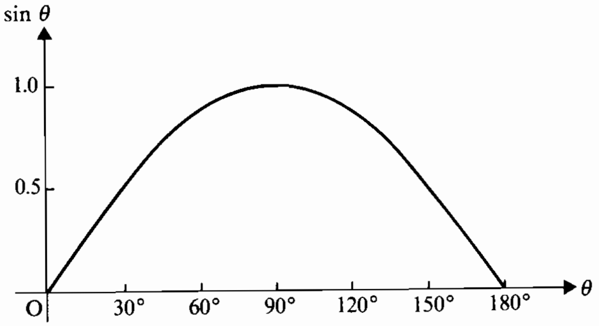
$$\cos(180° - \theta) = -\cos\theta$$

The Sine of an Obtuse Angle

If we start by listing, and plotting, the values given by a calculator for the sines of angles from 0 to 180°, we have

θ	0	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$ (to 2 d.p.)	0	0.5	0.71	0.87	1	0.87	0.71	0.5	0

and



This graph is called a sine curve; notice that it looks symmetrical about a vertical line through 90°

Again relationships can be observed between the sines of pairs of angles, for example

$$\sin 30° = 0.5$$

and $\quad \sin 150° = 0.5$

i.e. $\quad \sin 150° = \sin 30° \qquad (150° + 30° = 180°)$

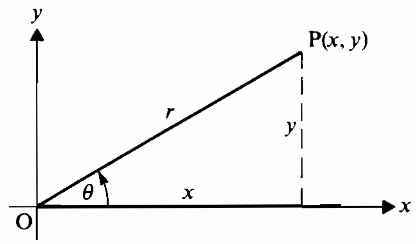
also $\quad \sin 60° = 0.87$

and $\quad \sin 120° = 0.87$

i.e. $\quad \sin 120° = \sin 60° \qquad (120° + 60° = 180°)$

This time it looks as if $\sin\theta = \sin(180° - \theta)$ and we shall now check this relationship from a definition similar to that used for the cosine of a general angle.

GENERAL DEFINITION OF THE SINE RATIO

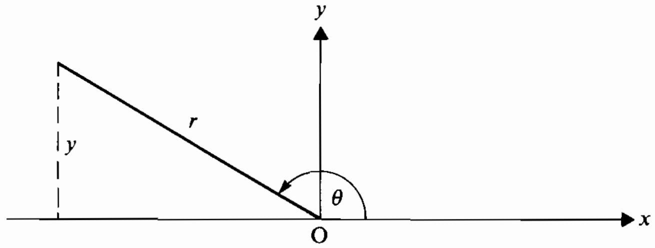


Using the same axes and symbols as we used when defining the cosine ratio, the sine ratio is defined by

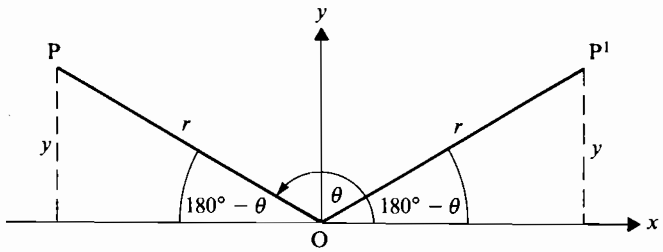
$$\sin\theta = \frac{y}{r}$$

$\left(\text{Again this is consistent with the concept } \sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}\right)$

When θ is obtuse, both r and y are positive, so the sine ratio of an obtuse angle is positive.



Also, the line OP′, which represents a rotation of $(180° - \theta)$, is the reflection of OP in the y-axis. Therefore the values of y at P and P′ are equal, i.e. y/r has the same value at P and at P′.

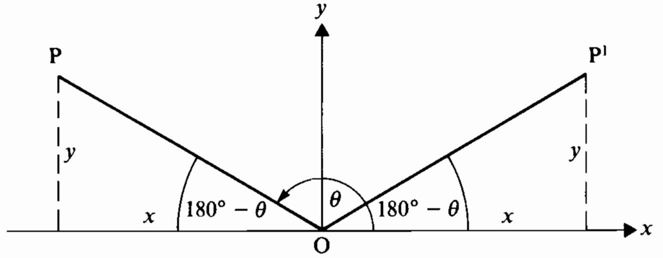


This proves that, in general,

$$\sin(180° - \theta) = \sin\theta$$

GENERAL DEFINITION OF THE TANGENT RATIO

Using the same notation, the tangent ratio is defined by $\tan\theta = \dfrac{y}{x}$



Reasoning similar to that used when considering $\sin\theta$ and $\cos\theta$ shows that

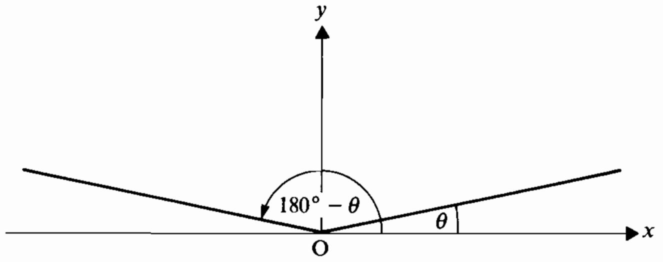
$$\tan(180° - \theta) = -\tan\theta$$

Examples 7b

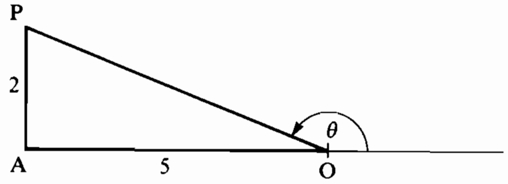
1. If $\sin\theta = \frac{1}{5}$ find two possible values for θ

As given by a calculator, the angle with a sine of 0.2 is 11.5°

But $\sin\theta = \sin(180° - \theta)$ so $\sin 11.5° = \sin(180° - 11.5°)$
if $\sin\theta = \frac{1}{5}$, two values of θ are 11.5° and 168.5°



2. Use the information in the diagram to find $\cos\theta$ and $\tan\theta$



In $\triangle OPA$ $\quad OP^2 = 4 + 25 = 29$ (Pythagoras)

and $\quad A\hat{O}P = (180° - \theta)$

$$\cos(180° - \theta) = \frac{OA}{OP} = \frac{5}{\sqrt{29}}$$

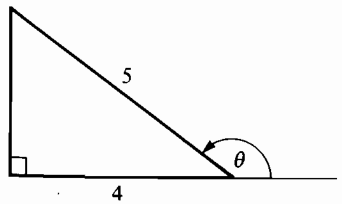
$$\cos\theta = -\cos(180° - \theta) = -\frac{5}{\sqrt{29}}$$

$$\tan\theta = -\tan(180° - \theta) = -\frac{AP}{OA} = -\tfrac{2}{5}$$

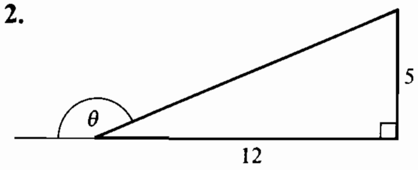
EXERCISE 7b

In each question from 1 to 4, find $\sin\theta$, $\cos\theta$ and $\tan\theta$, giving unknown lengths in surd form when necessary.

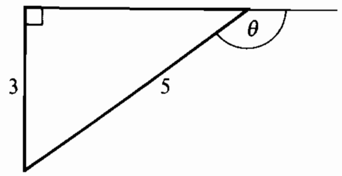
1.



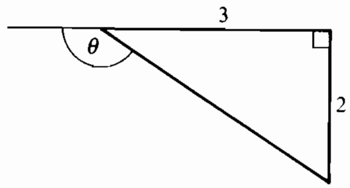
2.



3.



4.



In each question from 5 to 16, find X where X is an angle from 0 to 180°

5. $\sin X = \sin 80°$ **6.** $\tan X = -\tan 120°$

7. $\cos X = -\cos 75°$ **8.** $\sin X = \sin 128°$

9. $\tan 45° = -\tan X$ **10.** $\cos 30° = -\cos X$

11. $\sin X = \sin 81°$ **12.** $-\cos 123 = \cos X$

13. $\sin 90° = \sin X$ **14.** $\tan X = -\tan 100°$

15. $\cos 91° = -\cos X$ **16.** $\cos 0 = -\cos X$

The unknown angles in questions 17 to 23 are in the range 0 to 180°

17. If $\sin A = \frac{3}{5}$ find two possible values of $\cos(180° - A)$

18. If $\cos X = -\frac{12}{13}$ find $\sin X$

19. If $\sin\theta = \frac{4}{5}$ find, to the nearest degree, two possible values of θ

20. Given that $\sin A = 0.5$ and $\cos A = -0.8660$, find $\angle A$

21. Find the angle P if $\cos P = -0.7071$ and $\sin P = 0.7071$

22. Is there an angle X for which
(a) $\cos X = 0$ and $\sin X = 1$
(b) $\sin X = 0$ and $\cos X = 1$
(c) $\cos X = 0$ and $\sin X = -1$
(d) $\tan X = 0$ and $\sin X = 0$?

23. If $\cos A = -\cos B$, what is the relationship between $\angle A$ and $\angle B$

24. Draw a diagram to show the angle T for which $\tan T = \frac{3}{4}$. Draw on your diagram an angle with a tangent of $-\frac{3}{4}$.

25. Within the range $0 < \theta < 180°$, are there angles for which $\sin\theta$ and $\cos\theta$ are
(a) both positive (b) both negative?

FINDING UNKNOWN SIDES AND ANGLES IN A TRIANGLE

Triangles are involved in many practical measurements (e.g. surveying) so it is important to be able to make calculations from limited data about a triangle.

Although a triangle has three sides and three angles, it is not necessary to know all of these in order to define a particular triangle. If enough information about a triangle is known, the remaining sides and angles can be calculated. This is called *solving* the triangle and it requires the use of one of a number of formulae.

The two relationships that are used most frequently are the sine rule and the cosine rule.

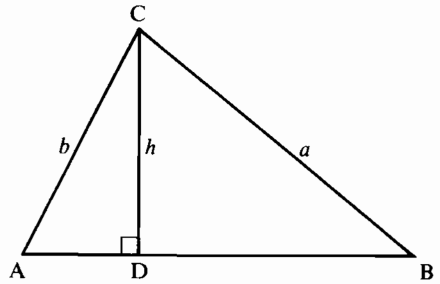
When working with a triangle ABC the side opposite to $\angle A$ is denoted by a, the side opposite to $\angle B$ by b and so on.

THE SINE RULE

In a triangle ABC, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof

Consider a triangle ABC in which there is no right angle.



A line drawn from C, perpendicular to AB, divides triangle ABC into two right-angled triangles, CDA and CDB.

In $\triangle$CDA $\quad \sin A = h/b \Rightarrow h = b\sin A$

In $\triangle$CDB $\quad \sin B = h/a \Rightarrow h = a\sin B$

Therefore $\quad a\sin B = b\sin A$

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

We could equally well have divided $\triangle$ABC into two right-angled triangles by drawing the perpendicular from A to BC (or from B to AC). This would have led to a similar result,

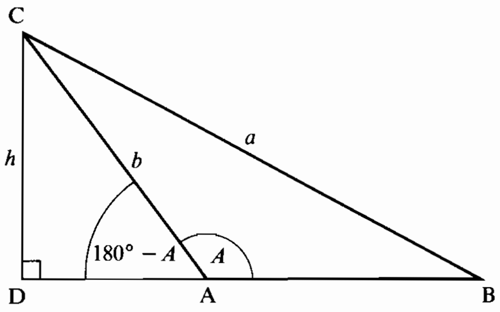
i.e. $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

By combining the two results we produce the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note that this proof is equally valid when $\triangle$ABC contains an obtuse angle.

Suppose that $\angle$A is obtuse.



This time $h = b\sin(180° - A)$ but, as $\sin(180° - A) = \sin A$, we see that once again $h = b\sin A$

In all other respects the proof given above is unaltered, showing that the sine rule applies to any triangle.

Using the Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

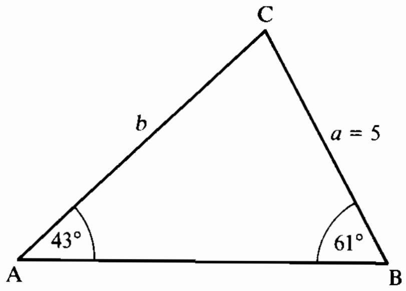
This rule is made up of three separate fractions, only two of which can be used at a time. We select the two which contain three known quantities and only one unknown.

Note that, when the sine rule is being used to find an unknown angle, it is more conveniently written in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Examples 7c

1. In $\triangle$ABC, BC = 5 cm, A = 43° and B = 61°. Find the length of AC.



$\angle$A, $\angle$B and b are known and b is required, so the two fractions we select from the sine rule are

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

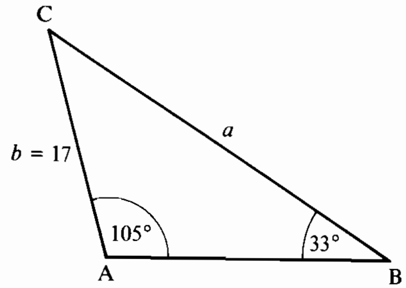
i.e. $$\frac{5}{\sin 43°} = \frac{b}{\sin 61°}$$

$$\Rightarrow \qquad b = \frac{5\sin 61°}{\sin 43°} = 6.412$$

Therefore AC = 6.41 cm correct to 3 s.f.

---

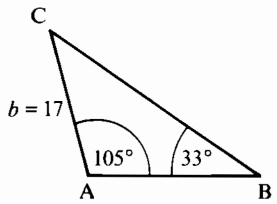
2. In ABC, AC = 17 cm, $\angle$A = 105° and $\angle$B = 33°. Find AB.



The two sides involved are b and c, so before the sine rule can be used we must find C.

$$\angle A + \angle B + \angle C = 180° \quad \Rightarrow \quad \angle C = 42°$$

Now from the sine rule we can use



$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{17}{\sin 33°} = \frac{c}{\sin 42°}$$

i.e. $$c = \frac{17 \times 0.6691}{0.5446} = 20.88$$

Therefore AB = 20.9 cm correct to 3 s.f.

EXERCISE 7c

1. In $\triangle$ABC, AB = 9 cm, $\angle$A = 51° and $\angle$C = 39°
   Find BC.

2. In $\triangle$XYZ, $\angle$X = 27°, YZ = 6.5 cm and $\angle$Y = 73°
   Find ZX.

3. In $\triangle$PQR, $\angle$R = 52°, $\angle$Q = 79° and PR = 12.7 cm.
   Find PQ.

4. In $\triangle$ABC, AC = 9.1 cm, $\angle$A = 59° and $\angle$B = 62°
   Find BC.

5. In $\triangle$DEF, DE = 174 cm, $\angle$D = 48° and $\angle$F = 56°
   Find EF.

6. In $\triangle$XYZ, $\angle$X = 130°, $\angle$Y = 21° and XZ = 53 cm.
   Find YZ.

7. In $\triangle$PQR, $\angle$Q = 37°, $\angle$R = 101° and PR = 4.3 cm.
   Find PQ.

8. In $\triangle$ABC, BC = 73 cm, $\angle$A = 54° and $\angle$C = 99°
   Find AB.

9. In $\triangle$LMN, LN = 637 cm, $\angle$M = 128° and $\angle$N = 46°
   Find LM.

10. In $\triangle$XYZ, XY = 92 cm, $\angle$X = 59° and $\angle$Y = 81°
    Find XZ.

11. In $\triangle$PQR, $\angle$Q = 64°, $\angle$R = 38° and PR = 15 cm.
    Find QR.

12. In $\triangle$ABC, AB = 24 cm, $\angle$A = 132° and $\angle$C = 22°
    Find AC.

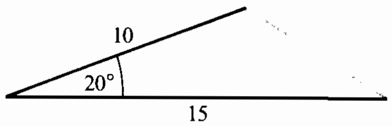
13. In $\triangle$XYZ, $\angle$X = 49°, XY = 98 cm and $\angle$Z = 100°
    Find XZ.

14. In $\triangle$ABC, AB = 10 cm, BC = 9.1 cm and AC = 17 cm.
    Can you use the sine rule to find $\angle$A? If you answer YES, write down the two parts of the sine rule that you would use. If you answer NO, give your reason.

The Ambiguous Case

Consider a triangle specified by two sides and one angle.

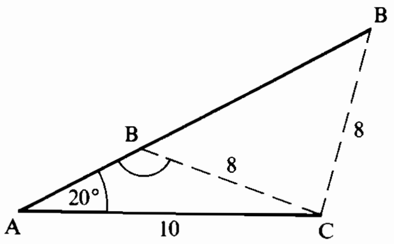
If the angle is between the two sides there is only one possible triangle, e.g.



If, however, the angle is not between the two given sides it is sometimes possible to draw two triangles from the given data.

Consider, for example, a triangle ABC in which $\angle$A = 20°, b = 10 and a = 8

The two triangles with this specification are shown in the diagram; in one of them B is an acute angle, while in the other one, B is obtuse.



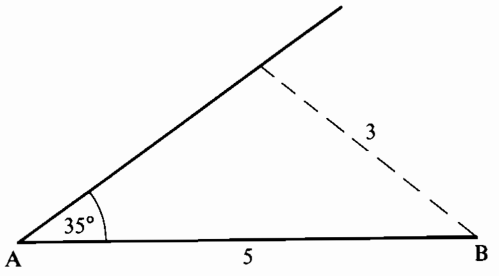
Therefore, when two sides and an angle of a triangle are given,

it is essential to check whether the obtuse angle is possible.

The following worked examples illustrate this special case.

Examples 7d

1. In the triangle ABC, find C given that AB = 5 cm, BC = 3 cm and A = 35°



We know a, c and ∠A so the sine rule can be used to find ∠C.

As we are looking for an angle, the form we use is

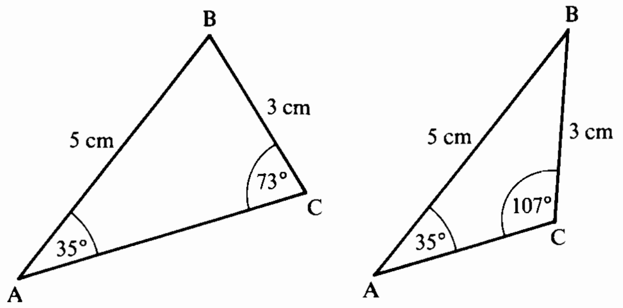
$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin 35°}{3} = \frac{\sin C}{5}$$

Hence $$\sin C = \frac{5 \times 0.5736}{3} = 0.9560$$

One angle whose sine is 0.9560 is 73° but there is also an obtuse angle with the same sine, i.e. 107°

If C = 107, then A + C = 107 + 35 = 142
⇒ B = 180 − 142 = 38

So in this case ∠C = 107° *is* an acceptable solution and we have two possible triangles.

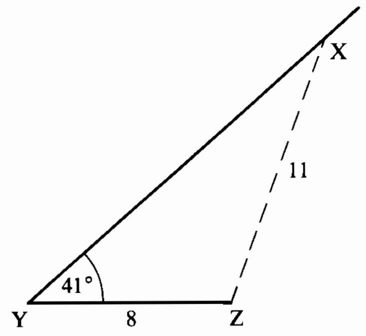


Therefore ∠C is either 73° or 107°

The reader should *not* assume that there are *always* two possible angles when the sine rule is used to find a second angle in a triangle. The next example shows that this is not so.

Examples 7d (continued)

2. In the triangle XYZ, ∠Y = 41°, XZ = 11 cm and YZ = 8 cm. Find ∠X.



Using the part of the sine rule that involves x, y, ∠X and ∠Y we have

$$\frac{\sin X}{x} = \frac{\sin Y}{y} \Rightarrow \frac{\sin X}{8} = \frac{\sin 41°}{11}$$

Hence $$\sin X = \frac{8 \times 0.6561}{11} = 0.4771$$

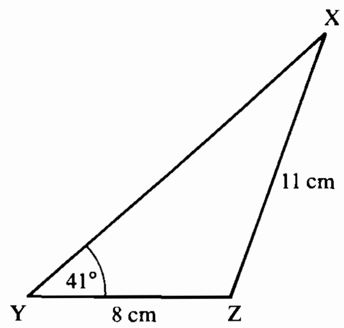
The two angles with a sine of 0.4771 are 28° and 152°

Checking to see whether 152° is a possible value for ∠X we see that

∠X + ∠Y = 152° + 41° = 193°

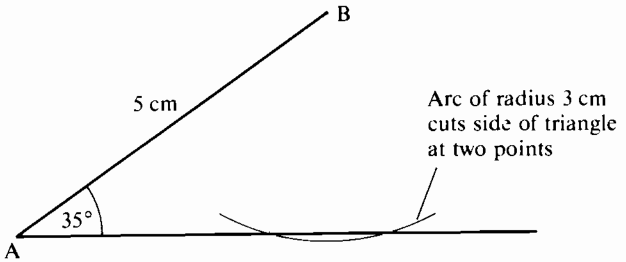
This is greater than 180°, so it is not possible for the angle X to have the value 152°.

In this case then, there is only one possible triangle containing the given data, i.e. the triangle in which ∠X = 28°

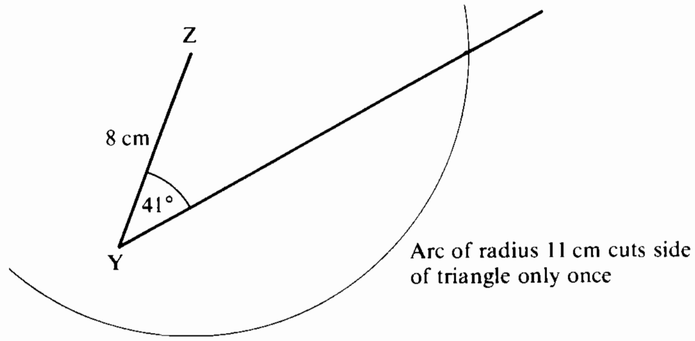


It is interesting to notice how the two different situations that arose in Examples 1 and 2 above can be illustrated by the construction of the triangles with the given data.

When AB = 5 cm, BC = 3 cm and ∠A = 35° we have



When XZ = 11 cm, YZ = 8 cm and ∠Y = 41° we have



EXERCISE 7d

In each of the following questions, find the angle indicated by the question mark, giving two values in those cases where there are two possible triangles. Illustrate your solution to each question.

	AB	BC	CA	∠A	∠B	∠C
1.		2.9 cm	6.1 cm	?	40°	
2.	5.7 cm		2.3 cm		20°	?
3.	21 cm	36 cm		29.5°		?
4.		2.7 cm	3.8 cm	?	54°	
5.	4.6 cm		7.1 cm		?	33°
6.	9 cm	7 cm		?		40°

THE COSINE RULE

When solving a triangle, the sine rule cannot be used unless the data given includes one side and the angle opposite to that side. If, for example, a, b and C are given then in the sine rule we have

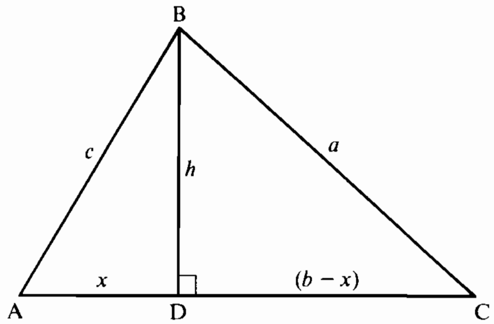
$$\frac{ⓐ}{\sin A} = \frac{ⓑ}{\sin B} = \frac{c}{\sin Ⓒ}$$

and it is clear that no pair of fractions contains only one unknown quantity.

Some other method is therefore needed in such circumstances and the one we use is called the *cosine rule*. This rule states that

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Proof



Let ABC be a non-right-angled triangle in which BD is drawn perpendicular to AC. Taking x as the length of AD, the length of CD is $(b - x)$. Then, using h as the length of BD, we can use Pythagoras' theorem to find h in each of the right-angled triangles BDA and BDC, i.e.

$$h^2 = c^2 - x^2 \quad \text{and} \quad h^2 = a^2 - (b - x)^2$$

Therefore $\quad c^2 - x^2 = a^2 - (b - x)^2$

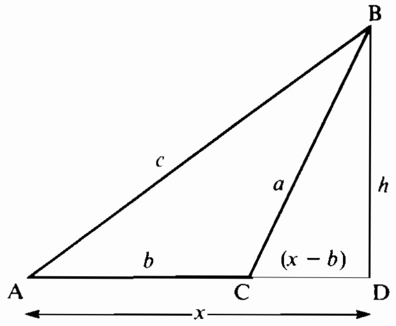
$\Rightarrow \quad c^2 - x^2 = a^2 - b^2 + 2bx - x^2$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2bx$

But $x = c\cos A$

Therefore $\quad a^2 = b^2 + c^2 - 2bc\cos A$

The proof is equally valid for an obtuse-angled triangle, as is shown overleaf.



In this case the length of CD is $(x-b)$, so in $\triangle$BCD we have

$$h^2 = a^2 - (x-b)^2 = a^2 - x^2 + 2bx - b^2$$

This is identical to the expression found for h^2 above.

The remainder of the proof above is unchanged so we have now proved that, in *any* triangle,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

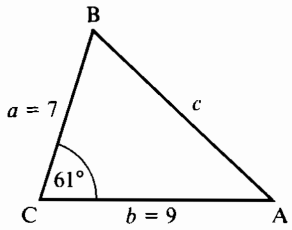
When the altitude is drawn from A or from C similar expressions for the other sides of a triangle are obtained, i.e.

$$b^2 = c^2 + a^2 - 2ca\cos B$$

and $$c^2 = a^2 + b^2 - 2ab\cos C$$

Examples 7e

1. In $\triangle$ABC, BC = 7 cm, AC = 9 cm and C = 61°. Find AB.



Using the cosine rule, starting with c^2, we have

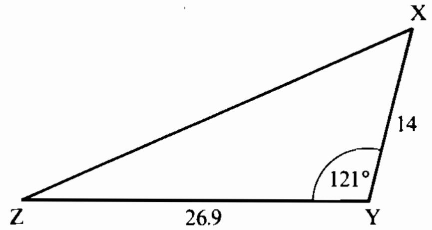
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$\Rightarrow$ $$c^2 = 7^2 + 9^2 - (2)(7)(9)(0.4848)$$

$\Rightarrow$ $$c = 8.302$$

Hence AB = 8.30 cm correct to 3 s.f.

2. XYZ is a triangle in which $\angle$Y = 121°, XY = 14 cm and YZ = 26.9 cm. Find XZ.



Using $y^2 = z^2 + x^2 - 2zx\cos Y$ gives

$$y^2 = (14)^2 + (26.9)^2 - (2)(14)(26.9)(-0.5150)$$

Note that, because Y is an obtuse angle, it has a negative cosine. Extra care therefore has to be taken with the sign of the term $-2zx\cos Y$. The best way to avoid mistakes is to enclose the cosine in brackets as shown.

Hence $y^2 = 1307.51 \Rightarrow y = 36.16$

Therefore XZ = 36.2 cm correct to 3 s.f.

EXERCISE 7e

In each question use the data given for $\triangle$PQR to find the length of the third side.

	PQ	QR	RP	P	Q	R
1.		8 cm	4.6 cm			39°
2.	11.7 cm		9.2 cm	75°		
3.	29 cm	37 cm			109°	
4.		2.1 cm	3.2 cm			97°
5.	135 cm		98 cm	48°		
6.	4.7 cm	8.1 cm			138°	
7.		44 cm	62 cm			72°
8.	19.4 cm		12.6 cm	167°		

Using the Cosine Rule to Find an Angle

So far the cosine rule has been used only to find an unknown side of a triangle. When we want to find an unknown angle, it is advisable to rearrange the formula to some extent.
The version of the cosine rule that starts with c^2,

i.e. $$c^2 = a^2 + b^2 - 2ab\cos C$$

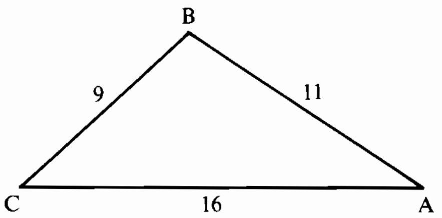
can be written as $$2ab\cos C = a^2 + b^2 - c^2$$

and further as $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The reader should find this last form quite easy to remember if it is noted that the side opposite to the angle being found, c^2 in this case, appears only once as the last term in the formula. Some readers however may prefer to work from the basic cosine formula for all calculations, carrying out any necessary manipulation in each problem as it arises.

Examples 7f

1. If, in $\triangle$ABC, $a = 9$, $b = 16$ and $c = 11$, find, to the nearest degree, the largest angle in the triangle.



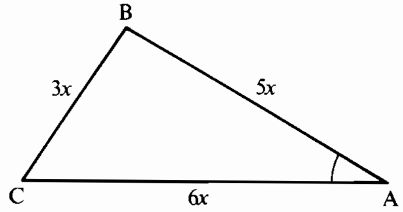
The largest angle in a triangle is opposite to the longest side, so in this question we are looking for angle B and we use

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$
$$= \frac{121 + 81 - 256}{(2)(11)(9)}$$
$$= -0.2727$$

The negative sign shows that $\angle$B is obtuse.

Hence B = 106° and this is the largest angle in $\triangle$ABC.

2. The sides a, b, c of a triangle ABC are in the ratio 3:6:5 Find the smallest angle in the triangle.



The actual lengths of the sides are not necessarily 3, 6 and 5 units so we represent them by $3x$, $6x$ and $5x$. The smallest angle is A (opposite to the smallest side).

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{36x^2 + 25x^2 - 9x^2}{60x^2}$$
$$= \frac{52}{60}$$
$$= 0.8667$$

Therefore the smallest angle in $\triangle$ABC is 30°

EXERCISE 7f

1. In $\triangle$XYZ, XY = 34 cm, YZ = 29 cm and ZX = 21 cm. Find the smallest angle in the triangle.

2. In $\triangle$PQR, PQ = 1.3 cm, QR = 1.8 cm and RP = 1.5 cm. Find $\angle$Q.

3. In $\triangle$ABC, AB = 51 cm, BC = 37 cm and CA = 44 cm. Find $\angle$A.

4. Find the largest angle in $\triangle$XYZ given that $x = 91$, $y = 77$ and $z = 43$

5. What is the size of (a) the smallest, (b) the largest angle in $\triangle$ABC if $a = 13$, $b = 18$ and $c = 7$?

6. In $\triangle PQR$ the sides PQ, QR and RP are in the ratio 2:1:2 Find $\angle P$.

7. ABCD is a quadrilateral in which AB = 5 cm, BC = 8 cm, CD = 11 cm, DA = 9 cm and angle ABC = 120° Find the length of AC and the size of the angle ADC.

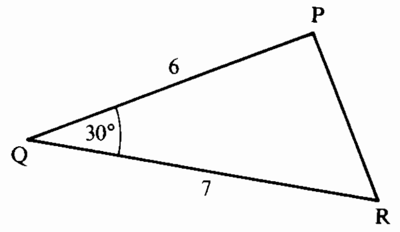
GENERAL TRIANGLE CALCULATIONS

If three independent facts are given about the sides and/or angles of a triangle and further facts are required, a choice must be made between using the sine rule or the cosine rule for the first step.

As the sine rule is easier to work out, it is preferred to the cosine rule whenever the given facts make this possible, i.e. whenever an angle and the opposite side are known. (Remember that if two angles are given, then the third angle is also known.)

The cosine rule is used only when the sine rule is not suitable and it is never necessary to use it more than once in solving a triangle.

Suppose, for example, that the triangle PQR given below is to be solved.



Only one angle is known and the side opposite to it is not given. We must therefore use the cosine rule first to find the length of PR.

Once we know q as well as $\angle Q$, the sine rule can be used to find either of the remaining angles, the third angle then following from the sum of the angles in the triangle.

EXERCISE 7g

Each of the following questions refers to a triangle ABC. Fill in the blank spaces in the table.

	$\angle A$	$\angle B$	$\angle C$	a	b	c
1.		80°	50°			68 cm
2.			112°	15.7 cm	13 cm	
3.	41°	69°		12.3 cm		
4.	58°				131 cm	87 cm
5.		49°	94°		206 cm	
6.	115°		31°			21 cm
7.	59°	78°		17 cm		
8.		48°	80°		31.3 cm	
9.	77°				19 cm	24 cm
10.		125°		14 cm		20 cm

11. A tower stands on level ground. From a point P on the ground, the angle of elevation of the top of the tower is 26° Another point Q is 3 m vertically above P and from this point the angle of elevation of the top of the tower is 21° Find the height of the tower.

12. A survey of a triangular field, bounded by straight fences, found the three sides to be of lengths 100 m, 80 m and 65 m. Find the angles between the boundary fences.

MIXED EXERCISE 7

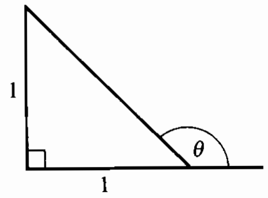
1. Find the value, between 0° and 180°, of $\angle A$ if
   (a) $\cos A = -\cos 64°$ (b) $\sin 94° = \sin A$

2. If $\angle X$ is acute and $\sin X = \frac{7}{25}$, find $\cos(180° - X)$

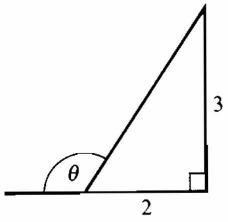
3. Given that $\sin A = \frac{5}{8}$, find $\tan A$ in surd form if
   (a) $\angle A$ is acute (b) $\angle A$ is obtuse

4. Find, in surd form, $\sin\theta$ and $\cos\theta$, given

(a)



(b)



5. Given that $\sin X = \frac{12}{13}$ and X is obtuse, find $\cos X$.

6. In $\triangle$ABC, BC = 11 cm, $\angle$B = 53° and $\angle$A = 76°, find AC.

7. In $\triangle$PQR, $p = 3$, $q = 5$ and R = 69°, find r

8. In $\triangle$XYZ, XY = 8 cm, YZ = 7 cm and ZX = 10 cm, find $\angle$Y.

9. In $\triangle$ABC, AB = 7 cm, BC = 6 cm and $\angle$A = 44°, find all possible values of $\angle$ACB.

10. Find the angles of a triangle whose sides are in the ratio 2:4:5

11. Use the cosine formula, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, to show that

(a) $\angle$A is acute if $a^2 < b^2 + c^2$

(b) $\angle$A is obtuse if $a^2 > b^2 + c^2$

CHAPTER 8

TRIANGLES

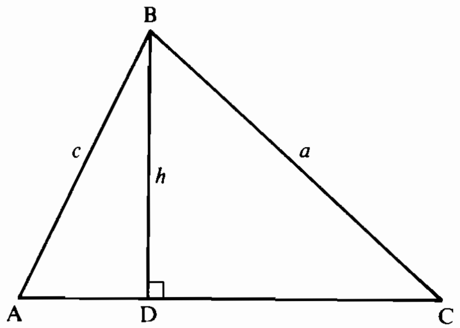
THE AREA OF A TRIANGLE

The simplest way to find the area of a triangle is to use the formula

$$\text{Area} = \tfrac{1}{2}\,\text{base} \times \text{perpendicular height}$$

Clearly this is of immediate use only when the perpendicular height is known. It can be adapted, however, to cover other cases.

Consider the triangle shown below, in which b, c, and A are known



The line BD, drawn from B perpendicular to AC, is the height, h, of the triangle, so the area of the triangle is $\frac{1}{2}bh$

In the triangle ADB, $\quad \sin A = \dfrac{h}{c} \Rightarrow h = c\sin A$

Therefore the area of triangle ABC is

$$\tfrac{1}{2}bc\sin A$$

Drawing the perpendicular heights from A to C give similar expressions, i.e.

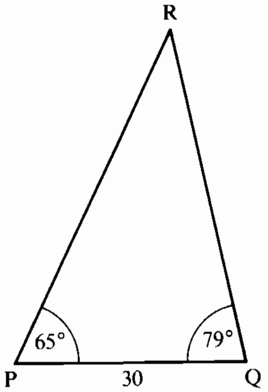
$$\text{Area of triangle ABC} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B$$

Each of these formulae can be expressed in the 'easy to remember' form

$$\text{Area} = \tfrac{1}{2}\,\text{product of two sides} \times \text{sine of included angle}$$

Example 8a

Find the area of triangle PQR, given that $P = 65°$, $Q = 79°$ and $PQ = 30$ cm.



The given facts do not include two sides and the included angle so we must first find another side. To do this the sine rule can be used and we need angle R.

$$\angle R = 180° - 65° - 79° = 36°$$

From the sine rule, $\dfrac{p}{\sin P} = \dfrac{r}{\sin R}$

$$\Rightarrow \qquad p = \frac{30 \times \sin 65°}{\sin 36°} = 46.26$$

i.e. QR = 46.3 cm (correct to 3 s.f.).

Now we can use area $PQR = \frac{1}{2}pr\sin Q$

$$\tfrac{1}{2}pr\sin Q = \tfrac{1}{2} \times 46.26 \times 30 \times \sin 79 = 681.2$$

So the area of triangle PQR is 681 cm^2 (corr to 3 s.f.).

EXERCISE 8a

Find the area of each triangle given in Questions 1 to 5.

1. $\triangle XYZ$; $XY = 180$ cm, $YZ = 145$ cm, $\angle Y = 70°$
2. $\triangle ABC$; $AB = 75$ cm, $AC = 66$ cm, $\angle A = 62°$
3. $\triangle PQR$; $QR = 69$ cm, $PR = 49$ cm, $\angle R = 85°$
4. $\triangle XYZ$; $x = 30$, $y = 40$, $\angle Z = 49°$
5. $\triangle PQR$; $p = 9$, $r = 11$, $\angle Q = 120°$
6. In triangle ABC, $AB = 6$ cm, $BC = 7$ cm and $CA = 9$ cm. Find $\angle A$ and the area of the triangle.

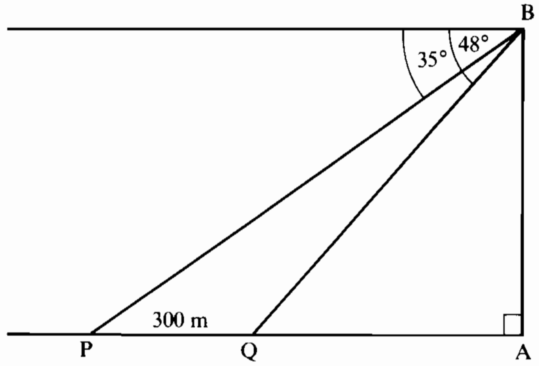
7. $\triangle PQR$ is such that $\angle P = 60°$, $\angle R = 50°$ and $QR = 12$ cm. Find PQ and the area of the triangle.
8. In $\triangle XYZ$, $XY = 150$ cm, $YZ = 185$ cm and the area is 11 000 cm^2. Find $\angle Y$ and XZ.
9. The area of triangle ABC is 36.4 cm^2. Given that $AC = 14$ cm and $\angle A = 98°$, find AB.

PROBLEMS

Many practical problems which involve distances and angles can be illustrated by a diagram. Often, however, this diagram contains too many lines, dimensions, etc. to be clear enough to work from. In these cases we can draw a second figure by extracting a triangle (or triangles) in which three facts about sides and/or angles are known. The various methods given in Chapter 7 can then be used to analyse this triangle and so to solve the problem.

Examples 8b

1. Two boats, P and Q, are 300 m apart. The base, A, of a lighthouse is in line with PQ. From the top, B, of the lighthouse the angles of depression of P and Q are found to be 35° and 48°. Write down the values of the angles BQA, PBQ and BPQ and find, correct to the nearest metre, the height of the lighthouse.

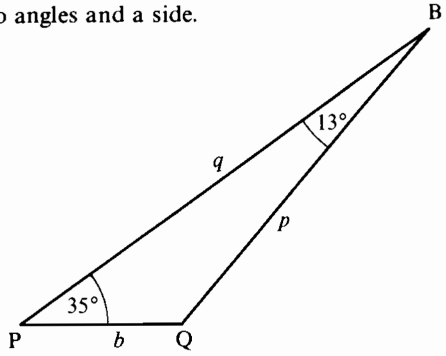


$\angle BQA = 48°$ (alternate angles)

$\angle PBQ = 48° - 35° = 13°$

$\angle BPQ = 35°$ (alternate angles)

Now we can extract $\triangle PBQ$, knowing two angles and a side.



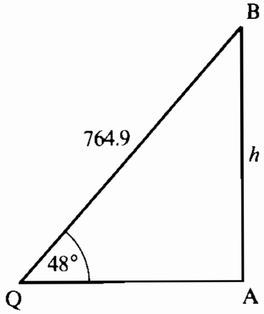
From the sine rule,

$$\frac{p}{\sin P} = \frac{b}{\sin B}$$

$$\therefore \quad p = \frac{(300)(\sin 35°)}{\sin 13°}$$

$$= 764.9$$

We can now use the right-angled triangle ABQ



$$h = p \sin 48°$$

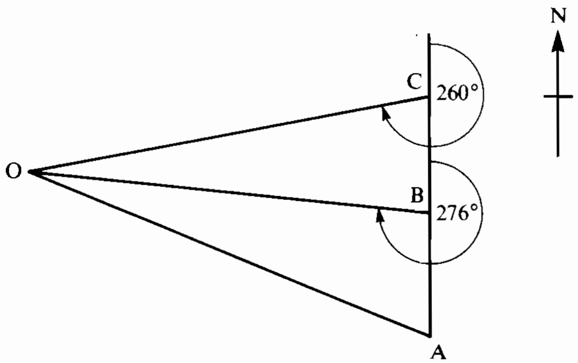
$$= (764.9)(\sin 48°)$$

$$= 568.4$$

Correct to the nearest metre the height of the light-house is 568 m

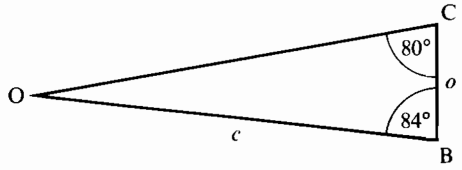
---

2. A traveller pitches camp in a desert. He knows that there is an oasis in the distance, but cannot see it. Wishing to know how far away it is, he measures 250 m due north from his starting point, A, to a point B where he can see the oasis, O, and finds that its bearing is 276°. He then measures a further 250 m due north to point C from which the bearing of the oasis is 260°. Find how far from the oasis he has camped.



$\angle OCB = 260° - 180° = 80°$ and $\angle OBC = 360° - 276° = 84°$

As two angles and a side are known, $\triangle OBC$ can be used.

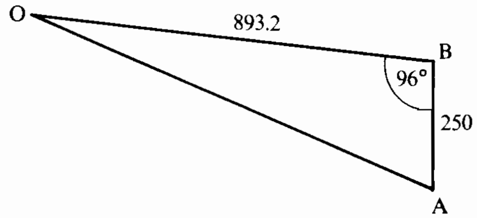


$$\angle BOC = 180° - 80° - 84° = 16°$$

From the sine rule, $\dfrac{c}{\sin C} = \dfrac{o}{\sin O}$

$$\Rightarrow \quad c = \frac{250 \times \sin 80°}{\sin 16°} = 893.2$$

Now in $\triangle ABO$, $\angle ABO = 276° - 180° = 96°$ and we also know OB and AB. As two sides and the included angle are known, it is the cosine rule that must be used.



$$OA^2 = OB^2 + AB^2 - 2 \times OB \times AB \times \cos ABO$$

$$= (893.2)^2 + (250)^2 - 2 \times 893.2 \times 250 \times \cos 96°$$

$$= 906\,989$$

$$\Rightarrow \quad OA = 952.4$$

To the nearest metre the initial distance from the oasis was 952 m

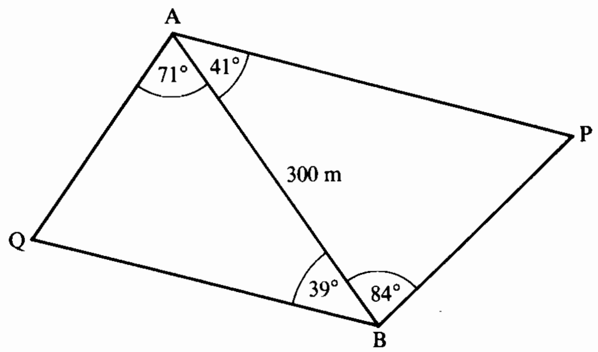
---

EXERCISE 8b

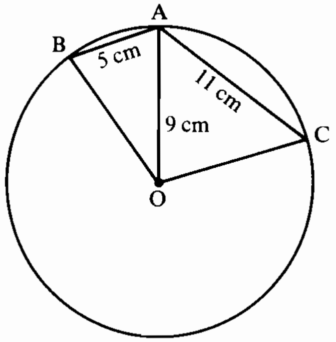
1. In a quadrilateral PQRS, PQ = 6 cm, QR = 7 cm, RS = 9 cm, $\angle PQR = 115°$ and $\angle PRS = 80°$. Find the length of PR. Considering it as split into two separate triangles, find the area of the quadrilateral PQRS.

2. A light aircraft flies from an airfield, A, a distance of 50 km on a bearing of 049° to a town, B. The pilot then changes course and flies on a bearing of 172° to a landing strip, C, 68 km from B. How far is the landing strip from the airfield?

3. In a surveying exercise, P and Q are two points on land which is inaccessible. To find the distance PQ, a line AB of length 300 metres is marked out so that P and Q are on opposite sides of AB. The directions of P and Q relative to the line AB are then measured and are shown in the diagram. Calculate the length of PQ. (Hint. Find AP and AQ.)

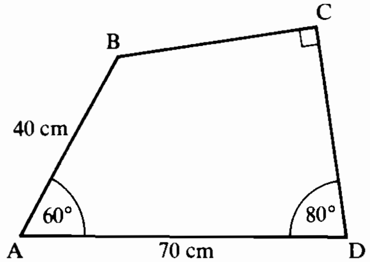


4.



AB, of length 5 cm, and AC, of length 11 cm, are two chords of a circle with centre O and radius 9 cm. Find each of the angles BAO and CAO and hence calculate the area of the triangle ABC.

5.



The diagram shows the cross section of a beam of length 2 m. Calculate

(a) the length of BD

(b) the angle ADB

(c) the length of CD

(d) the area of the cross section

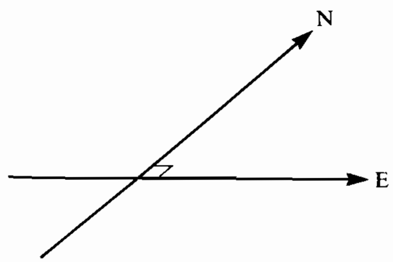
(e) the volume of the beam.

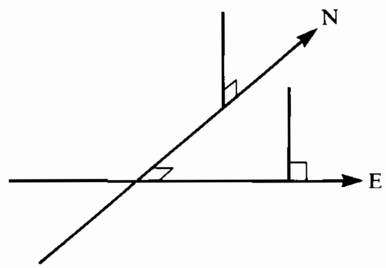
THREE-DIMENSIONAL PROBLEMS

One of the difficulties which many people experience with this topic, arises when attempting to illustrate a three-dimensional situation on a two-dimensional diagram. The following hints may help in producing a clear representation of the 3-D problem from which appropriate calculations can be made.

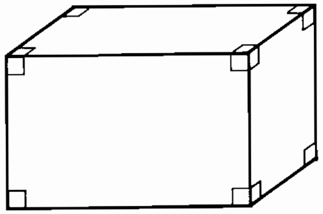
1) Vertical lines should be drawn vertically on the page.

2) Lines in the East-West direction should be drawn horizontally on the page. North-South lines are shown as inclined at an acute angle to the East direction.



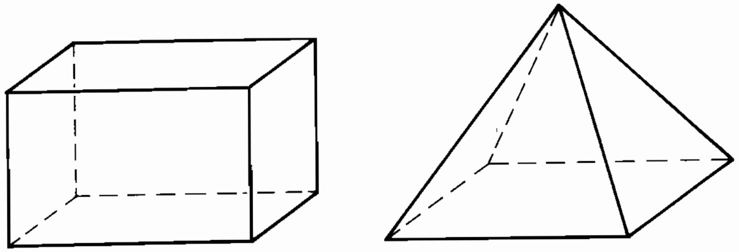


3) All angles that are 90° in three dimensions should be marked as right angles on the diagram, particularly those that do not *appear* to be 90°



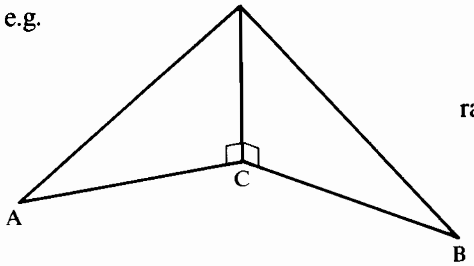
4) Perspective drawing is rarely used, so parallel lines are drawn parallel in the diagram.

5) When viewing a 3-D object, some of its sides are usually not visible. It is helpful to indicate these by broken lines.

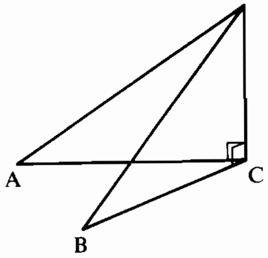


6) In a situation involving two points in the foreground and an object in the background it is usually clearer to draw the object *between* the two points.

e.g.



rather than

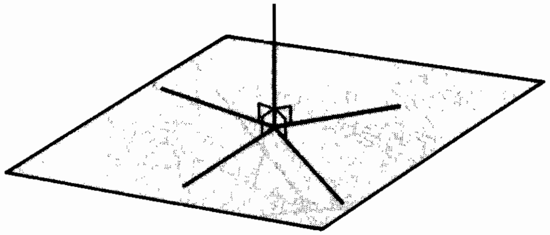


7) It is often helpful to draw a separate diagram showing each individual triangle in which calculations are needed.

The following facts and definitions should also be known.

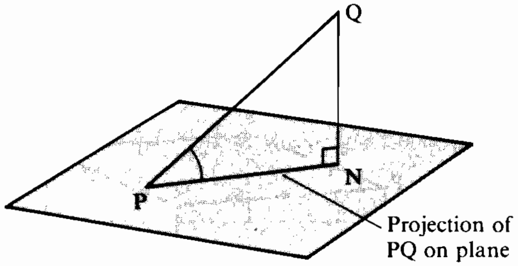
1) Two non-parallel planes meet in a line called the common line.

2) A line that is perpendicular to a plane is also perpendicular to every line in that plane

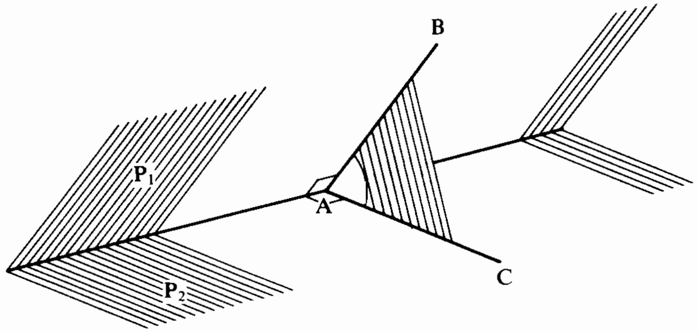


and if a line is perpendicular to two non-parallel lines in a plane, then it is perpendicular to the plane.

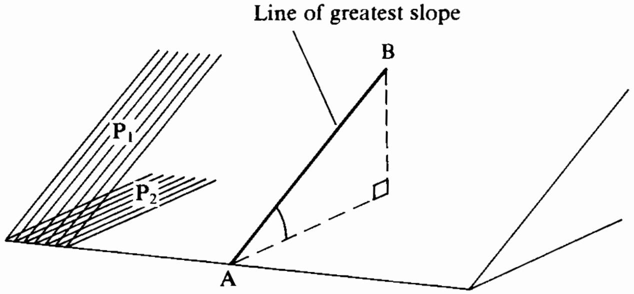
3) The angle between a line and a plane is defined as the angle between that line and its projection on the plane. (Its projection can be thought of as the shadow of the line cast on the plane by a beam of light shining at right-angles to the plane.)



4) The angle between two planes is defined as follows. From any point A, on the common line of two planes P_1 and P_2, lines AB and AC are drawn, one in each plane, perpendicular to the common line. Then angle BAC is the angle between the two planes.

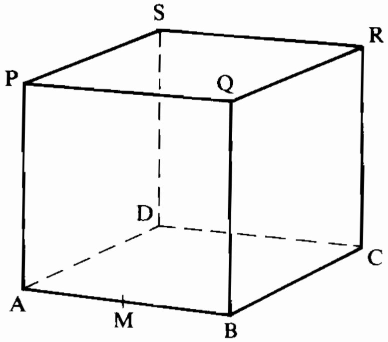


5) If, in fact 4, one of the planes, P_2 say, is horizontal, then AB is called *a line of greatest slope* of the plane P_1

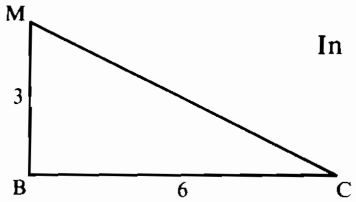


Examples 8c

1. The diagram shows a cube of side 6 cm. M is the midpoint of AB. Find
   (a) the length of MC (b) the length of MR
   (c) to the nearest degree, the angle between MR and the plane ABCD.



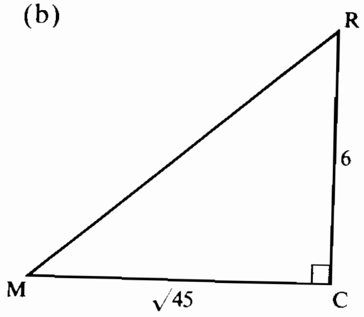
(a)



In $\triangle$MBC, $(MC)^2 = 6^2 + 3^2$
$= 45$
$\Rightarrow$ $MC = \sqrt{45}$

Therefore the length of MC is 6.71 cm (correct to 3.s.f).

(b)

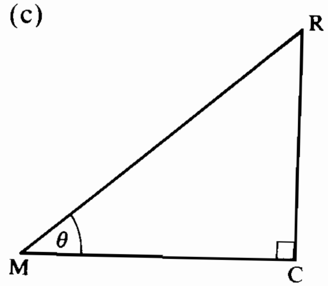


In $\triangle$MCR, $(MR)^2 = (MC)^2 + 6^2$
$= 45 + 36$
$= 81$
$\Rightarrow$ $MR = 9$

Therefore the length of MR is 9 cm.

To find the angle between RM and the plane ABCD, we need the projection of RM on the plane. As RC is perpendicular to ABCD, it follows that CM is the projection of RM on that plane. So the angle we are looking for is RMC.

(c)



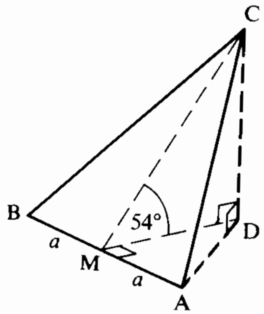
In $\triangle$RMC, $\sin\theta = \frac{6}{9} = 0.6667$
$\Rightarrow$ $\theta = 41.8°$

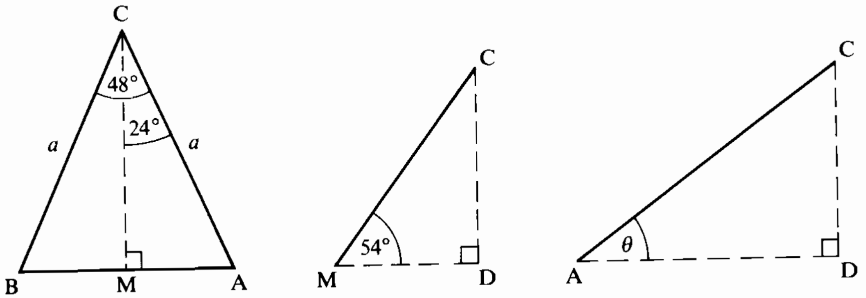
Therefore, to the nearest degree, the angle between RM and the plane ABCD is 42°

---

2. The base AB of an isosceles triangle ABC is horizontal. The plane containing the triangle is inclined to the horizontal at 54° If the angle ACB is 48°, find the angle between AC and the horizontal plane.

AB is the line common to the horizontal plane and the plane of the triangle and M is its midpoint. Because the triangle is isosceles, CM is perpendicular to AB. So CM is a line of greatest slope and therefore makes an angle of 54° with its projection, MD, on the horizontal plane.





Let the length of AC and BC be a

In $\triangle$CMA, $\quad CM = a\cos 24°$

In $\triangle$CDM, $\quad CD = CM\sin 54°$

$$= (a\cos 24°)(\sin 54°)$$

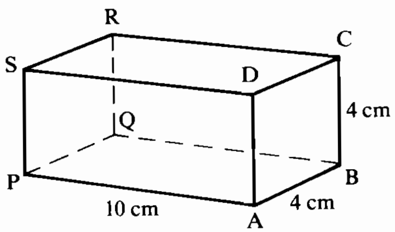
The angle between AC and the horizontal plane is the angle between AC and its projection AD on that plane. If this angle is θ then

$$\sin\theta = \frac{CD}{CA} = \frac{(a)(\cos 24°)(\sin 54°)}{a}$$

$$\Rightarrow \quad \theta = 47.7°$$

EXERCISE 8c

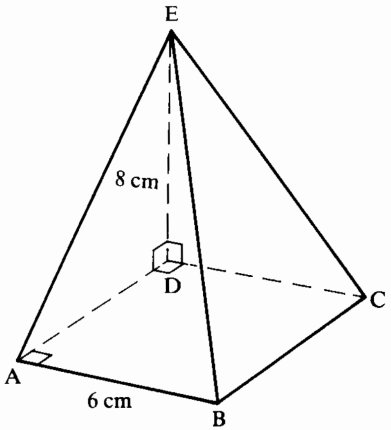
1.



In the cuboid shown above, ABCD is a square of side 4 cm and PA = 10 cm. Find the length of

(a) AC (b) AS (c) AQ (d) a diagonal of the cuboid.

2.



The pyramid ABCDE has a square base of side 6 cm. E is 8 cm vertically above D. Calculate

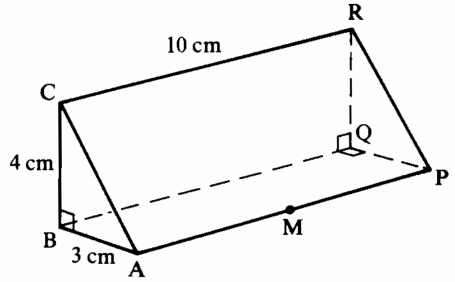
(a) the lengths of AD, BD and BE

(b) the angle between AE and the plane ABCD

(c) the angle between BE and the plane ABCD

(d) the angle between the planes EBC and ABCD.

3.



Given the triangular prism in the diagram, in which M is the midpoint of AP, find the following lengths and angles.

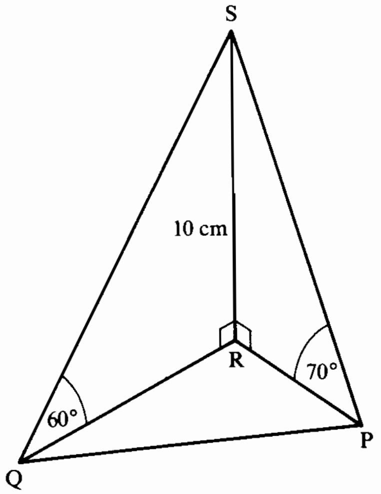
(a) RM (b) RA (c) QA

(d) the angle between RA and the plane ABQP

(e) the angle between RM and the plane ABQP.

4. Given a regular tetrahedron (i.e. a pyramid where each face is an equilateral triangle), find the cosine of the angle between two of the faces.

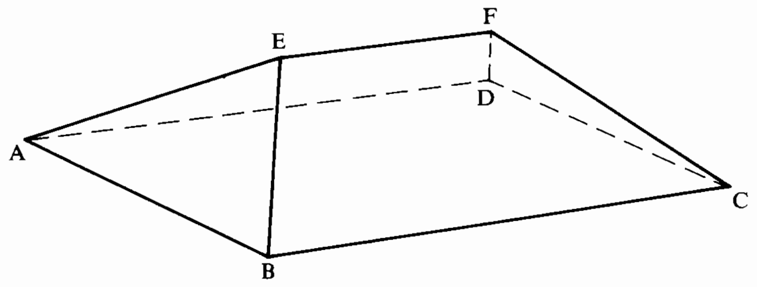
5.



Three points, P, Q and R lie in a plane. The line RS is perpendicular to the plane and is of length 10 cm. If angle SPR = 70°, angle SQR = 60° and PQ = 7 cm, calculate each of the angles in triangle PQR.

6. Find the angle between two diagonals of a cube.

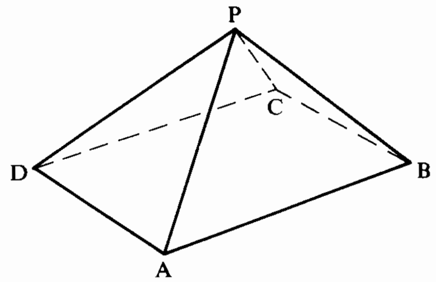
7.



The diagram shows a roof whose base is a rectangle of length 15 m and width 9 m. Each end face is an isosceles triangle inclined to the horizontal at an angle, α, and each long face is a trapezium inclined to the horizontal at an angle β
If $\tan\alpha = 2$ and $\tan\beta = \frac{4}{3}$, calculate

(a) the height of the ridge (EF) above the base
(b) the length of the ridge
(c) the angle between AE and the horizontal
(d) the total surface area of the roof.

8.

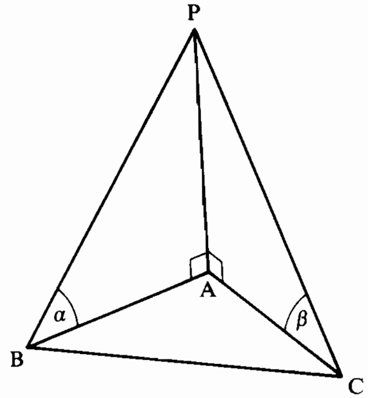


The diagram shows a solid figure in which ABCD is a horizontal rectangle. AB = 13 cm, BC = 8 cm, AP = DP = 9 cm and BP = CP = 7 cm. Calculate

(a) the length of AC
(b) the height of P above the plane ABCD
(c) the angle between AP and the horizontal
(d) the angle between the faces APB and ABCD.

9. An aircraft is noted simultaneously by three observers, A, B and C, stationed in a horizontal straight line. AB and BC are each 200 m and the noted angles of elevation of the aircraft from A and C are 25° and 40° respectively. What is the height of the aircraft? Find also the angle of elevation of the aircraft from B.

10.



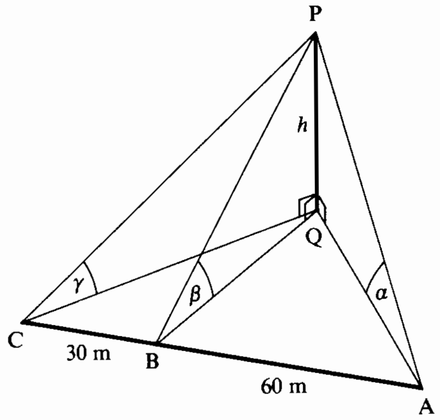
ABC is a horizontal triangle in which BC = 10 m. P is a point 12 m vertically above A. The angles of elevation of P from B and C are α and β, where $\tan\alpha = 1$ and $\tan\beta = \frac{6}{7}$. Find the angle between the planes PBC and ABC.

Harder Problems

Certain problems in three dimensions are rather more demanding than those seen up to now. An example of such a problem is given below and, in the following Mixed Exercise, Questions 8 to 10 are also a little harder. It is recommended that they be used at a later date for revision.

Example 8d

A, B and C are points on a horizontal line such that AB = 60 m and BC = 30 m. The angles of elevation, from A, B and C respectively, of the top of a clock tower are α, β, and γ, where $\tan\alpha = \frac{1}{13}$, $\tan\beta = \frac{1}{15}$ and $\tan\gamma = \frac{1}{20}$. The foot of the tower is at the same level as A, B and C. Find the height of the tower.



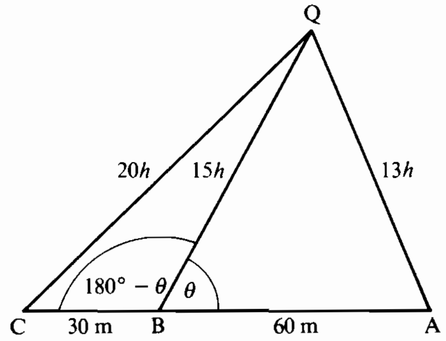
If the height of the tower, PQ, is h then

$$h = QA\tan\alpha = QB\tan\beta = QC\tan\gamma$$

i.e. $$h = \frac{QA}{13} = \frac{QB}{15} = \frac{QC}{20}$$

$\Rightarrow$ QA = $13h$, QB = $15h$, QC = $20h$

Now considering the base triangle ABCQ, we have



Using the cosine rule in $\triangle$ABQ gives

$$\cos\theta = \frac{(60)^2 + (15h)^2 - (13h)^2}{2(60)(15h)}$$

and using the cosine rule in $\triangle$CBQ gives

$$\cos(180° - \theta) = -\cos\theta = \frac{(30)^2 + (15h)^2 - (20h)^2}{2(30)(15h)}$$

$$\therefore \quad \frac{(60)^2 + (15h)^2 - (13h)^2}{2(60)(15h)} = \frac{(20h)^2 - (30)^2 - (15h)^2}{2(30)(15h)}$$

$$\Rightarrow \quad 3600 + 56h^2 = 2(175h^2 - 900)$$

$$\Rightarrow \quad 5400 = 294h^2$$

Hence $$h = 4.285$$

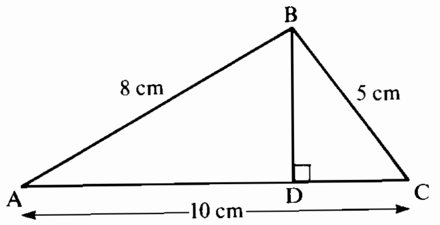
The height of the clock tower is 4.29 m (correct to 3 s.f.).

MIXED EXERCISE 8

1. In $\triangle$PQR, PQ = 11 cm, PR = 14 cm and QPR = 100°. Find the area of the triangle.

2. The area of ABC is 9 cm^2. If AB = AC = 6 cm, find sin A. Are there two possible triangles? Give a reason for your answer.

3.



Given the information in the diagram,

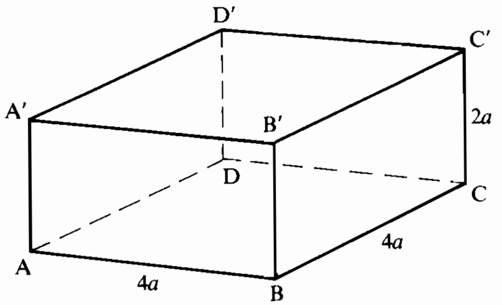
(a) find $\angle ABC$

(b) find the area of $\triangle ABC$

(c) *hence* find the length of BD.

4. Triangle PQR, in which PRQ = 120°, lies in a horizontal plane and X is a point 6 cm vertically above R. If XQR = 45° and XPR = 60°, find the lengths of the sides of $\triangle PQR$. Find also the area of this triangle.

5.



Given the cuboid shown in the diagram, find

(a) the angle between AC and the plane ABB′A′

(b) the angle between the planes ACD′ and ABCD.

6. Two rectangular panels, ABCD and ABEF, each measure 1.5 m by 2 m. They are hinged along the edge AB which is 2 m long. If the angle between their planes is 60°, find the angle between the diagonals AC and AE.

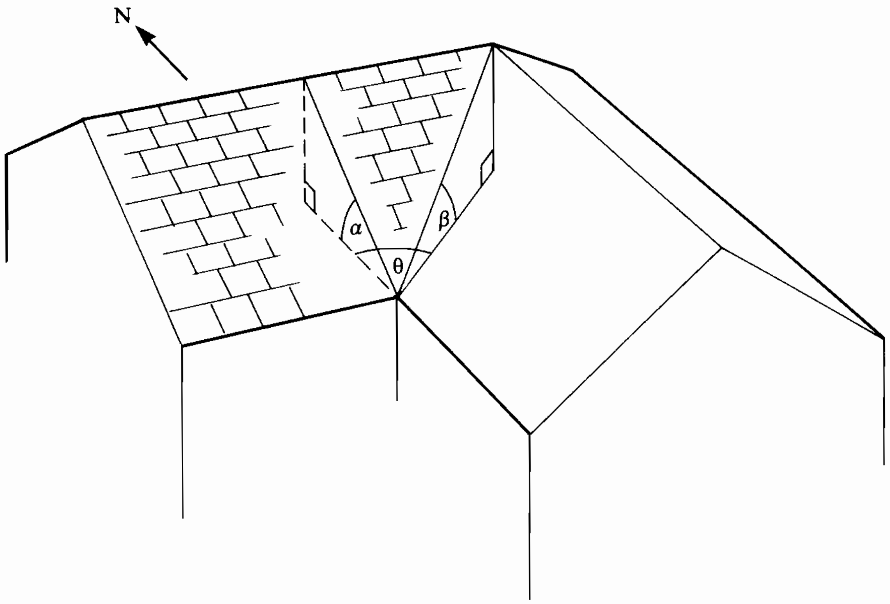
7. A river running due east has straight parallel banks. A vertical post stands with its base, P, on the north bank of the river. On the south bank are two surveyors, A who is to the east, and B who is to the west of the post. A and B are at a distance $\frac{2}{7}a$ apart and the angle APB is 150° The angles of elevation from A and B of the top, Q, of the post are 45° and 30° Find, in terms of a, the width of the river and the height of the post.

The remaining questions are a little more demanding.

8. ABCD is a tetrahedron whose horizontal base ABC is an equilateral triangle. The angle between each pair of slant edges is θ where $\tan\theta = \frac{5}{12}$ and the length of these edges is a. Find the height of D above ABC.

9. Two lamp standards are each of height h m. They are d m apart on level ground where a man of height t m is also standing. Each light casts a shadow of the man on the ground. Prove that, no matter where the man stands, the distance between the ends of his two shadows is $\dfrac{dt}{h-t}$ m. (Hint. Look for similar triangles.)

10. The roof of a south-facing house slopes down at an angle α to the horizontal. A gulley at the end of the roof is in the direction θ east of north. If the gulley is inclined to the horizontal at an angle β, show that $\tan\beta = \tan\alpha\cos\theta$



CHAPTER 9

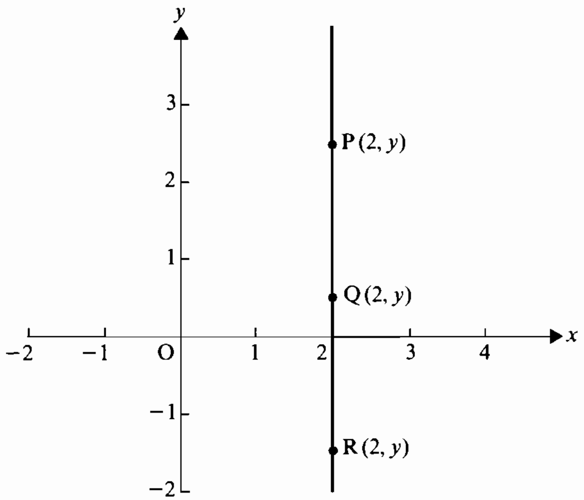
STRAIGHT LINES 1

THE MEANING OF EQUATIONS

The Cartesian frame of reference provides a means of defining the position of any point in a plane. This plane is called the xy-plane.

In general x and y are independent variables. This means that they can each take any value independently of the value of the other unless some restriction is placed on them.

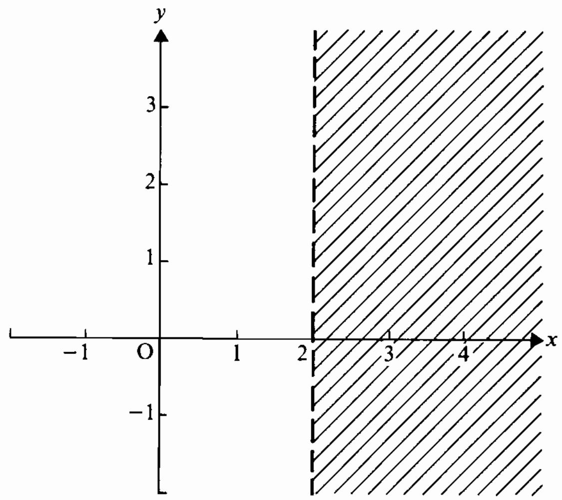
Consider the case when the value of x is restricted to 2



As the value of y is not restricted, the condition above gives a set of points which form a straight line parallel to the y-axis and passing through P, Q and R as shown. Therefore the condition that x is equal to 2, i.e. $x = 2$, defines the line through P, Q and R, i.e.

> in the context of the xy-plane, the equation $x = 2$ defines the line shown in the diagram. Further, $x = 2$ is called *the equation of this line* and we can refer briefly to *the line* $x = 2$

Now consider the set of points for which the condition is $x > 2$



All the points to the right of the line $x = 2$ have an x-coordinate that is greater than 2

So the inequality $x > 2$ defines the shaded region of the xy-plane shown above.
Similarly, the inequality $x < 2$ defines the region left unshaded in the diagram.

Note that the region defined by $x > 2$ does not include the line $x = 2$

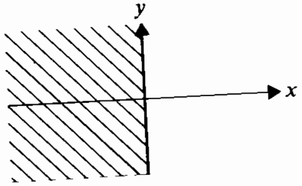
When a region does not include a boundary line this is drawn as a broken line. When the points on a boundary *are* included in a region, this boundary is drawn as a solid line.

Example 9a

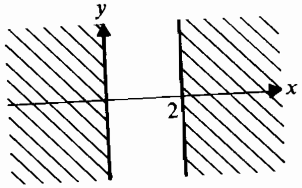
Draw a sketch of the region of the xy-plane defined by the inequalities $0 \leqslant x \leqslant 2$ and $0 < y < 4$

The relationship $0 \leqslant x \leqslant 2$ contains two inequalities which must be considered separately, i.e. $x \geqslant 0$ and $x \leqslant 2$. Similarly, $0 < y < 4$ contains two relationships, i.e. $y > 0$ and $y < 4$. The required region is found by considering each inequality in turn and shading the unwanted region. This leaves the required region clear.

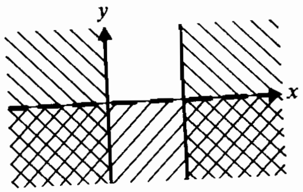
$x \geqslant 0$ defines both the line $x = 0$ (i.e. the y-axis) and the region to the right of the y-axis ($x > 0$) so we shade the region to the left of the y-axis.



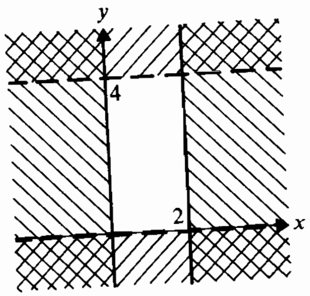
$x \leqslant 2$ defines both the line $x = 2$ and the region to the left of the line.



$y > 0$ defines the region above the x-axis, but does not include the x-axis.



$y < 4$ defines the region below the line $y = 4$ but does not include the line $y = 4$



Therefore the unshaded region in the bottom figure, including the solid boundary lines but not the broken ones, is the set of points that satisfy all the given inequalities.

Note that in this book, we use the convention of shading unwanted regions when dealing with more than one inequality in a plane. This convention is not universal and some problems ask for such a diagram to be drawn with the required region shaded. In this case we recommend that the reader follows the procedure used in the worked example and then redraws the diagram to shade the required region.

EXERCISE 9a

1. Draw a sketch showing the lines defined by the equations $x = 5$, $x = -3$, $y = 0$, $y = 6$

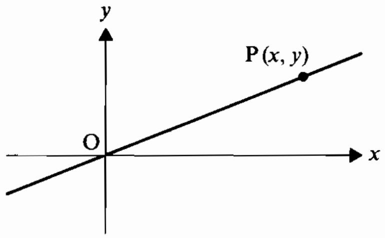
2. Draw a sketch showing the lines defined by the equations $y = -3$, $y = -10$, $x = 7$, $x = -5$

Draw a sketch showing the region of the xy-plane defined by the following inequalities.

3. $x > 3$	**4.** $y \leqslant 2$	**5.** $x \geqslant -8$
6. $0 < x < 5$	**7.** $-1 < y < 4$	**8.** $-2 \leqslant x < 2$
9. $0 \leqslant x \leqslant 5,\ -1 \leqslant y \leqslant 3$	**10.** $x \leqslant -3,\ x \geqslant 4,\ y < -2$	

THE EQUATION OF A STRAIGHT LINE

A straight line may be defined in many ways; for example, a line passes through the origin and has a gradient of $\frac{1}{2}$.



The point $P(x,y)$ is on this line if and only if the gradient of OP is $\frac{1}{2}$

In terms of x and y, the gradient of OP is $\dfrac{y}{x}$, so the statement above can be written in the form

$$P(x,y) \text{ is on the line if and only if } \frac{y}{x} = \frac{1}{2}, \text{ i.e. } 2y = x$$

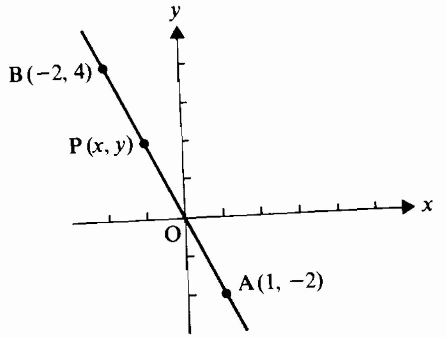
Therefore the coordinates of points on the line satisfy the relationship $2y = x$, and the coordinates of points that are not on the line do not satisfy this relationship.

$2y = x$ is called the equation of the line.

The equation of a line (straight or curved) is a relationship between the x and y coordinates of all points on the line and which is not satisfied by any other point in the plane.

Examples 9b

1. Find the equation of the line through the points $(1, -2)$ and $(-2, 4)$.



$P(x, y)$ is on the line if and only if the gradient of PA is equal to the gradient of AB (or PB).

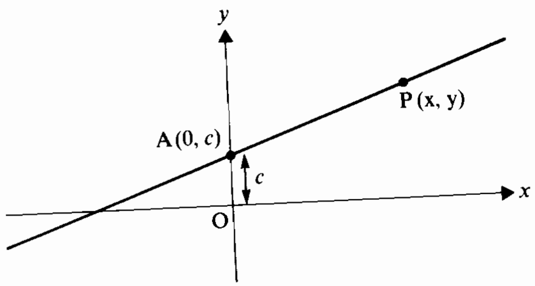
The gradient of PA is $\dfrac{y-(-2)}{x-1} = \dfrac{y+2}{x-1}$

The gradient of AB is $\dfrac{-2-4}{1-(-2)} = -2$

Therefore the coordinates of P satisfy the equation $\dfrac{y+2}{x-1} = -2$

i.e. $$y + 2x = 0$$

Consider the more general case of the line whose gradient is m and which cuts the y-axis at a directed distance c from the origin. Note that c is called the *intercept* on the y-axis.



Now $P(x, y)$ is on this line if and only if the gradient of AP is m

Therefore the coordinates of P satisfy the equation $\dfrac{y-c}{x-0} = m$

i.e. $$y = mx + c$$

This is the *standard form* for the equation of a straight line.

An equation of the form $y = mx + c$ represents a straight line with gradient m and intercept c on the y-axis.

Because the value of m and/or c may be fractional, this equation can be rearranged and expressed as $ax + by + c = 0$, i.e.

$$ax + by + c = 0$$

where a, b and c are constants, is the equation of a straight line.

Note that in this form c is *not* the intercept.

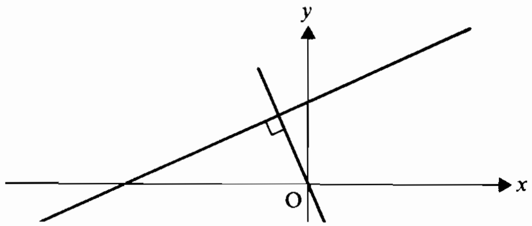
Examples 9b (continued)

2. Write down the gradient of the line $3x - 4y + 2 = 0$ and find the equation of the line through the origin which is perpendicular to the given line.

Rearranging $3x - 4y + 2 = 0$ in the standard form gives

$$y = \tfrac{3}{4}x + \tfrac{1}{2}$$

Comparing with $y = mx + c$ we see that the gradient (m) of the line is $\frac{3}{4}$ (and the intercept on the y-axis is $\frac{1}{2}$).



The gradient of the perpendicular line is given by $-\dfrac{1}{m} = -\dfrac{4}{3}$

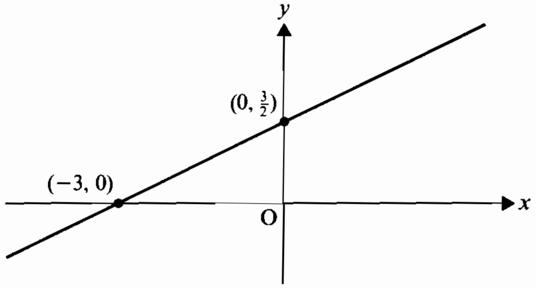
and it passes through the origin so the intercept on the y-axis is 0

Therefore its equation is $\quad y = -\frac{4}{3}x + 0$

$\Rightarrow \quad 4y + 3x = 0$

3. Sketch the line $x - 2y + 3 = 0$

This line can be located accurately in the xy-plane when we know two points on the line. We will use the intercepts on the axes as these can be found easily (i.e. $x = 0 \Rightarrow y = \frac{3}{2}$ and $y = 0 \Rightarrow x = -3$).



Notice that the diagrams in the worked examples are sketches, not accurate plots, but they show reasonably accurately the position of the lines in the plane.

EXERCISE 9b

1. Write down the equation of the line through the origin and with gradient
   (a) 2 (b) -1 (c) $\frac{1}{3}$ (d) $-\frac{1}{4}$ (e) 0 (f) ∞
   Draw a sketch showing all these lines on the same set of axes.

2. Write down the equation of the line passing through the given point and with the given gradient.
   (a) $(0, 1)$, $\frac{1}{2}$ (b) $(0, 0)$, $-\frac{2}{3}$ (c) $(-1, -4)$, 4
   Draw a sketch showing all these lines on the same set of axes.

3. Write down the equation of the line passing through the points
   (a) $(0, 0)$, $(2, 1)$ (b) $(1, 4)$, $(3, 0)$ (c) $(-1, 3)$, $(-4, -3)$

4. Write down the equation of the line passing through the origin and perpendicular to
   (a) $y = 2x + 3$ (b) $3x + 2y - 4 = 0$ (c) $x - 2y + 3 = 0$

5. Write down the equation of the line passing through $(2, 1)$ and perpendicular to
   (a) $3x + y - 2 = 0$ (b) $2x - 4y - 1 = 0$
   Draw a sketch showing all four lines on the same set of axes.

6. Write down the equation of the line passing through $(3, -2)$ and parallel to
   (a) $5x - y + 3 = 0$ (b) $x + 7y - 5 = 0$

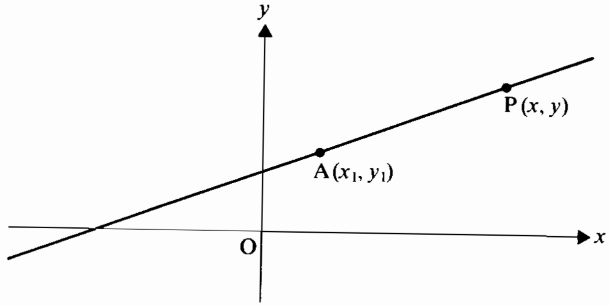
7. A$(1, 5)$ and B$(4, 9)$ are two adjacent vertices of a square. Find the equation of the line on which the side BC of the square lies. How long are the sides of this square?

Formulae for Finding the Equation of a Line

Straight lines play a major role in graphical analysis and it is important to be able to find their equations easily. This section gives two formulae derived from the commonest ways in which a straight line is defined.

The appropriate formula can then be used to write down the equation of a particular line.

The equation of a line with gradient m and passing through the point (x_1, y_1)

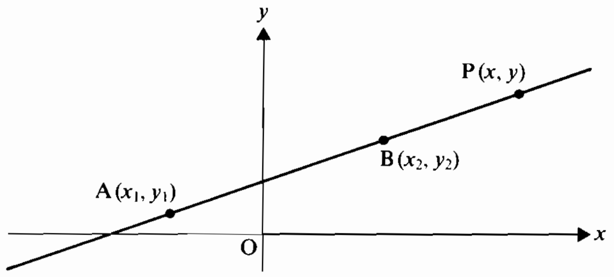


$P(x, y)$ is a point on the line if and only if the gradient of AP is m

i.e. $$\frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow \quad y - y_1 = m(x - x_1) \quad [1]$$

The equation of the line passing through (x_1, y_1) and (x_2, y_2)



The gradient of AB is $\dfrac{y_2 - y_1}{x_2 - x_1}$

so the formula given in [1] becomes

$$y - y_1 = \left[\frac{y_2 - y_1}{x_2 - x_1}\right](x - x_1) \qquad [2]$$

Examples 9c

1. Find the equation of the line with gradient $-\frac{1}{3}$ and passing through $(2, -1)$

Using [1] with $m = -\frac{1}{3}$, $x_1 = 2$ and $y_1 = -1$ gives

$$y - (-1) = -\tfrac{1}{3}(x - 2)$$

$\Rightarrow$ $$x + 3y + 1 = 0$$

Alternatively the equation of this line can be found from the standard form of the equation of a straight line, i.e. $y = mx + c$

Using $y = mx + c$ and $m = -\frac{1}{3}$ we have

$$y = -\tfrac{1}{3}x + c$$

The point $(2, -1)$ lies on this line so its coordinates satisfy the equation, i.e.

$$-1 = -\tfrac{1}{3}(2) + c \quad \Rightarrow \quad c = -\tfrac{1}{3}$$

Therefore $$y = -\tfrac{1}{3}x - \tfrac{1}{3}$$

$\Rightarrow$ $$x + 3y + 1 = 0$$

2. Find the equation of the line through the points $(1, -2)$, $(3, 5)$

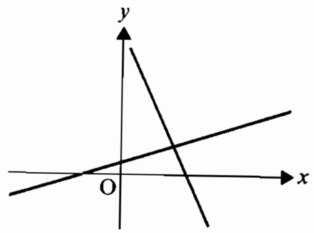
Using formula [2] with $x_1 = 1$, $y_1 = -2$, $x_2 = 3$ and $y_2 = 5$ gives

$$y - (-2) = \frac{-2 - 5}{1 - 3}(x - 1)$$

$\Rightarrow$ $$7x - 2y - 11 = 0$$

The worked examples in this book necessarily contain a lot of explanation but this should not mislead readers into thinking that their solutions must be equally long. The temptation to 'overwork' a problem should be avoided, particularly in the case of coordinate geometry problems which are basically simple. With practice, any of the methods illustrated above enable the equation of a straight line to be written down directly.

3. Find the equation of the line through $(1, 2)$ which is perpendicular to the line $3x - 7y + 2 = 0$



Expressing $3x - 7y + 2 = 0$ in standard form gives $y = \frac{3}{7}x + \frac{2}{7}$

Hence the given line has gradient $\frac{3}{7}$.

So the required line has a gradient of $-\frac{7}{3}$ and it passes through $(1, 2)$

Using $y - y_1 = m(x - x_1)$ gives its equation as

$$y - 2 = -\tfrac{7}{3}(x - 1) \quad \Rightarrow \quad 7x + 3y - 13 = 0$$

In the last example note that the line perpendicular to

$$3x - 7y + 2 = 0$$

has equation $\quad 7x + 3y - 13 = 0$

i.e. the coefficients of x and y have been transposed and the sign between the x and y terms has changed. This is a particular example of the general fact that

given a line with equation $ax + by + c = 0$ then the equation of any perpendicular line is $bx - ay + k = 0$

This property of perpendicular lines can be used to shorten the working of problems,
e.g. to find the equation of the line passing through $(2, -6)$ which is perpendicular to the line $5x - y + 3 = 0$, we can say that the required line has an equation of the form $y + 5x + k = 0$ and then use the fact that the coordinates $(2, -6)$ satisfy this equation to find the value of k.

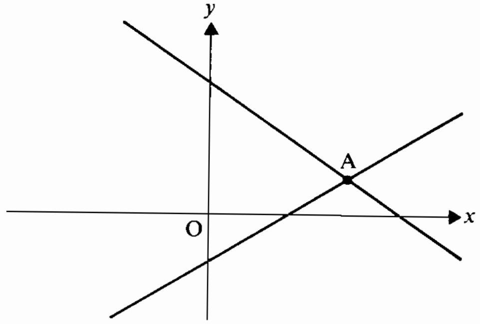
EXERCISE 9c

1. Find the equation of the line with the given gradient and passing through the given point.
   (a) $3, (4, 9)$ (b) $-5, (2, -4)$ (c) $\frac{1}{4}, (4, 0)$
   (d) $0, (-1, 5)$ (e) $-\frac{2}{5}, (\frac{1}{2}, 4)$ (f) $-\frac{3}{8}, (\frac{22}{5}, -\frac{5}{2})$
2. Find the equation of the line passing through the points
   (a) $(0, 1), (2, 4)$ (b) $(-1, 2), (1, 5)$ (c) $(3, -1), (3, 2)$
3. Determine which of the following pairs of lines are perpendicular.
   (a) $x - 2y + 4 = 0$ and $2x + y - 3 = 0$
   (b) $x + 3y - 6 = 0$ and $3x + y + 2 = 0$
   (c) $x + 3y - 2 = 0$ and $y = 3x + 2$
   (d) $y + 2x + 1 = 0$ and $x = 2y - 4$
4. Find the equation of the line through the point $(5, 2)$ and perpendicular to the line $x - y + 2 = 0$
5. Find the equation of the perpendicular bisector of the line joining
   (a) $(0, 0), (2, 4)$ (b) $(3, -1), (-5, 2)$ (c) $(5, -1), (0, 7)$
6. Find the equation of the line through the origin which is parallel to the line $4x + 2y - 5 = 0$

7. The line $4x - 5y + 20 = 0$ cuts the x-axis at A and the y-axis at B. Find the equation of the median through O of $\triangle$OAB.
8. Find the equation of the altitude through O of the triangle OAB defined in Question 7.
9. Find the equation of the perpendicular from $(5, 3)$ to the line $2x - y + 4 = 0$
10. The points A$(1, 4)$ and B$(5, 7)$ are two adjacent vertices of a parallelogram ABCD. The point C$(7, 10)$ is another vertex of the parallelogram. Find the equation of the side CD.

INTERSECTION

The point where two lines (or curves) cut is called a point of intersection.



If A is the point of intersection of the lines $\quad y - 3x + 1 = 0 \quad [1]$

and $\quad y + x - 2 = 0 \quad [2]$

then the coordinates of A satisfy both of these equations. A can be found by solving [1] and [2] simultaneously, i.e.

[2] − [1] $\Rightarrow \quad 4x - 3 = 0 \quad \Rightarrow \quad x = \frac{3}{4}$ and $y = \frac{5}{4}$

Therefore $(\frac{3}{4}, \frac{5}{4})$ is the point of intersection.

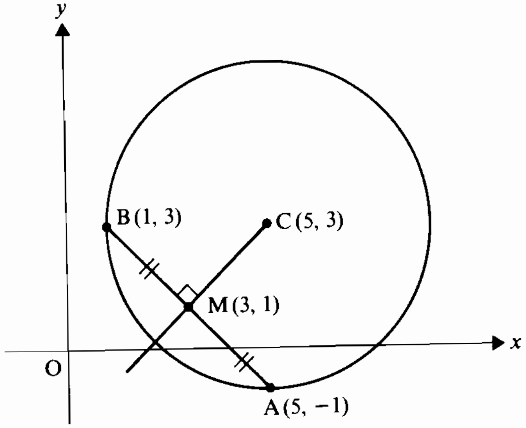
Note that the coordinates of A can also be found using a graphics calculator or a graph drawing package on a computer.

Example 9d

A circle has radius 4 and its centre is the point C(5, 3).

(a) Show that the points A(5, −1) and B(1, 3) are on the circumference of the circle.

(b) Prove that the perpendicular bisector of AB goes through the centre of the circle.



(a) From the diagram, BC = 4

$\therefore$ B is on the circumference.

Similarly AC = 4,

$\therefore$ A is on the circumference.

(b) The midpoint, M, of AB is $\left[\frac{5+1}{2}, \frac{-1+3}{2}\right]$ i.e. (3, 1)

The gradient of AB is $\frac{-1-3}{5-1} = -1$

If l is the perpendicular bisector of AB, its gradient is 1 and it goes through (3, 1).

$\therefore$ the equation of l is

$$y - 1 = 1(x - 3) \quad \Rightarrow \quad y = x - 2 \qquad [1]$$

In equation [1], when $x = 5$, $y = 3$

$\therefore$ the point (5, 3) is on l,

i.e. the perpendicular bisector of AB goes through C.

MIXED EXERCISE 9

1. Show that the triangle whose vertices are (1, 1), (3, 2) and (2, −1) is isosceles.

2. Find the area of the triangular region enclosed by the x and y axes and the line $2x - y - 1 = 0$

3. Find the coordinates of the triangular region enclosed by the lines $y = 0$, $y = x + 5$ and $x + 2y - 6 = 0$

4. Write down the equation of the perpendicular bisector of the line joining the points $(2, -3)$ and $(-\frac{1}{2}, 3\frac{1}{2})$

5. Find the equation of the line through A(5, 2) which is perpendicular to the line $y = 3x - 5$. Hence find the coordinates of the foot of the perpendicular from A to the line.

6. Find, in terms of a and b, the coordinates of the foot of the perpendicular from the point (a, b) to the line $x + 2y - 4 = 0$

7. The coordinates of a point P are $(t + 1, 2t - 1)$. Sketch the position of P when $t = -1$, 0, 1 and 2 Show that these points are collinear and write down the equation of the line on which they lie.

8. Write down the equation of the line which goes through (7, 3) and which is inclined at 45° to the positive direction of the x-axis.

9. Find the equation of the perpendicular bisector of the line joining the points (a, b) and $(2a, -3b)$

10. The centre of a circle is at the point C(3, 7) and the point A(5, 3) is on the circumference of the circle. Find
    (a) the radius of the circle,
    (b) the equation of the line through A that is perpendicular to AC.

11. The equations of two sides of a square are $y = 3x - 1$ and $x + 3y - 6 = 0$. If $(0, -1)$ is one vertex of the square find the coordinates of the other vertices.

12. The lines $y = 2x$, $2x + y - 12 = 0$ and $y = 2$ enclose a triangular region of the xy-plane. Find
    (a) the coordinates of the vertices of this region,
    (b) the area of this region.

CHAPTER 10

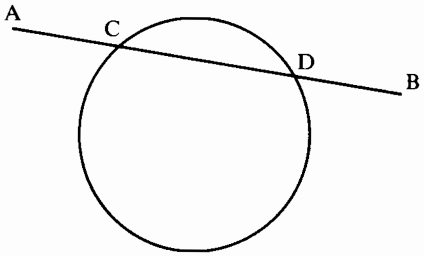
CIRCLE GEOMETRY

PARTS OF A CIRCLE

We start this chapter with a reminder of the language used to describe parts of a circle.

Part of the circumference is called an *arc*.
If the arc is less than half the circumference it is called a *minor arc*; if it is greater than half the circumference it is called a *major arc*.

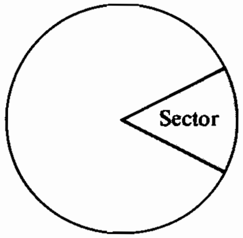
A straight line which cuts a circle in two distinct points is called a *secant*. The part of the line inside the circle is called a *chord*.

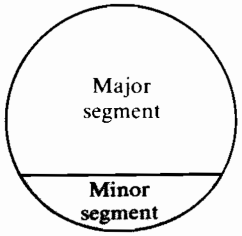


AB is a secant,
CD is a chord.

The area enclosed by two radii and an arc is called a *sector*.

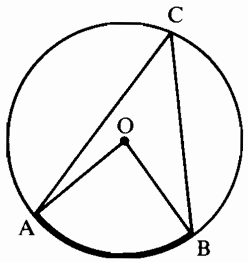
The area enclosed by a chord and an arc is called a *segment*. If the segment is less than half a circle it is called a *minor segment*; if it is greater than half a circle it is called a *major segment*.





THE ANGLE SUBTENDED BY AN ARC

Consider the points A, B and C on the circumference of a circle whose centre is O.



We say that $\angle ACB$ stands on the minor arc AB.

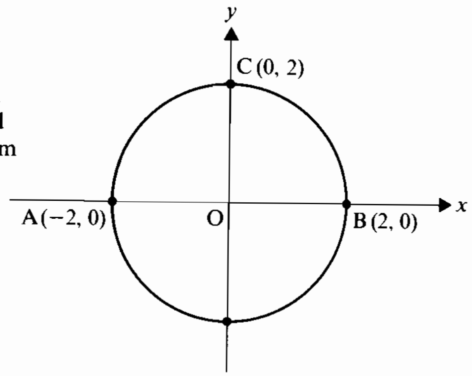
The minor arc AB is said to *subtend* the angle ACB at the circumference (and the angle is *subtended* by the arc).

In the same way, the arc AB is said to subtend the angle AOB at the centre of the circle.

Example 10a

A circle of radius 2 units which has its centre at the origin, cuts the x-axis at the points A and B and cuts the y-axis at the point C. Prove that $\angle ACB = 90°$

All the information given in the question, and gleaned from the known properties of the figure, can be marked in the diagram as shown. The diagram can then be referred to as justification for steps taken in the solution.



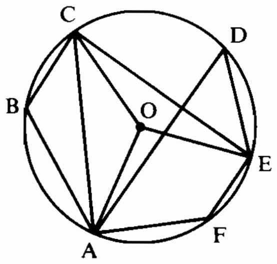
From the diagram, the gradient of AC is $\dfrac{2-0}{0-(-2)} = 1$

and the gradient of BC is $\dfrac{2-0}{0-2} = -1$

$\therefore$ (gradient of AC) $\times$ (gradient of BC) $= -1$

i.e. AC is perpendicular to BC $\Rightarrow$ $\angle ACB = 90°$

EXERCISE 10a

1. 

Name the angles subtended
(a) at the circumference by the minor arc AE
(b) at the circumference by the major arc AE
(c) at the centre by the minor arc AC
(d) at the circumference by the major arc AC
(e) at the centre by the minor arc CE
(f) at the circumference by the minor arc CD
(g) at the circumference by the minor arc BC.

2. AB is a chord of a circle, centre O, and M is its midpoint. The radius from O is drawn through M. Prove that OM is perpendicular to AB.

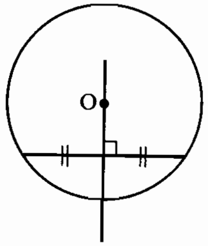
3. C(5, 3) is the centre of a circle of radius 5 units.
(a) Show that this circle cuts the x-axis at A(1, 0) and B(9, 0)
(b) Prove that the radius that is perpendicular to AB goes through the midpoint of AB.
(c) Find the angle subtended at C by the minor arc AB.
(d) The point D is on the major arc AB and DC is perpendicular to AB. Find the coordinates of D and hence find the angle subtended at D by the minor arc AB.

4. A and B are two points on the circumference of a circle centre O. C is a point on the major arc AB. Draw the lines AC, BC, AO, BO and CO, extending the last line to a point D inside the sector AOB. Prove that $\angle$AOD is twice $\angle$ACO and that $\angle$BOD is twice angle $\angle$BCO. Hence show that the angle subtended by the minor arc AB at the centre of the circle is twice the angle that it subtends at the circumference of the circle.

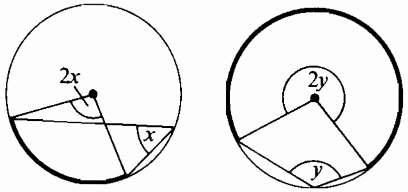
ANGLES IN A CIRCLE

The solutions to questions in the last exercise illustrate two important results.

1) The perpendicular bisector of a chord of a circle goes through the centre of the circle.

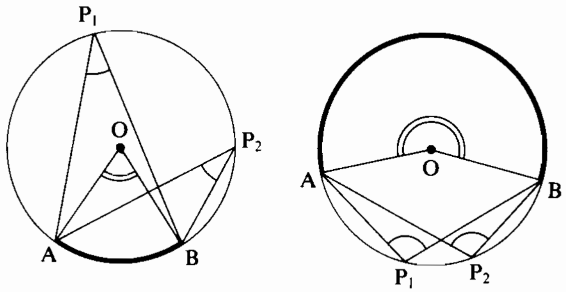


2) The angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference by the same arc.



Further important results follow from the last fact.

3)



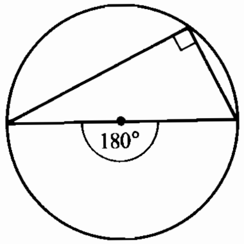
In both diagrams, $\angle AOB = 2\angle P_1 = 2\angle P_2$

So it follows that $\angle P_1 = \angle P_2$

i.e.

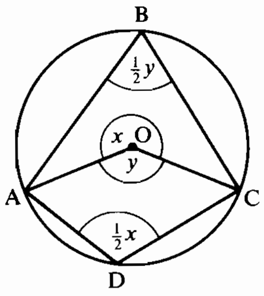
all angles subtended at the circumference by the same arc are equal.

4) A semicircle subtends an angle of 180° at the centre of the circle; therefore it subtends an angle half that size, i.e. 90°, at any point on the circumference. This angle is called the angle in a semicircle.



Hence the angle in a semicircle is 90°

5) If all four vertices of a quadrilateral ABCD lie on the circumference of a circle, ABCD is called a *cyclic quadrilateral.*



In the diagram, O is the centre of the circle.

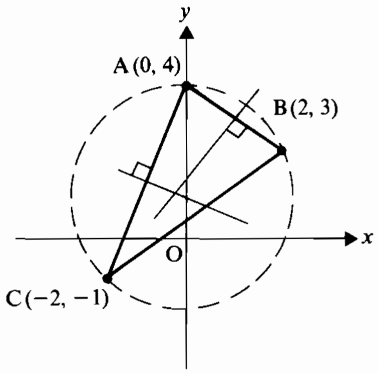
$\therefore$ $\angle ADC = \frac{1}{2}x$ and $\angle ABC = \frac{1}{2}y$

But $x + y = 360°$, therefore $\angle ADC + \angle ABC = 180°$

i.e. the opposite angles of a cyclic quadrilateral are supplementary

Example 10b

A circle circumscribes a triangle whose vertices are at the points A(0, 4), B(2, 3) and C(−2, −1). Find the centre of the circle.



When a circle circumscribes a figure, the vertices of the figure lie on the circumference of the circle. The centre of the circle can be found by locating the point of intersection of the perpendicular bisectors of two chords.

The gradient of AC is $\dfrac{4-(-1)}{0-(-2)} = \dfrac{5}{2}$

and its midpoint is $\left[\dfrac{0-2}{2}, \dfrac{4-1}{2}\right] \Rightarrow (-1, \frac{3}{2})$,

$\therefore$ the gradient of the perpendicular bisector of AC is $-\frac{2}{5}$ and its equation is

$$y = -\tfrac{2}{5}x + \tfrac{11}{10} \quad\Rightarrow\quad 4x + 10y - 11 = 0 \qquad [1]$$

Similarly the gradient of AB is $-\frac{1}{2}$ and its midpoint is $(1, \frac{7}{2})$

$\therefore$ the gradient of the perpendicular bisector of AB is 2 and its equation is

$$y = 2x + \tfrac{3}{2} \quad\Rightarrow\quad 4x - 2y + 3 = 0 \qquad [2]$$

Solving equations [1] and [2] simultaneously gives

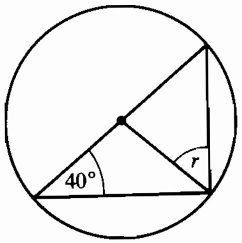
$$12y - 14 = 0 \quad\Rightarrow\quad y = \tfrac{7}{6} \text{ and } x = -\tfrac{1}{6}$$

Therefore the centre of the circle is the point $(-\frac{1}{6}, \frac{7}{6})$.

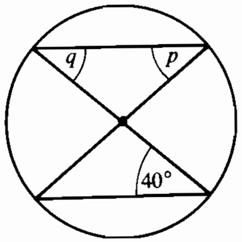
EXERCISE 10b

1. Find the size of each marked angle.

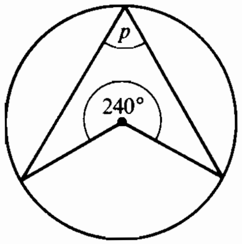
(a)



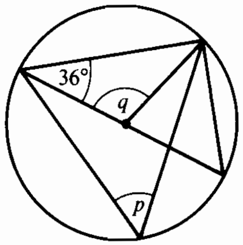
(b)



(c)



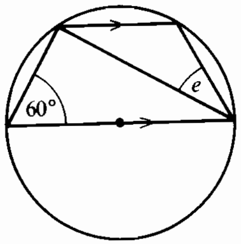
(d)



2. AB is a diameter of a circle centre O. C is a point on the circumference. D is a point on AC such that OD bisects ∠AOC. Prove that OD is parallel to BC.

3. A triangle has its vertices at the points A(1, 3), B(5, 1) and C(7, 5). Prove that △ABC is right-angled and hence find the coordinates of the centre of the circumcircle of △ABC.

4. Find the size of the angle marked e in the diagram.



5. AB and CD are two chords of a circle that cut at E. (E is not the centre of the circle.) Show that △s ACE and BDE are similar.

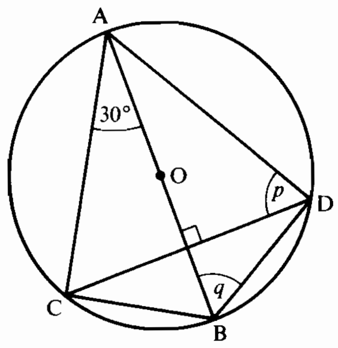
6. A circle with centre O circumscribes an equilateral triangle ABC. The radius drawn through O and the midpoint of AB meets the circumference at D. Prove that △ADO is equilateral.

7. The line joining A(5, 3) and B(4, −2) is a diameter of a circle. If P(a, b) is a point on the circumference find a relationship between a and b

8. ABCD is a cyclic quadrilateral. The side CD is produced to a point E outside the circle. Show that ∠ABC = ∠ADE.

9. A triangle has its vertices at the points A(1, 3), B(−2, 5) and C(4, −2). Find the coordinates of the centre, and correct to 3 s.f., the radius of the circle that circumscribes △ABC.

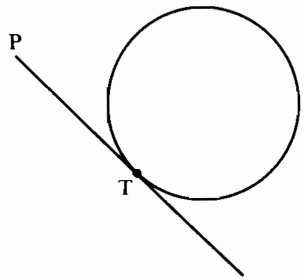
10. In the diagram, O is the centre of the circle and CD is perpendicular to AB. If ∠CAB = 30° find the size of each marked angle.



TANGENTS TO CIRCLES

If a line and a circle are drawn on a plane then there are three possibilities for the position of the line in relation to the circle. The line can miss the circle, or it can cut the circle in two distinct points, or it can touch the circle at one point. In the last case the line is called a *tangent* and the point at which it touches the circle is called the *point of contact*.

The length of a tangent drawn from a point to a circle is the distance from that point to the point of contact.



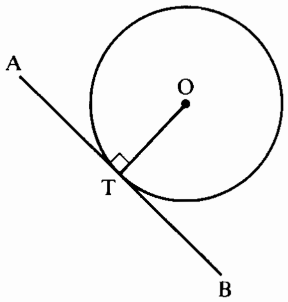
T is the point of contact.

PT is the length of the tangent from P.

Properties of Tangents to Circles

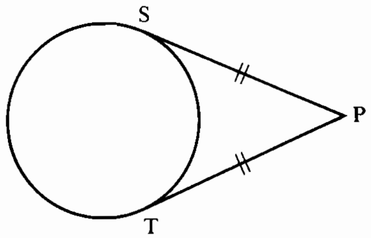
There are two important and useful properties of tangents to circles.

A tangent to a circle is perpendicular to the radius drawn through the point of contact, i.e. AB is perpendicular to OT.



The two tangents drawn from an external point to a circle are equal in length,

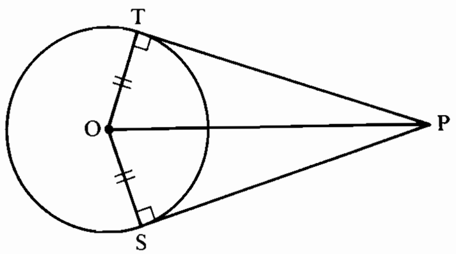
i.e. PS = PT



The second property is proved in the following worked example.

Examples 10c

1. PS and PT are two tangents drawn from a point P to a circle whose centre is O. Prove that PT = PS.



In △s OTP, OSP $\quad \angle T = \angle S = 90°$

OS = OT (radii)

and OP is common

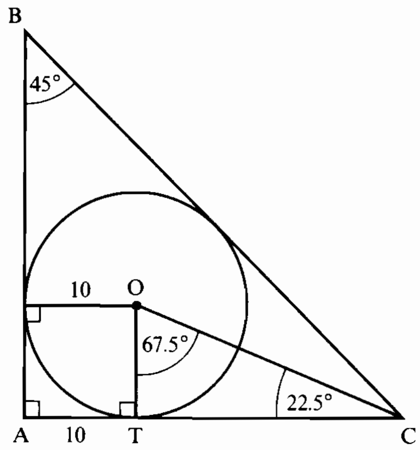
∴ △s OTP, OSP are congruent.

Hence PT = PS.

Another useful property follows from the last example, namely

when two tangents are drawn from a point to a circle, the line joining that point to the centre of the circle bisects the angle between the tangents.

2. A circle of radius 10 units is circumscribed by a right-angled isosceles triangle. Find the lengths of the sides of the triangle.



A circle is *circumscribed* by a figure when all the sides of the figure touch the circle. Note also that the circle is inscribed in the figure.

From the diagram

in △OTC, $\quad TC = 10 \tan 67.5°$

$= 24.14$

$AT = 10$

∴ $\quad AC = 34.14 = AB$

in △ABC, $\quad BC = \sqrt{(34.14^2 + 34.14^2)}$ (Pythagoras)

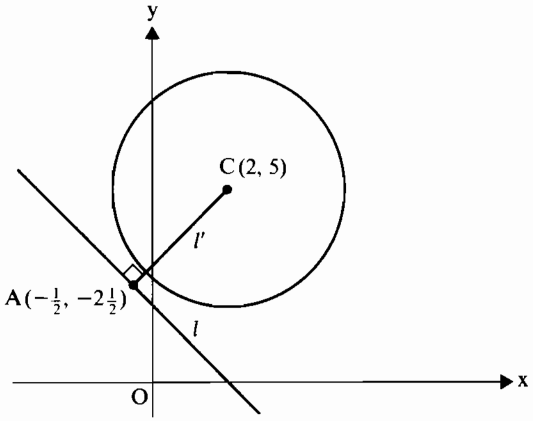
$= 48.28$

∴ correct to 3 s.f. the lengths of the sides of the triangle are 34.1 units, 34.1 units and 48.3 units.

3. The centre of a circle of radius 3 units is the point C(2, 5). The equation of a line, l, is $x + y - 2 = 0$

   (a) Find the equation of the line through C, perpendicular to l

   (b) Find the distance of C from l and hence determine whether l is a tangent to the circle.



(a) The line l' is perpendicular to $x + y - 2 = 0$

so its equation is $x - y + k = 0$

The point (2, 5) lies on l'

$\therefore \quad 2 - 5 + k = 0 \quad \Rightarrow \quad k = 3$

$\therefore$ the equation of l' is $x - y + 3 = 0$

(b) To find the distance of C from l we need the coordinates of A, the point of intersection of l and l'.

Adding the equations of l and l' gives $2x + 1 = 0$

$\Rightarrow \quad x = -\frac{1}{2}$ and $y = \frac{5}{2}$

so A is the point $(-\frac{1}{2}, \frac{5}{2})$

$\therefore$ CA $= \sqrt{[\{2 - (-\frac{1}{2})\}^2 + \{5 - \frac{5}{2}\}^2]} = 3.54$ to 3 s.f.

For the line to be a tangent, CA would have to be 3 units exactly (i.e. equal to the radius).

CA > 3, therefore l is not a tangent.

EXERCISE 10c

1. The two tangents from a point to a circle of radius 12 units are each of length 20 units. Find the angle between the tangents.

2. Two circles with centres C and O have radii 6 units and 3 units respectively and the distance between O and C is less than 9 units. AB is a tangent to both circles, touching the larger circle at A and the smaller circle at B, where AB is of length 4 units. Find the length of OC.

3. The two tangents from a point A to a circle touch the circle at S and at T. Find the angle between one of the tangents and the chord ST given that the radius of the circle is 5 units and that A is 13 units from the centre of the circle.

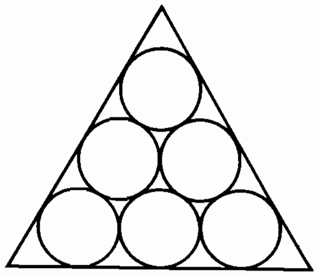
4. An equilateral triangle of side 25 cm circumscribes a circle. Find the radius of the circle.

5. AB is a diameter of the circle and C is a point on the circumference. The tangent to the circle at A makes an angle of 30° with the chord AC. Find the angles in $\triangle$ABC.

6. The centre of a circle is at the point C(4, 8) and its radius is 3 units. Find the length of the tangents from the origin to the circle.

7. A circle touches the y-axis at the origin and goes through the point A(8, 0). The point C is on the circumference. Find the greatest possible area of $\triangle$OAC.

8. A triangular frame is made to enclose six identical spheres as shown. Each sphere has a radius of 2 cm. Find the lengths of the sides of the frame.



9. The line $y = 3x - 4$ is a tangent to the circle whose centre is the point $(5, 2)$. Find the radius of the circle.

10. A circle of radius 6 units has its centre at the point $(9, 0)$. If the two tangents from the origin to the circle are inclined to the x-axis at angles α and β, find $\tan\alpha$ and $\tan\beta$

11. A, B and C are three points on the circumference of a circle. The tangent to the circle at A makes an angle α with the chord AB. The diameter through A cuts the circle again at D and D is joined to B. Prove that $\angle ACB = \alpha$

12. The equations of the sides of a triangle are $y = 3x$, $y + 3x = 0$ and $3y - x + 12 = 0$ Find the coordinates of the circumcircle of this triangle.

13. The line $x - 2y + 4 = 0$ is a tangent to the circle whose centre is the point $C(-1, 2)$.
   (a) Find the equation of the line through C that is perpendicular to the line $x - 2y + 4 = 0$
   (b) Hence find the coordinates of the point of contact of the tangent and the circle.

14. The point $A(6, 8)$ is on the circumference of a circle whose centre is the point $C(3, 5)$. Find the equation of the tangent that touches the circle at A.

CHAPTER 11

RADIANS, ARCS AND SECTORS

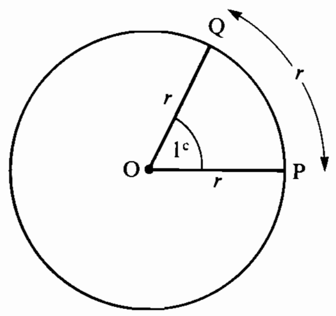
ANGLE UNITS

An angle is a measure of rotation and the units which have been used up to now are the revolution and the degree. It is interesting to note why the number of degrees in a revolution was taken as 360. The ancient Babylonian mathematicians, in the belief that the length of the solar year was 360 days, divided a complete revolution into 360 parts, one for each day as they thought. We now know they did not have the length of the year quite right but the number they used, 360, remains as the number of degrees in one revolution.

Part of an angle smaller than a degree is usually given as a decimal part but until recently the common practice was to divide a degree into 60 minutes $(60')$ and each minute into 60 seconds $(60'')$. Limited use is still made of this system.

Now we consider a different unit of rotation which is of great importance in much of the mathematics that follows.

THE RADIAN



If O is the centre of a circle and an arc PQ is drawn so that its length is equal to the radius of the circle then the angle POQ is called a *radian* (one radian is written 1 rad or 1^c), i.e.

An arc equal in length to the radius of a circle subtends an angle of 1 radian at the centre.

It follows that the number of radians in a complete revolution is the number of times the radius divides into the circumference.

Now the circumference of a circle is of length $2\pi r$

Therefore the number of radians in a revolution is

$$\frac{2\pi r}{r} = 2\pi$$

i.e. $\quad 2\pi \text{ radians} = 360°$

Further $\quad \pi \text{ radians} = 180°$

and $\quad \frac{1}{2}\pi \text{ radians} = 90°$

When an angle is given in terms of π it is usual to omit the radian symbol, i.e. we would write $180° = \pi$ (not $180° = \pi^c$)

If an angle is a simple fraction of 180°, it is easily expressed in terms of π

e.g. $\quad 60° = \frac{1}{3} \text{ of } 180° = \frac{1}{3}\pi$

and $\quad 135° = \frac{3}{4} \text{ of } 180° = \frac{3}{4}\pi$

Conversely, $\quad \frac{7}{6}\pi = \frac{7}{6} \text{ of } 180° = 210°$

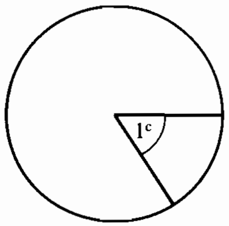
and $\quad \frac{2}{3}\pi = \frac{2}{3} \text{ of } 180° = 120°$

Angles that are not simple fractions of 180°, or of π, can be converted by using the relationship $\pi = 180°$ and the value of π from a calculator,

e.g. $\quad 73° = \frac{73}{180} \times \pi = 1.27^c$ (correct to 3 s.f.)

and $\quad 2.36^c = \frac{2.36}{\pi} \times 180° = 135°$ (correct to the nearest degree)

It helps in visualising the size of a radian to remember that 1 radian is just a little less than 60°.
($180° = \pi \text{ rad} = 3.142 \text{ rad} \quad \Rightarrow \quad 1 \text{ rad} = 180°/3.142 \approx 57°$)



EXERCISE 11a

1. Express each of the following angles in radians as a fraction of π.
   45°, 150°, 30°, 270°, 225°, 22.5°, 240°, 300°, 315°

2. Without using a calculator express each of the following angles in degrees.
   $\frac{1}{6}\pi$, π, $\frac{1}{10}\pi$, $\frac{1}{4}\pi$, $\frac{5}{6}\pi$, $\frac{1}{12}\pi$, $\frac{1}{8}\pi$, $\frac{4}{3}\pi$, $\frac{1}{9}\pi$, $\frac{3}{2}\pi$, $\frac{4}{9}\pi$

3. Use a calculator to express each of the following angles in radians.
   35°, 47.2°, 93°, 233°, 14.1°, 117°, 370°

4. Use a calculator to express each of the following angles in degrees.
   1.7 rad, 3.32 rad, 1 rad, 2.09 rad, 5 rad, 6.283 19 rad

MENSURATION OF A CIRCLE

The reader will already be familiar with the formulae for the circumference and the area of a circle, i.e.

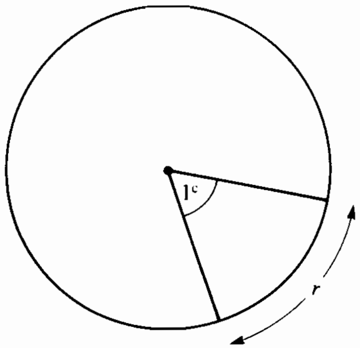
$$\text{Circumference} = 2\pi r \quad \text{and} \quad \text{Area} = \pi r^2$$

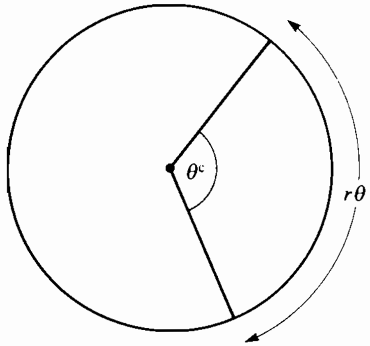
Now that we have defined a radian, these formulae can be used to derive other results.

The Length of an Arc

Consider an arc which subtends an angle θ at the centre of a circle, *where θ is measured in radians*

From the definition of a radian, the arc which subtends an angle of 1 radian at the centre of the circle is of length r. Therefore an arc which subtends an angle of θ radians at the centre is of length $r\theta$





The Area of a Sector

The area of a sector can be thought of as a fraction of the area of the whole circle.

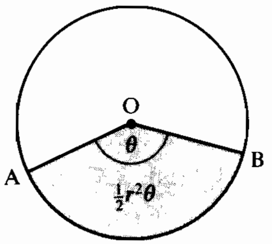
Consider a sector containing an angle of θ radians at the centre of the circle.

The complete angle at the centre of the circle is 2π, hence

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector} = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \tfrac{1}{2}r^2\theta$$



We now have two important facts about a circle in which an arc AB subtends an angle θ at the centre of the circle

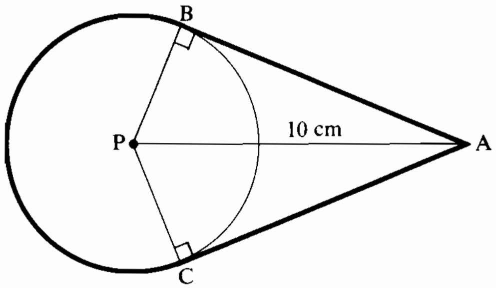
The length of arc AB $= r\theta$

The area of sector AOB $= \frac{1}{2}r^2\theta$

When solving problems involving arcs and sectors, answers are usually given in terms of π. If a numerical answer is required it will be asked for specifically.

Examples 11b

1. An elastic belt is placed round the rim of a pulley of radius 5 cm. One point on the belt is pulled directly away from the centre, P, of the pulley, until it is at A, 10 cm from P. Find the length of the belt that is in contact with the rim of the pulley.



The belt leaves the pulley at B and at C. At these two points the belt is a tangent to the rim, so AB is perpendicular to the radius BP. Similarly, AC and PC are perpendicular.

In $\triangle$ABP $\quad$ BP = 5 cm, AP = 10 cm and $\angle ABP = 90°$

Therefore $\quad \cos APB = \frac{5}{10} = \frac{1}{2}$

$\Rightarrow \quad \angle APB = 60° = \frac{1}{3}\pi$

Similarly $\quad \angle APC = \frac{1}{3}\pi$

The angle subtended at P by the major arc BC is given by

$$2\pi - \angle BPA - \angle CPA = 2\pi - \tfrac{1}{3}\pi - \tfrac{1}{3}\pi = \tfrac{4}{3}\pi$$

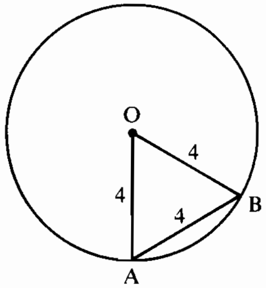
The length of an arc is given by $r\theta$, therefore

the length of the major arc BC is $5 \times \frac{4}{3}\pi = \frac{20}{3}\pi$

i.e. the length of belt in contact with the pulley is $\frac{20}{3}\pi$ cm.

---

2. AB is a chord of a circle with centre O and radius 4 cm. AB is of length 4 cm and divides the circle into two segments. Find, correct to two decimal places, the area of the minor segment.



Each side of $\triangle$ABC is 4 cm,
$\therefore$ ABC is an equilateral triangle
$\therefore$ each angle is 60°,

$$\text{Area of sector AOB} = \tfrac{1}{2}r^2\theta$$
$$= \tfrac{1}{2}(4^2)(\tfrac{1}{3}\pi)$$
$$= \tfrac{8}{3}\pi$$

$$\text{Area of minor segment} = \text{area of sector AOB} - \text{area of } \triangle AOB$$
$$= \tfrac{8}{3}\pi - \tfrac{1}{2}(4)(4)(\sin 60°)$$
$$= 8.378 - 6.928$$
$$= 1.450$$

The area of the minor segment is 1.45 cm² correct to 2 d.p.

EXERCISE 11b

In questions 1 to 10, s is the length of an arc subtending an angle θ at the centre of a circle of radius r, and a is the area of the corresponding sector. Complete the table.

	s (cm)	θ	r (cm)	a (cm^2)
1.		30°	4	
2.		$\frac{5}{6}\pi$	10	
3.	15	π		
4.	20	$\frac{4}{5}\pi$		
5.		135°	8	
6.			2	π
7.			5	12
8.		$\frac{1}{6}\pi$		3π
9.	5π			20π

10. Calculate, in degrees, the angle subtended at the centre of a circle of radius 2.7 cm by an arc of length 6.9 cm.

11. Calculate, in radians, the angle at the centre of a circle of radius 83 mm contained in a sector of area 974 mm^2.

12. The diameter of the moon is about 3445 km and the distance between the moon and earth is about 382 100 km. Find the angle subtended at a point on the earth's surface by the moon (give your answer as a decimal part of a degree to 2 d.p.).

13. In a circle with centre O and radius 5 cm, AB is a chord of length 8 cm. Find

(a) the area of triangle AOB

(b) the area of the sector AOB (in square centimetres, correct to 3 s.f.).

14. A chord of length 10 mm divides a circle of radius 7 mm into two segments. Find the area of each segment.

15. A chord PQ, of length 12.6 cm, subtends an angle of $\frac{2}{3}\pi$ at the centre of a circle. Find

(a) the length of the arc PQ

(b) the area of the minor segment cut off by the chord PQ.

16. A curve in the track of a railway line is a circular arc of length 400 m and radius 1200 m. Through what angle does the direction of the track turn?

17. Two discs, of radii 5 cm and 12 cm, are placed, partly overlapping, on a table. If their centres are 13 cm apart find the perimeter of the 'figure-eight' shape.

18. Two circles, each of radius 14 cm, are drawn with their centres 20 cm apart. Find the length of their common chord. Find also the area common to the two circles.

The next three questions are a little more demanding.

19. A chord of a circle subtends an angle of θ radians at the centre of the circle. The area of the minor segment cut off by the chord is one eighth of the area of the circle.
Prove that $4\theta = \pi + 4\sin\theta$

20. A chord PQ of length $6a$ is drawn in a circle of radius $10a$. The tangents to the circle at P and Q meet at R. Find the area enclosed by PR, QR and the minor arc PQ.

21. Two discs are placed, in contact with each other, on a table. Their radii are 4 cm and 9 cm. An elastic band is stretched round the pair of discs. Calculate

(a) the angle subtended at the centre of the smaller disc by the arc that is in contact with the elastic band.

(b) the length of the part of the band that is in contact with the smaller disc.

(c) the length of the part of the band that is in contact with the larger disc.

(d) the total length of the stretched band.

(*Hint.* The straight parts of the stretched band are common tangents to the two circles.)

CHAPTER 12

FUNCTIONS

MAPPINGS

SG

If, on a calculator, the number 2 is entered and then the x^2 button is pressed, the display shows the number 4

We say that 2 is mapped to 2^2 or $2 \to 2^2$

Under the same rule, i.e. squaring the input number,
$3 \to 9$, $25 \to 625$, $0.5 \to 0.25$, $-4 \to 16$ and in fact,

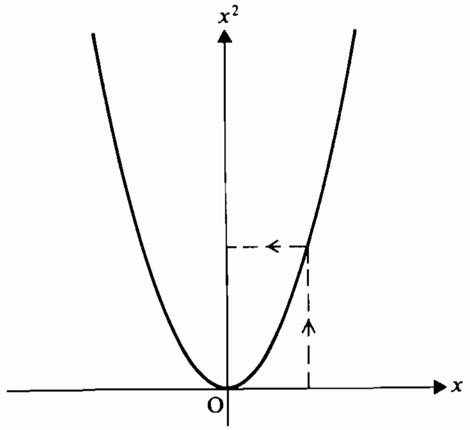
(any real number) $\to$ (the square of that number)

The last statement can be expressed more briefly as

$$x \to x^2$$

where x is any real number.

This mapping can be represented graphically by plotting values of x^2 against values of x



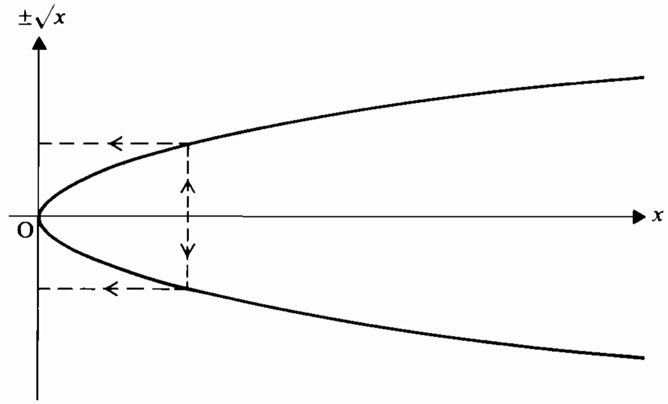
The graph, and knowledge of what happens when we square a number, show that one input number gives just one output number.

Now consider the mapping which maps a number to its square roots; the rule by which, for example, $4 \to +2$ and -2

This rule gives a real output only if the input number is greater than zero (negative numbers do not have real square roots). This mapping can now be written in general terms as

$$x \to \pm\sqrt{x} \quad \text{for} \quad x \geqslant 0$$

The graphical representation of this mapping is shown below.



This time we notice that one input value gives two output values.

From these two examples, we can see that a mapping is a rule for changing a number to another number or numbers.

FUNCTIONS

Under the first mapping, $x \to x^2$, one input number gives one output number. However, for the second mapping, $x \to \pm\sqrt{x}$, one input number gives two output numbers.

We use the word *function* for any rule that gives the same kind of result as the first mapping, i.e. one input value gives one output value.

A function is a rule which maps a single number to another single number.

The second mapping does not satisfy this condition so we cannot call it a function.

Consider again what we can now call the function for which $x \to x^2$
Using f for 'function' and the symbol : to mean 'such that',
we can write $f: x \to x^2$

We use the notation $f(x)$ to represent the output values of the function.

e.g.

$$\text{for } f: x \to x^2, \text{ we have } f(x) = x^2$$

Examples 12a

1. Determine whether these mappings are functions,

(a) $x \to \frac{1}{x}$ (b) $x \to y$ where $y^2 - x = 0$

(a) For any value of x, except $x = 0$, $\frac{1}{x}$ has a single value,

therefore $x \to \frac{1}{x}$ is a function provided that $x = 0$ is excluded.

Note that $\frac{1}{0}$ is meaningless, so to make this mapping a function we have to exclude 0 as an input value. The function can be described by $f(x) = \frac{1}{x}, \ x \neq 0$

(b) If, for example, we input $x = 4$, then the output is the value of y when

$$y^2 - 4 = 0 \Rightarrow y = 2 \text{ and } y = -2$$

therefore an input gives more than one value for the output,
so $x \to y$ where $y^2 - x = 0$ is not a function.

2. If $f(x) = 2x^2 - 5$, find $f(3)$ and $f(-1)$

As $f(x)$ is the output of the mapping, $f(3)$ is the output when 3 is the input, i.e. $f(3)$ is the value of $2x^2 - 5$ when $x = 3$

$$f(3) = 2(3)^2 - 5 = 13$$

$$f(-1) = 2(-1)^2 - 5 = -3$$

EXERCISE 12a

1. Determine which of these mappings are functions.

(a) $x \to 2x - 1$ (b) $x \to x^3 + 3$ (c) $x \to \frac{1}{x-1}$

(d) $x \to k$ where $k^2 = x$ (e) $x \to \sqrt{x}$

(f) $x \to$ the length of the line from the origin to $(0, x)$

(g) $x \to$ the greatest integer less than or equal to x

(h) $x \to$ the height of a triangle whose area is x

2. If $f(x) = 5x - 4$ find $f(0)$, $f(-4)$

3. If $f(x) = 3x^2 + 25$ find $f(0)$, $f(8)$

4. If $f(x) =$ the value of x correct to the nearest integer, find $f(1.25)$, $f(-3.5)$, $f(12.49)$

5. If $f(x) = \sin x$, find $f(\frac{1}{2}\pi)$, $f(\frac{2}{3}\pi)$

DOMAIN AND RANGE

We have assumed that we can use any real number as an input for a function unless some particular numbers have to be excluded because they do not give real numbers as output.

The set of inputs for a function is called the *domain* of the function.

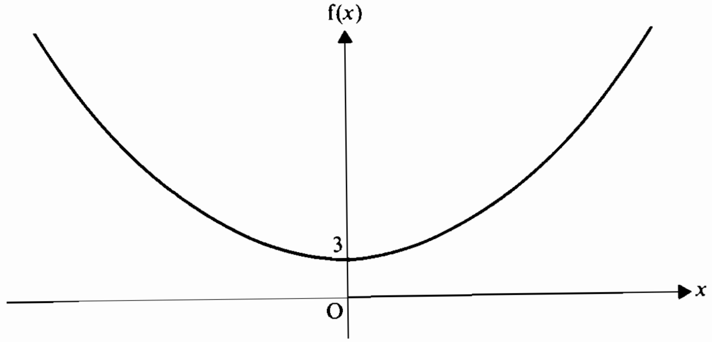
The domain does not have to contain all possible inputs; it can be as wide, or as restricted, as we choose to make it. Hence to define a function fully, the domain must be stated.

If the domain is not stated, we assume that it is the set of all real numbers ($\mathbb{R}$).

Consider the mapping $x \to x^2 + 3$

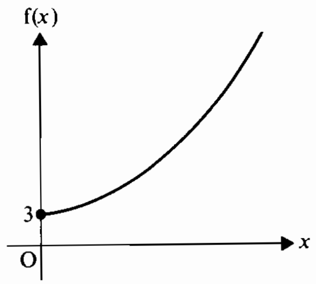
We can define a function f for this mapping over any domain we choose. Some examples, together with their graphs are given overleaf.

1) $f(x) = x^2 + 3$ for $x \in \mathbb{R}$



2) $f(x) = x^2 + 3$ for $x \geqslant 0$

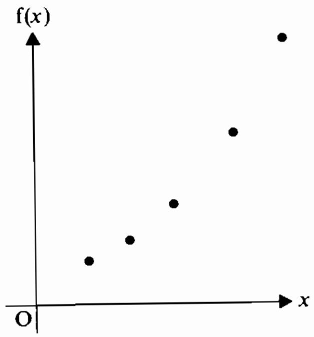
Note that the point on the curve where $x = 0$ is included and we denote this on the curve by a solid circle.



If the domain were $x > 0$, then the point would not be part of the curve and we indicate this fact by using an open circle.

3) $f(x) = x^2 + 3$ for $x \in \{1, 2, 3, 4, 5\}$

This time the graphical representation consists of just five discrete (i.e. separate) points.



For each domain, there is a corresponding set of output numbers.

The set of output numbers is called the *range* or *image-set* of the function.

Thus for the function defined in **(1)** above, the range is $f(x) \geqslant 3$ and for the function given in **(2)**, the range is also $f(x) \geqslant 3$ For the function defined in **(3)**, the range is the set $\{4, 7, 12, 19, 28\}$.

Sometimes a function can be made up from more than one mapping, where each mapping is defined for a different domain. This is illustrated in the next worked example.

Example 12b

The function, f, is defined by $f(x) = x^2$ for $x \leqslant 0$
and $f(x) = x$ for $x > 0$

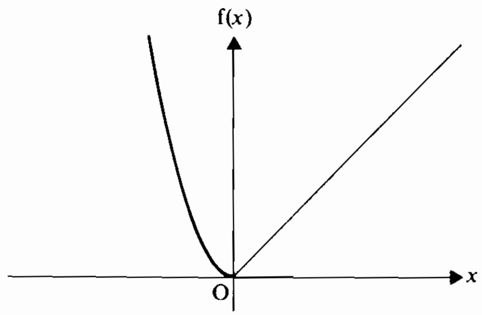
(a) Find f(4) and f(−4) (b) Sketch the graph of f

(c) Give the range of f

(a) For $x > 0$, $f(x) = x$, $\therefore$ $f(4) = 4$

For $x \leqslant 0$, $f(x) = x^2$, $\therefore$ $f(-4) = (-4)^2 = 16$

(b) To sketch the graph of a function, we can apply our knowledge of lines and curves in the xy-plane to the equation $y = f(x)$. In this way we can interpret $f(x) = x$ for $x > 0$, as that part of the line $y = x$ which corresponds to positive values of x



(c) The range of f is $f(x) \geqslant 0$

EXERCISE 12b

1. Find the range of each of the following functions.
   (a) $f(x) = 2x - 3$ for $x \geqslant 0$
   (b) $f(x) = x^2 - 5$ for $x \leqslant 0$
   (c) $f(x) = 1 - x$ for $x \leqslant 1$
   (d) $f(x) = 1/x$ for $x \geqslant 2$

2. Draw a sketch graph of each function given in Question 1.

3. The function f is such that $f(x) = -x$ for $x < 0$
   and $f(x) = x$ for $x \geqslant 0$
   (a) Find the value of $f(5)$, $f(-4)$, $f(-2)$ and $f(0)$
   (b) Sketch the graph of the function.

4. The function f is such that $f(x) = x$ for $0 \leqslant x \leqslant 5$
   and $f(x) = 5$ for $x > 5$
   (a) Find the value of $f(0)$, $f(2)$, $f(4)$, $f(5)$ and $f(7)$
   (b) Sketch the graph of the function.
   (c) Give the range of the function.

5. In Utopia, the tax on earned income is calculated as follows. The first £20 000 is tax free and remaining income is taxed at 20%.
   (a) Find the tax payable on an earned income of £15 000 and of £45 000
   (b) Taking x as the number of pounds of earned income and y as the number of pounds of tax payable, define a function f such that $y = f(x)$. Draw a sketch of the function and state the domain and range.

CURVE SKETCHING

When functions have similar definitions they usually have common properties and graphs of the same form. If the common characteristics of a group of functions are known, the graph of any one particular member of the group can be sketched without having to plot points.

Quadratic Functions

The general form of a quadratic function is

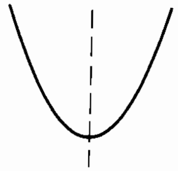
$$f(x) = ax^2 + bx + c \quad \text{for } x \in \mathbb{R}$$

where a, b and c are constants and $a \neq 0$

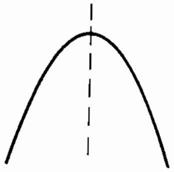
When a graphics calculator, or a computer, is used to draw the graphs of quadratic functions for a variety of values of a, b and c, the basic shape of the curve is always the same. This shape is called a *parabola*.

Every parabola has an axis of symmetry which goes through the vertex, i.e. the point where the curve turns back upon itself.

If the coefficient of x^2 is positive, i.e. $a > 0$, then $f(x)$ has a least value, and the parabola looks like this.



If the coefficient of x^2 is negative, i.e. $a < 0$, then $f(x)$ has a greatest value and the curve is this way up.



These properties of the graph of a quadratic function can be proved algebraically.

For $f(x) = ax^2 + bx + c$, 'completing the square' on the RHS and simplifying, gives

$$f(x) = \left[\frac{4ac - b^2}{4a}\right] + a\left[x + \frac{b}{2a}\right]^2 \qquad [1]$$

Now the first bracket is constant and, as the second bracket is squared, its value is zero when $x = -\dfrac{b}{2a}$ and greater than zero for all other values of x.

Hence

when a is positive,

$$f(x) = ax^2 + bx + c \text{ has a least value when } x = -\frac{b}{2a}$$

and when a is negative,

$$f(x) = ax^2 + bx + c \text{ has a greatest value when } x = -\frac{b}{2a}$$

Further, taking values of x that are symmetrical about $x = -\dfrac{b}{2a}$,
e.g. $x = \pm k - \dfrac{b}{2a}$, we see from [1] that

$$f\left(k - \frac{b}{2a}\right) = f\left(-k - \frac{b}{2a}\right) = \left[\frac{4ac - b^2}{4a}\right] + ak^2$$

i.e. the value of $f(x)$ is symmetrical about $x = -\dfrac{b}{2a}$

These properties can now be used to draw *sketches* of the graphs of quadratic functions.

Examples 12c

1. Find the greatest or least value of the function given by $f(x) = 2x^2 - 7x - 4$ and hence sketch the graph of $f(x)$.

$f(x) = 2x^2 - 7x - 4 \quad \Rightarrow \quad a = 2,\ b = -7$ and $c = -4$

As $a > 0$, $f(x)$ has a least value

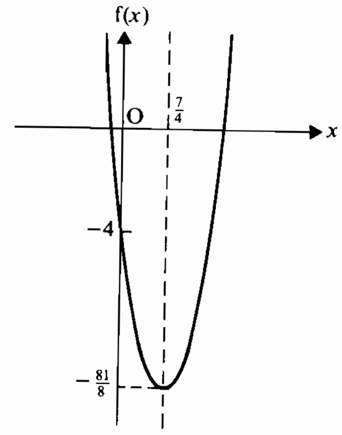
and this occurs when $x = -\dfrac{b}{2a} = \dfrac{7}{4}$

$\therefore$ the least value of $f(x)$ is $f(\frac{7}{4}) = 2(\frac{7}{4})^2 - 7(\frac{7}{4}) - 4$

$$= -\tfrac{81}{8}$$

We now have one point on the graph of $f(x)$ and we know that the curve is symmetrical about this value of x. However, to locate the curve more accurately we need another point; we use $f(0)$ as it is easy to find.

$f(0) = -4$



2. Draw a quick sketch of the graph of $f(x) = (1 - 2x)(x + 3)$

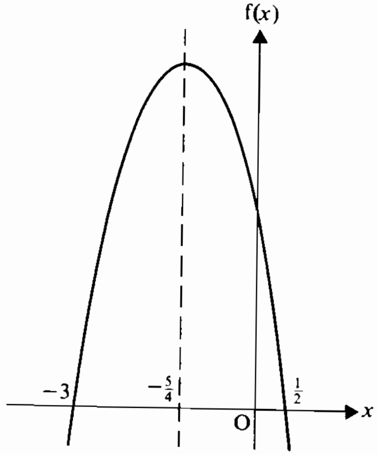
The coefficient of x^2 is negative, so $f(x)$ has a greatest value.
The curve cuts the x-axis when $f(x) = 0$

When $f(x) = 0$, $\quad (1 - 2x)(x + 3) = 0$

$\Rightarrow \quad x = \frac{1}{2}$ or -3

The average of these values is $-\frac{5}{4}$, so the curve is symmetrical about $x = -\frac{5}{4}$

We now have enough information to draw a quick sketch, but note that this method is suitable only when the quadratic function factorises.



EXERCISE 12c

1. Find the greatest or least value of $f(x)$ where $f(x)$ is
   (a) $x^2 - 3x + 5$ (b) $2x^2 - 4x + 5$ (c) $3 - 2x - x^2$

2. Find the range of f where $f(x)$ is
   (a) $7 + x - x^2$ (b) $x^2 - 2$ (c) $2x - x^2$

3. Sketch the graph of each of the following quadratic functions, showing the greatest or least value and the value of x at which it occurs.
   (a) $x^2 - 2x + 5$ (b) $x^2 + 4x - 8$ (c) $2x^2 - 6x + 3$
   (d) $4 - 7x - x^2$ (e) $x^2 - 10$ (f) $2 - 5x - 3x^2$

4. Draw a quick sketch of each of the following functions.
   (a) $(x-1)(x-3)$ (b) $(x+2)(x-4)$ (c) $(2x-1)(x-3)$
   (d) $(1+x)(2-x)$ (e) $x^2 - 9$ (f) $3x^2$

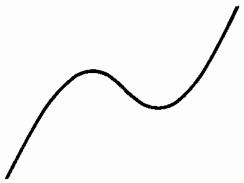
CUBIC FUNCTIONS

The general form of a cubic function is

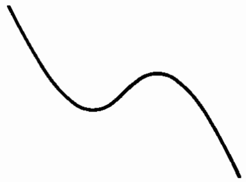
$$f(x) = ax^3 + bx^2 + cx + d$$

where a, b, c and d, are constants and $a \neq 0$

Investigating the curve $y = ax^3 + bx^2 + cx + d$ for a variety of values of a, b, c and d shows that the shape of the curve is

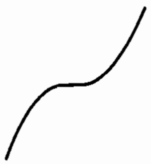


and

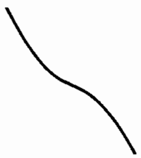


when $a > 0$ when $a < 0$

Sometimes there are no turning points and the curve looks like this



or



POLYNOMIAL FUNCTIONS

The general form of a polynomial function is

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants, n is a positive integer and $a_n \neq 0$

Examples of polynomials are

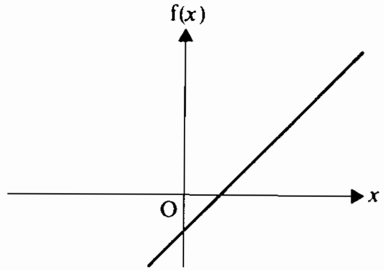
$$f(x) = 3x^4 - 2x^3 + 5, \quad f(x) = x^5 - 2x^3 + x, \quad f(x) = x^2$$

The *order* of a polynomial is the highest power of x in the function. Thus, the order of $x^4 - 7$ is 4, and the order of $2x - 1$ is 1

We have already investigated the graphs of polynomials of order 1,

e.g. $f(x) = 2x - 1$

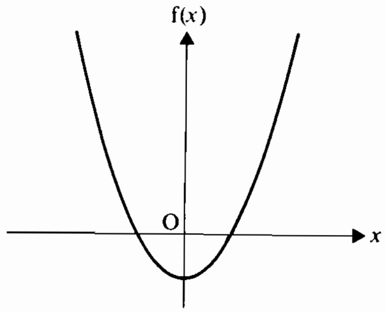
which gives a straight line,



and of order 2,

e.g. $f(x) = x^2 - 4$

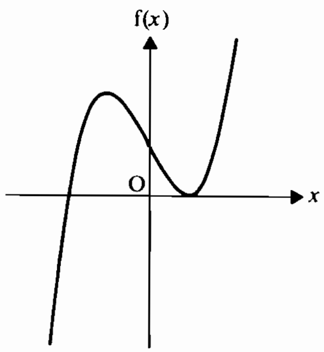
which gives a parabola,



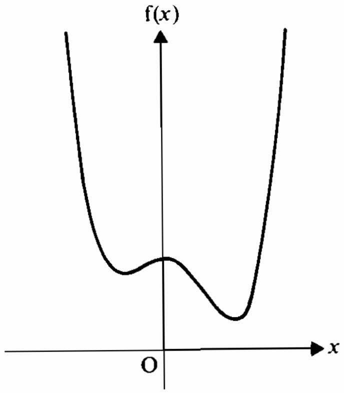
and of order 3,

e.g. $f(x) = x^3 - 2x + 1$

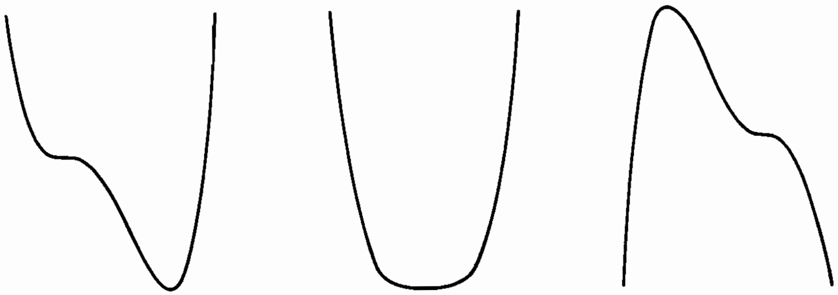
which gives a cubic curve.



The shape of the curve
$f(x) = x^4 - 3x^3 + 2x^2 + 1$
looks like this.



Experimenting with the curves of other polynomial functions of order 4 shows that, in general, the curve has three turning points although some or all of these may merge,
e.g.



$a > 0$ $\quad$ $a > 0$ $\quad$ $a < 0$

RATIONAL FUNCTIONS

A rational function is one in which both numerator and denominator are polynomial.

Examples of rational functions of x are

$$\frac{1}{x}, \quad \frac{x}{x^2 - 1}, \quad \frac{3x^2 + 2x}{x - 1}$$

Now consider the familiar function $f(x) = 1/x$ and its graph. From its form we can infer various properties of $f(x)$

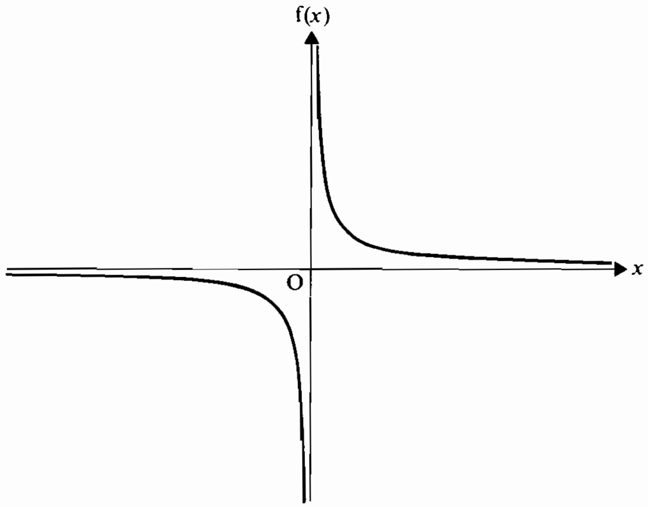
1) As the value of x increases, the value of $f(x)$ gets closer to zero,
e.g. when $x = 100$, $f(x) = 1/100$
and when $x = 1000$, $f(x) = 1/1000$
We write this as $x \to \infty$, $f(x) \to 0$

Similarly as the value of x decreases, i.e. as $x \to -\infty$, the value of $f(x)$ again gets closer to zero, i.e. $f(x) \to 0$

2) $f(x)$ does not exist when $x = 0$, so this value of x must be excluded from the domain of f
x can get as close as we like to zero however, and can approach zero in two ways.

If $x \to 0$ from above (i.e. from positive values, $--|_0\!\longleftarrow--$)
then $f(x) \to \infty$

If $x \to 0$ from below (i.e. from negative values, $--\!\longrightarrow|_0--$)
then $f(x) \to -\infty$



Notice that, as $x \to \pm\infty$, the curve gets closer to the x-axis but does not cross it. Also, as $x \to 0$, the curve approaches the y-axis but again does not cross it.
We say that the x-axis and the y-axis are *asymptotes* to the curve.

EXPONENTIAL FUNCTIONS

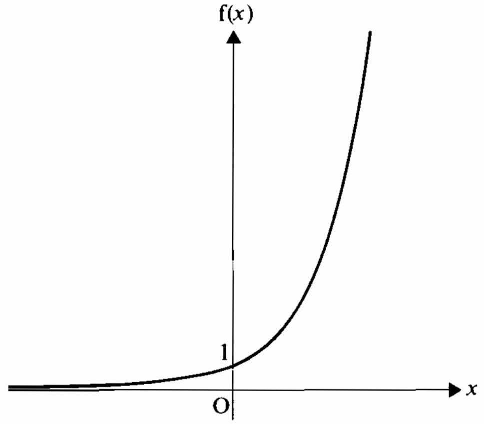
Exponent is another word for index or power.

An exponential function is one where the variable is in the index.

For example, 2^x, 3^{-x}, 10^{x+1} are exponential functions of x

Consider the function $f(x) = 2^x$ for which a table of corresponding values of x and $f(x)$ and a graph are given below.

x	$-\infty \leftarrow \dots$	-10	-1	$-\frac{1}{10}$	0	$\frac{1}{10}$	1	10	$\dots \rightarrow \infty$
$f(x)$	$0 \leftarrow \dots$	$\frac{1}{1024}$	$\frac{1}{2}$	0.93	1	1.07	2	1024	$\dots \rightarrow \infty$



From these we see that

1) 2^x has a real value for all real values of x and 2^x is positive for all values of x, i.e. the range of f is $f(x) > 0$

2) As $x \rightarrow -\infty$, $f(x) \rightarrow 0$, i.e. the x-axis is an asymptote.

3) As x increases, $f(x)$ increases at a rapidly accelerating rate.

Note also that the curve crosses the y-axis at $(0, 1)$, i.e. $f(0) = 1$. In fact for any function of the form $f(x) = a^x$, where a is a constant and greater than 1, $f(0) = 1$, and the curve representing it is similar in shape to that for 2^x

Example 12d

Sketch the graph of the function given by $f(x) = \dfrac{1}{2-x}$

$f(x) = \dfrac{1}{2-x}$ does not exist when $x = 2$, so the curve $y = f(x)$ does not cross the line $x = 2$

As $x \rightarrow 2$ from above, $2-x$ is negative and approaches zero,

$$\text{so} \quad \frac{1}{2-x} \rightarrow -\infty$$

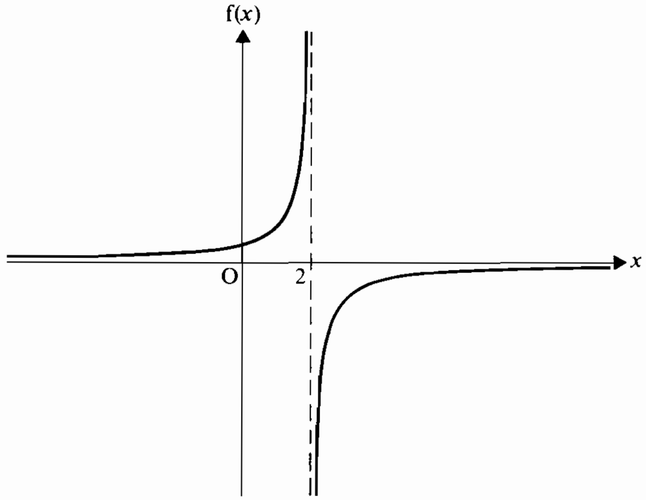
As $x \rightarrow 2$ from below, $2-x$ is positive and approaches zero,

$$\text{so} \quad \frac{1}{2-x} \rightarrow \infty$$

Therefore the line $x = 2$ is an asymptote.

As $x \rightarrow \infty$, $\dfrac{1}{2-x} \rightarrow 0$ from below
and as $x \rightarrow -\infty$, $\dfrac{1}{2-x} \rightarrow 0$ from above
$\therefore$ the x-axis is an asymptote.

As this is similar to $f(x) = \dfrac{1}{x}$, we now have enough information to sketch the graph.



EXERCISE 12d

1. Draw sketch graphs of the following functions.

   (a) 3^x (b) $\dfrac{1}{2x}$ (c) 4^{2x} (d) $\dfrac{1}{x-3}$

2. Write down the values of $f(x) = (\frac{1}{2})^x$ corresponding to $x = -4$, -3, -2, -1, 0, 1, 2, 3 and 4. From these values deduce the behaviour of $f(x)$ as $x \to \pm\infty$ and hence sketch the graph of the function.

3. What value of x must be excluded from the domain of $f(x) = \dfrac{1}{x+2}$? Describe the behaviour of $f(x)$ as x approaches this value from above and from below. Describe also the behaviour of $f(x)$ as $x \to \pm\infty$. Use this information to sketch the graph of $f(x)$

4. By following a procedure similar to that given in Question 3 draw sketch graphs of the following functions.

   (a) $-\dfrac{1}{x}$ (b) $\dfrac{1}{1-2x}$ (c) $\dfrac{2}{x+1}$ (d) $1+\dfrac{1}{x}$

5. Find the values of x where the curve $y = f(x)$ cuts the x-axis and sketch the curve when

   (a) $f(x) = x(x-1)(x+1)$ (b) $f(x) = x(x-1)(x+1)(x-2)$

   (c) $f(x) = (x^2-1)(2-x)$ (d) $f(x) = (x^2-1)(4-x^2)$

SIMPLE TRANSFORMATION OF CURVES

Transformations of curves are best appreciated if they can be 'seen', so this section starts with an investigative approach using a graphics calculator or a computer with a graph-drawing package. This exercise is not essential and all the necessary conclusions are drawn analytically in the next part of the text.

EXERCISE 12e

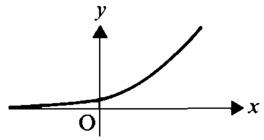
You will need a graphics calculator or computer for this exercise.

1. (a) On the screen, draw the graph of $y = 2^x$. Superimpose the graphs of $y = 2^x + 2$ and $y = 2^x - 1$ Clear the screen and again draw the graph of $y = 2^x$. This time superimpose the graph of $y = 2^x + c$ for a variety of values of c
   (b) Describe the transformation that maps the graph of $f(x) = 2^x$ to the graph of $g(x) = 2^x + c$
   (c) Repeat (a) and (b) for other simple functions, e.g., x^2, x^3, $1/x$

2. Use a procedure similar to that described in Question 1 to investigate the relationship between the graphs of $f(x)$ and $f(x+c)$

3. Investigate the relationship between the graphs of
   (a) $f(x)$ and $-f(x)$ (b) $f(x)$ and $f(-x)$

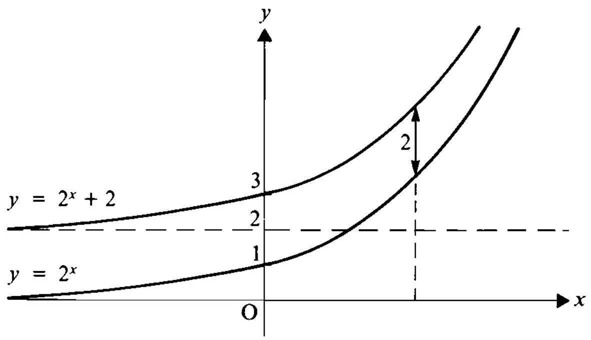
TRANSLATIONS



Consider the function f where $f(x) = 2^x$
The graph of this function is the curve $y = 2^x$

1) Now consider the function g where $g(x) = f(x) + 2$

Comparing $f(x) = 2^x$ with $g(x) = 2^x + 2$ we can see that for a particular value of x, the value of $g(x)$ is 2 units greater than the corresponding value of $f(x)$. Therefore, for equal values of x, points on the curve $y = g(x)$ are two units above points on the curve $y = f(x)$, i.e. the curve $y = 2^x + 2$ is a translation of the curve $y = 2^x$ by two units in the positive direction of the y-axis.



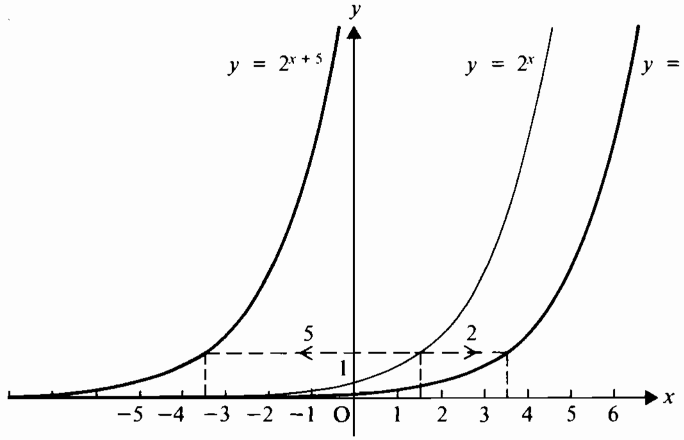
In general, for any function f, the curve $y = f(x) + c$ is the translation of the curve $y = f(x)$ by c units parallel to the y-axis.

2) Consider the function $g(x) = 2^{x-2}$

Comparing $f(x) = 2^x$ with $g(x) = 2^{x-2}$ we can see that the values of $f(x)$ and $g(x)$ are the same when the input value to $g(x)$ is 2 units greater than the input value for $f(x)$, i.e. $f(a) = g(a+2)$

Therefore, for equal values of y, points on the curve $y = 2^{x-2}$ are 2 units to the *right* of points on the curve $y = 2^x$, i.e. the curve $y = 2^{x-2}$ is a translation of the curve $y = 2^x$ by 2 units in the positive direction of the x-axis.

Similarly, considering $h(x) = 2^{x+5}$, the values of $f(x)$ and $h(x)$ are the same when the input to $h(x)$ is five units less than the input to $f(x)$. Thus for equal values of y, points on the curve $y = 2^{x+5}$ are 5 units to the *left* of points on the curve $y = 2^x$



In general, the curve $y = f(x+c)$ is a translation of the curve $y = f(x)$ by c units parallel to the x-axis.

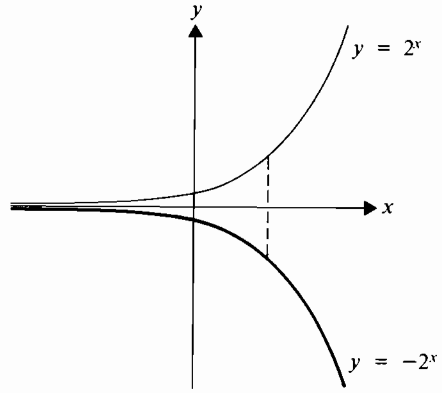
If $c > 0$, the translation is in the negative direction of the x-axis and if $c < 0$, the translation is in the positive direction of the x-axis.

REFLECTIONS

1) Consider the function $g(x) = -f(x)$

For a given value of x, $g(x)$ is equal to $-f(x)$. Therefore for equal values of x, points on the curve $y = -2^x$ are the reflection in the

x-axis of points on the curve $y = 2^x$, i.e. the curve $y = -f(x)$ is the reflection in the x-axis of the curve $y = f(x)$

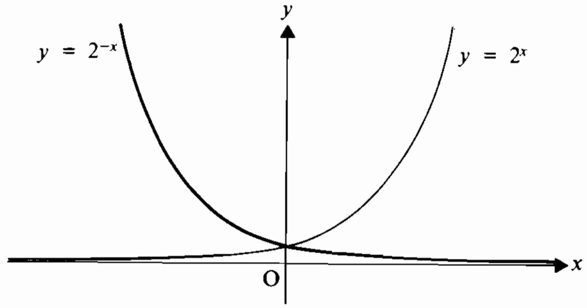


In general, the curve $y = -f(x)$ is the reflection of the curve $y = f(x)$ in the x-axis.

2) Consider the function $g(x) = 2^{-x}$

Comparing $f(x) = 2^x$ with $g(x) = 2^{-x}$ we see that $f(x)$ and $g(x)$ have the same value when the inputs to $g(x)$ and $f(x)$ are equal in value but opposite in sign, i.e. $g(a) = f(-a)$

Therefore points with the same y-coordinates on the curves $y = 2^{-x}$ and $y = 2^x$, are symmetrical about $x = 0$, i.e. the curve $y = 2^{-x}$ is the reflection of the curve $y = 2^x$ in the y-axis.

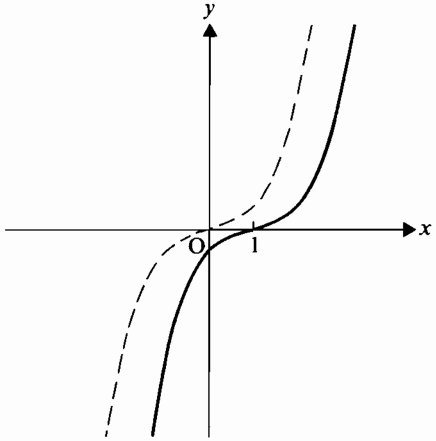


In general, the curve $y = f(-x)$ is the reflection of the curve $y = f(x)$ in the y-axis.

Example 12f

Sketch the curve $y = (x-1)^3$.

The shape and position of the curve $y = x^3$ is known. If $f(x) = x^3$, then $(x-1)^3 = f(x-1)$, so the curve $y = (x-1)^3$ is a translation of the first curve by one unit in the positive direction of the x-axis.



EXERCISE 12f

Sketch each of the following curves.

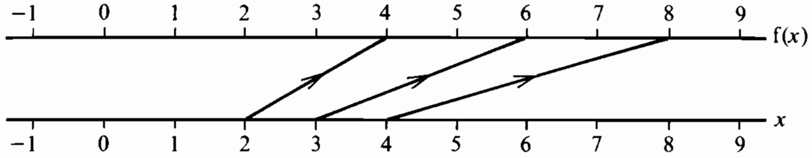
1. $y = -x^2$
2. $y = -\dfrac{1}{x}$
3. $y = -3^x$
4. $y = 1 + \dfrac{1}{x}$
5. $y = 2^x - 3$
6. $y = \dfrac{1}{x} - 2$
7. $y = (x-4)^4$
8. $y = x^2 - 9$
9. $y = \dfrac{1}{x-2}$
10. On the same set of axes sketch the graphs of $f(x) = x^3$, $f(x) = (x+1)^3$, $f(x) = -(x+1)^3$ and $f(x) = 2-(x+1)^3$
11. On the same set of axes sketch the lines $y = 2x - 1$ and $y = \frac{1}{2}(x+1)$. Describe a transformation which maps the first line to the second line.
12. Repeat Question 11 for the curves $y = 1 + \dfrac{1}{x}$ and $y = \dfrac{1}{x-1}$

13. Find the coordinates of the reflection of the point $(2, 5)$ in the line $y = x$
14. P' is the reflection of the point $P(a, b)$ in the line $y = x$. Find the coordinates of P' in terms of a and b

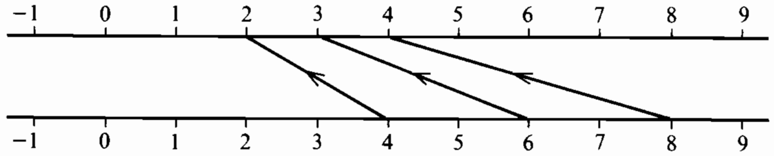
INVERSE FUNCTIONS

Consider the function f where $f(x) = 2x$ for $x \in \{2, 3, 4\}$

Under this function, the domain $\{2, 3, 4\}$ maps to the image-set $\{4, 6, 8\}$ and this is illustrated by the arrow diagram.



It is possible to reverse this mapping, i.e. we can map each member of the image-set back to the corresponding member of the domain by halving each member of the image-set.



This procedure can be expressed algebraically, i.e.
for $x \in \{4, 6, 8\}$, $x \to \frac{1}{2}x$ maps 4 to 2, 6 to 3 and 8 to 4

This reverse mapping is a function in its own right and it is called the *inverse* function of f where $f(x) = 2x$

Denoting this inverse function by f^{-1} we can write $f^{-1}(x) = \frac{1}{2}x$
In fact, $f(x) = 2x$ can be reversed for all real values of x and the procedure for doing this is a function.

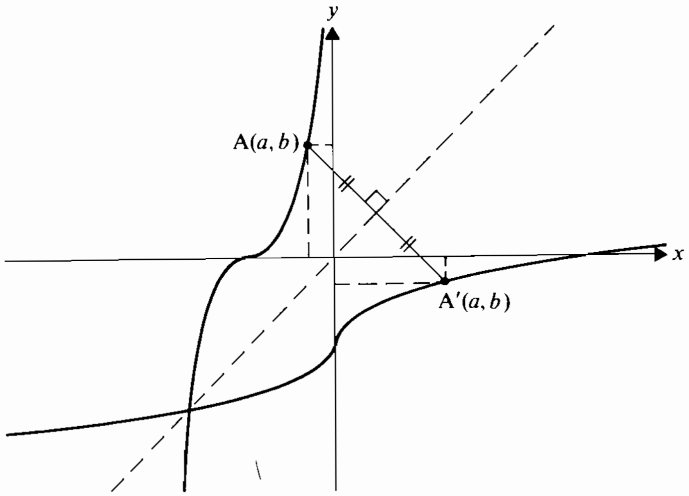
Therefore, for $f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$ is such that f^{-1} reverses f for all real values of x, i.e. f^{-1} maps the output of f to the input of f.

In general, for any function f,

> if there exists a function, g, that maps the output of f back to its input, i.e. $g: f(x) \rightarrow x$, then this function is called the inverse of f and it is denoted by f^{-1}.

THE GRAPH OF A FUNCTION AND ITS INVERSE

Consider the curve that is obtained by reflecting $y = f(x)$ in the line $y = x$. The reflection of a point $A(a, b)$ on the curve $y = f(x)$, is the point A' whose coordinates are (b, a), i.e. interchanging the x and y coordinates of A gives the coordinates of A'.

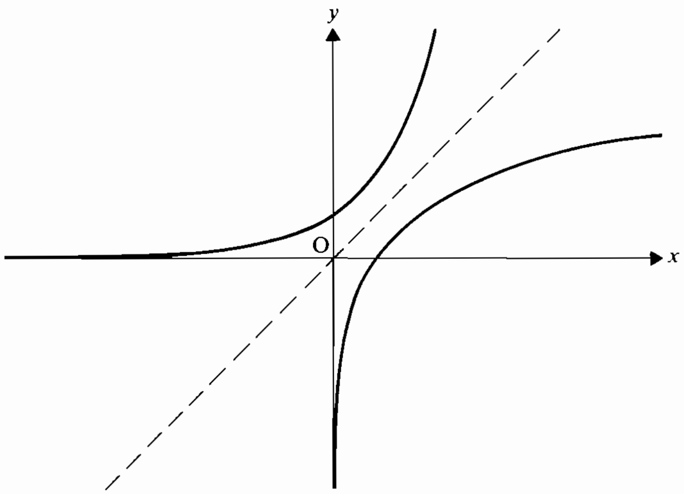


We can therefore obtain the equation of the reflected curve by interchanging x and y in the equation $y = f(x)$

Now the coordinates of A on $y = f(x)$ can be written as $[a, f(a)]$. Therefore the coordinates of A' on the reflected curve are $[f(a), a]$, i.e. the equation of the reflected curve is such that the output of f is mapped to the input of f.

Hence if the equation of the reflected curve can be written in the form $y = g(x)$, then g is the inverse of f, i.e. $g = f^{-1}$.

To illustrate these properties, consider the curve $y = 2^x$ and its reflection in the line $y = x$



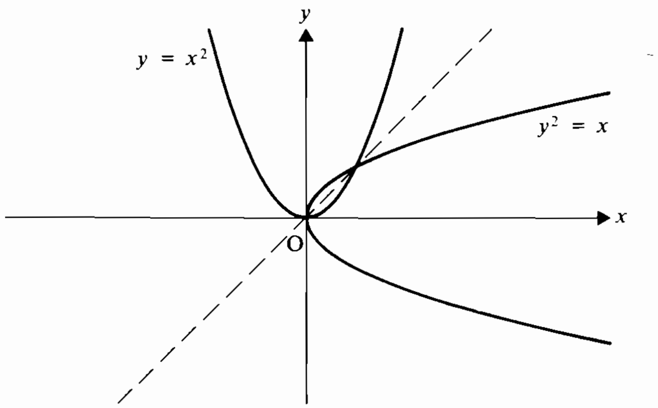
The equation of the reflected curve is given by $x = 2^y$

Using the 'log' notation introduced in Chapter 3, we can write this equation in the form $y = \log_2 x$

Therefore for the function $f(x) = 2^x$ the inverse function is given by $f^{-1}(x) = \log_2 x$

Any curve whose equation can be written in the form $y = f(x)$ can be reflected in the line $y = x$. However this reflected curve may not have an equation that can be written in the form $y = f^{-1}(x)$

Consider the curve $y = x^2$ and its reflection in the line $y = x$



The equation of the image curve is $x = y^2 \Rightarrow y = \pm\sqrt{x}$ and $x \to \pm\sqrt{x}$ is not a function.

(We can see this from the diagram as, on the reflected curve, one value of x maps to two values of y. So in this case y cannot be written as a function of x.)

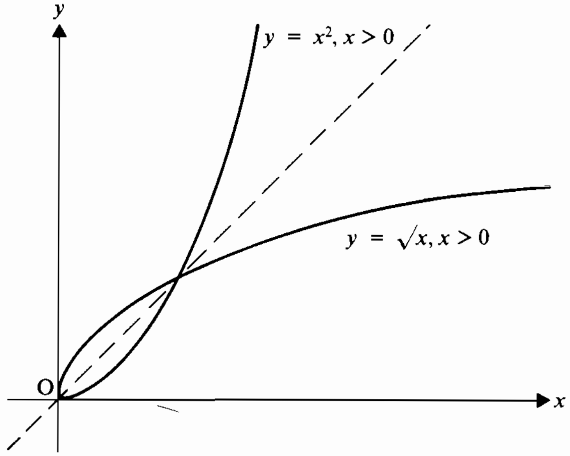
Therefore the function $f:x \to x^2$ does not have an inverse, i.e.

not every function has an inverse.

If we change the definition of f to $f:x \to x^2$ for $x \in \mathbb{R}^+$ then the inverse mapping

is $\quad x \to \sqrt{x}$ for $x \in \mathbb{R}^+$ and this is a function, i.e.

$$f^{-1}(x) = \sqrt{x} \text{ for } x \in \mathbb{R}^+$$



To summarise:

The inverse of a function undoes the function, i.e. it maps the output of a function back to its input.

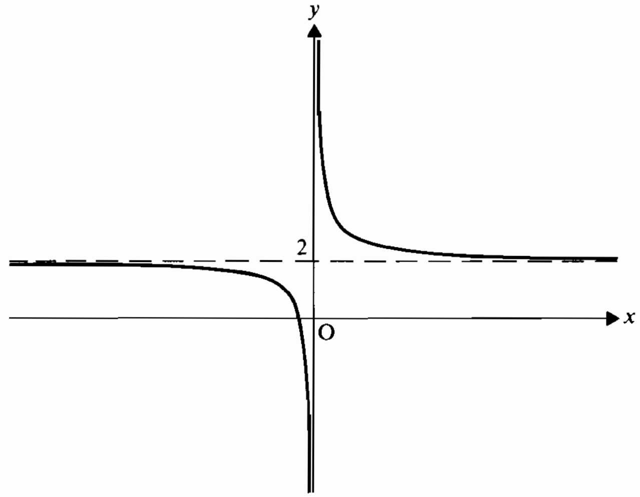
The inverse of the function f is written f^{-1}

Not all functions have an inverse.

When the curve whose equation is $y = f(x)$ is reflected in the line $y = x$, the equation of the reflected curve is $x = f(y)$
If this equation can be written in the form $y = g(x)$ then g is the inverse of f, i.e. $g(x) = f^{-1}(x)$

Examples 12g

1. Determine whether there is an inverse of the function f given by
$$f(x) = 2 + \frac{1}{x}$$
If f^{-1} exists, express it as a function of x



From the sketch of $f(x) = 2 + \frac{1}{x}$, we see that one value of $f(x)$ maps to one value of x, therefore the reverse mapping is a function.

The equation of the reflection of $y = 2 + \frac{1}{x}$ can be written as

$$x = 2 + \frac{1}{y} \Rightarrow y = \frac{1}{x-2}$$

$\therefore$ when $f(x) = 2 + \frac{1}{x}$, $\quad f^{-1}(x) = \frac{1}{x-2}$

2. Find $f^{-1}(4)$ when $f(x) = 5x - 1$

If $y = f(x)$, i.e. $y = 5x - 1$,

then $\quad x = 5y - 1 \Rightarrow y = \frac{1}{5}(x + 1)$

i.e. $\quad f^{-1}(x) = \frac{1}{5}(x + 1)$

$\therefore \quad f^{-1}(4) = \frac{1}{5}(4 + 1) = 1$

EXERCISE 12g

1. Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

(a) $f(x) = 3x - 1$ (b) $f(x) = 2^{-x}$ (c) $f(x) = (x-1)^3$

(d) $f(x) = 2 - x$ (e) $f(x) = \dfrac{1}{x-3}$ (f) $f(x) = \dfrac{1}{x}$

2. Which of the functions given in Question 1 are their own inverses?

3. Determine whether f has an inverse function, and if it does, find it when

(a) $f(x) = x + 1$ (b) $f(x) = x^2 + 1$ (c) $f(x) = x^3 + 1$

(d) $f(x) = x^2 - 4,\ x \geqslant 0$ (e) $f(x) = (x+1)^4,\ x \geqslant -1$

4. The function f is given by $f(x) = 1 - \dfrac{1}{x}$. Find

(a) $f^{-1}(4)$ (b) the value of x for which $f^{-1}(x) = 2$

(c) any values of x for which $f^{-1}(x) = x$

5. If $f(x) = 3^x$, find

(a) $f(2)$ (b) $f^{-1}(9)$ (c) $f^{-1}(\frac{1}{3})$

COMPOUND FUNCTIONS

Consider the two functions f and g given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \frac{1}{x}$$

These two functions can be combined in several ways.

1) They can be added or subtracted,

i.e. $f(x) + g(x) = x^2 + \dfrac{1}{x}$ and $f(x) - g(x) = x^2 - \dfrac{1}{x}$

2) They can be multiplied or divided,

i.e. $f(x)g(x) = (x^2)\left(\dfrac{1}{x}\right) = x$ and $\dfrac{f(x)}{g(x)} = \dfrac{x^2}{1/x} = x^3$

3) The output of f can be made the input of g,

i.e. $x \xrightarrow{f} x^2 \xrightarrow{g} \dfrac{1}{x^2}$ or $g[f(x)] = g(x^2) = \dfrac{1}{x^2}$

Therefore the function $x \to 1/x^2$ is obtained by taking the function g of the function f.

FUNCTION OF A FUNCTION

A compound function formed in the way described in **(3)** above is known as a *function of a function* and it is denoted by gf

For example, if $f(x) = 3^x$ and $g(x) = 1 - x$ then $gf(x)$ means the function g of the function $f(x)$,

i.e. $gf(x) = g(3^x) = 1 - 3^x$

Similarly $fg(x) = f(1-x) = 3^{(1-x)}$

Note that $gf(x)$ is *not* the same as $fg(x)$

EXERCISE 12h

1. If f, g and h are functions defined by $f(x) = x^2$, $g(x) = 1/x$, $h(x) = 1 - x$ find as a function of x

(a) fg (b) fh (c) hg (d) hf (e) gf

2. If $f(x) = 2x - 1$ and $g(x) = x^3$ find the value of

(a) gf(3) (b) fg(2) (c) fg(0) (d) gf(0)

3. Given that $f(x) = 2x$, $g(x) = 1 + x$ and $h(x) = x^2$, find as a function of x

(a) hg (b) fhg (c) ghf

4. The function $f(x) = (2-x)^2$ can be expressed as a function of a function. Find g and h as functions of x such that $gh(x) = f(x)$

5. Repeat Question 4 when $f(x) = (x+1)^4$

6. Express the function $f(x)$ as a combination of functions $g(x)$ and $h(x)$, and define $g(x)$ and $h(x)$, where $f(x)$ is

(a) $10^{(x+1)}$ (b) $1/(3x-2)^2$ (c) $2^x + x^2$

(d) $(2x+1)/x$ (e) $(5x-6)^4$ (f) $(x-1)(x^2-2)$

MIXED EXERCISE 12

1. A function f is defined by $f(x) = 1/(1-x),\ x \neq 1$
   (a) Why is 1 excluded from the domain of f?
   (b) Find the value of $f(-3)$
   (c) Sketch the curve $y = f(x)$
   (d) Find $f^{-1}(x)$ in terms of x and give the domain of f^{-1}

2. Find the greatest or least value of each of the following functions, stating the value of x at which they occur.
   (a) $f(x) = x^2 - 3x + 5$ (b) $f(x) = 2x^2 - 7x + 1$
   (c) $f(x) = (x-1)(x+5)$

3. If $f(x) = 10^x$, sketch the following curves on the same set of axes.
   (a) $y = f(x)$ (b) $y = f(x+3)$ (c) $y = f^{-1}(x)$

4. Given that $f(x) = 10^x$, $g(x) = x^2$ and $h(x) = 1/x$,
   (a) find fg(2), hg(3) and gf(−1)
   (b) find, in terms of x, hfg(x) and gfh(x)
   (c) if they exist, find $f^{-1}(x)$, $g^{-1}(x)$ and $h^{-1}(x)$
   (d) the value(s) of x for which $gh(x) = 9$
   (e) does the function $(gh)^{-1}$ exist?

5. Draw sketches of the following curves, showing any asymptotes.
   (a) $(x-2)(x-3)(x-4)$ (b) $y = \dfrac{1}{3-x}$ (c) $y = 2^{4-x}$

6. The function f is given by $f(x) = 2^{(3x-2)}$
   (a) find $g(x)$ and $h(x)$ such that $f = gh$
   (b) evaluate ff(2) and $f^{-1}(2)$

7. The functions f, g and h are defined by $f(x) = 2x$, $g(x) = 3x^2$ and $h(x) = x - 1$
   (a) Sketch the curves $y = g(x-3)$ and $y = gf^{-1}(x)$
   (b) Find the value(s) of x for which $f^{-1}(x) = g(x)$

CHAPTER 13

INEQUALITIES

MANIPULATING INEQUALITIES

An inequality compares two unequal quantities.

Consider, for example, the two real numbers 3 and 8 for which

$$8 > 3$$

The inequality remains true, i.e. the inequality sign is unchanged, when the same term is added or subtracted on both sides, e.g.

$$8+2 > 3+2 \quad\Rightarrow\quad 10 > 5$$

and $$8-1 > 3-1 \quad\Rightarrow\quad 7 > 2$$

The inequality sign is unchanged also when both sides are multiplied or divided by a positive quantity, e.g.

$$8 \times 4 > 3 \times 4 \quad\Rightarrow\quad 32 > 12$$

and $$8 \div 2 > 3 \div 2 \quad\Rightarrow\quad 4 > 1\tfrac{1}{2}$$

If, however, both sides are multiplied or divided by a *negative* quantity the inequality is no longer true. For example, if we multiply by -1, the LHS becomes -8 and the RHS becomes -3 so the correct inequality is now LHS < RHS, i.e.

$$8 \times -1 < 3 \times -1 \quad\Rightarrow\quad -8 < -3$$

Similarly, dividing by -2 gives $-4 < -1\tfrac{1}{2}$

These examples are illustrations of the following general rules.

Adding or subtracting a term, or multiplying or dividing both sides by a positive number, does not alter the inequality sign.

Multiplying or dividing both sides by a *negative* number reverses the inequality sign.

i.e. if a, b and k are real numbers, and $a > b$ then,

$a + k > b + k$ for *all* values of k

$ak > bk$ for *positive* values of k

$ak < bk$ for *negative* values of k

SOLVING LINEAR INEQUALITIES

When an inequality contains an unknown quantity, the rules given above can be used to 'solve' it. Whereas the solution of an equation is a value, or values, of the variable, the solution of an inequality is a range, or ranges of values, of the variable.

If the unknown quantity appears only in linear form, we have a *linear inequality* and the solution range has only *one boundary*.

Example 13a

Find the set of values of x that satisfy the inequality $x - 5 < 2x + 1$

$x - 5 < 2x + 1$

$\Rightarrow \quad x < 2x + 6$ adding 5 to each side

$\Rightarrow \quad -x < 6$ subtracting $2x$ from each side

$\Rightarrow \quad x > -6$ multiplying both sides by -1

Therefore the set of values of x satisfying the given inequality is

$$x > -6$$

EXERCISE 13a

Solve the following inequalities.

1. $x - 4 < 3 - x$	2. $x + 3 < 3x - 5$	3. $x < 4x + 9$
4. $7 - 3x < 13$	5. $x > 5x - 2$	6. $2x - 1 < x - 4$
7. $1 - 7x > x + 3$	8. $2(3x - 5) > 6$	9. $3(3 - 2x) < 2(3 + x)$

SOLVING QUADRATIC INEQUALITIES

A quadratic inequality is one in which the variable appears to the power 2, e.g. $x^2 - 3 > 2x$
The solution is a range or ranges of values of the variable with *two boundaries*.

If the terms in the inequality can be collected and factorised, a graphical solution is easy to find.

Example 13b

Find the range(s) of values of x that satisfy the inequality $x^2 - 3 > 2x$

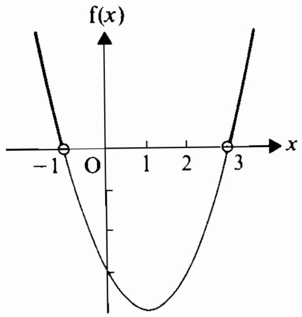
$$x^2 - 3 > 2x$$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x - 3)(x + 1) > 0$

or $\quad f(x) > 0$ where $f(x) = (x - 3)(x + 1)$

If we sketch the graph of $f(x)$ then $f(x) > 0$ where the graph is above the x-axis. The values of x corresponding to these portions of the graph satisfy $f(x) > 0$
The points where $f(x) = 0$, i.e. where $x = 3$ and -1 are not part of this solution and this is indicated on the sketch by open circles.



From the graph we see that the ranges of values of x which satisfy the given inequality are

$$x < -1 \text{ and } x > 3$$

Note that the solution in the above example was two separate ranges each with its own boundary. If however we consider the inequality $(x - 3)(x + 1) < 0$ the part of the graph of $f(x)$ for which $f(x) < 0$ is below the x-axis and the corresponding values of x are $-3 < x < 1$
This time there is only one range, but it still has two boundaries.

EXERCISE 13b

Find the ranges of values of x that satisfy the following inequalities.

1. $(x-2)(x-1) > 0$	**2.** $(x+3)(x-5) \geqslant 0$
3. $(x-2)(x+4) < 0$	**4.** $(2x-1)(x+1) \geqslant 0$
5. $x^2 - 4x > 3$	**6.** $4x^2 < 1$
7. $(2-x)(x+4) \geqslant 0$	**8.** $5x^2 > 3x + 2$
9. $(3-2x)(x+5) \leqslant 0$	**10.** $(x-1)^2 > 9$
11. $(x+1)(x+2) \leqslant 4$	**12.** $(1-x)(4-x) > x + 11$

RATIONAL FRACTIONS IN INEQUALITIES

Consider the inequality $\dfrac{x-2}{x+5} < 3$

The initial problem here is that we do not know whether $x+5$ is positive or negative. This prevents the apparently obvious step of multiplying both sides by $x+5$. A number of different ways of solving inequalities of this type are demonstrated in the worked examples that follow. No one method is ideal in all cases and the reader is advised to consider a variety of approaches before deciding how to solve a particular example.

Examples 13c

1. Find the range of values of x for which $\dfrac{x-2}{x-5} > 3$

Although we cannot multiply both sides by $x-5$ because its sign is not known, we *can* multiply both sides by $(x-5)^2$ which cannot be negative.

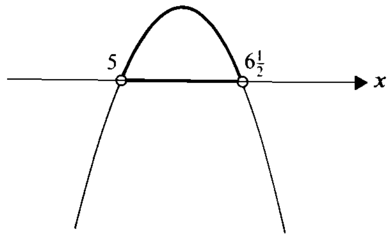
$$\frac{x-2}{x-5} > 3$$

$$\therefore \quad (x-2)(x-5) > 3(x-5)^2$$

$$\Rightarrow \quad (x-5)\{(x-2) - 3(x-5)\} > 0$$

Note that $(x-5)$ must not be cancelled because we do not know its sign.

$$\Rightarrow \quad (x-5)(13-2x) > 0$$



$\therefore$ the required range is $5 < x < 6\frac{1}{2}$

2. Find the possible values of x for which $\dfrac{(x-2)(x+3)(x-4)}{x-1} < 0$

$$f(x) = \frac{(x-2)(x+3)(x-4)}{x-1} < 0$$

The critical values of x in this inequality are -3, 1, 2 and 4 so we construct a table in which the columns are separated by these values, i.e.

	$x < -3$	$-3 < x < 1$	$1 < x < 2$	$2 < x < 4$	$x > 4$
$x+3$	$-$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$+$
f(x)	$+$	$-$	$+$	$-$	$+$

Therefore $\dfrac{(x-2)(x+3)(x-4)}{x-1} < 0$ if $-3 < x < 1$ or $2 < x < 4$

3. What values of x satisfy the inequality $\dfrac{(x-2)^2-8}{5-4x} > 1$?

$$\frac{(x-2)^2-8}{5-4x} > 1 \quad \Rightarrow \quad \frac{(x-2)^2-8}{5-4x} - 1 > 0$$

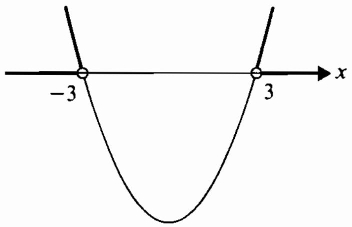
$$\Rightarrow \quad \frac{(x-2)^2-8-(5-4x)}{(5-4x)} > 0$$

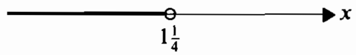
$$\Rightarrow \quad \frac{x^2-9}{5-4x} > 0$$

$$\Rightarrow \quad \frac{(x-3)(x+3)}{5-4x} > 0$$

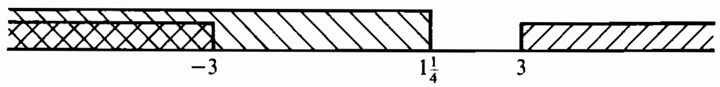
i.e. $\dfrac{f(x)}{g(x)} > 0$ where $f(x) = (x-3)(x+3)$ and $g(x) = 5-4x$

This fraction is positive if $f(x)$ and $g(x)$ have the same sign.





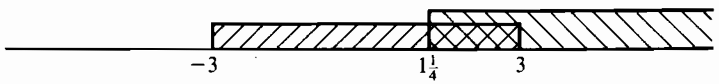
$f(x) > 0$ if $x < -3$ or $x > 3$ and $g(x) > 0$ if $x < 1\frac{1}{4}$



$\therefore$ $f(x)$ and $g(x)$ are *both* positive if $x < -3$

Similarly,

$f(x) < 0$ if $-3 < x < 3$ and $g(x) < 0$ if $x > 1\frac{1}{4}$

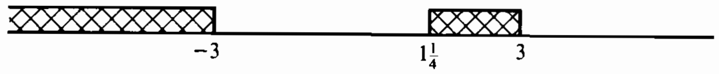


$\therefore$ $f(x)$ and $g(x)$ are *both* negative if $1\frac{1}{4} < x < 3$

$\therefore$ $\dfrac{x^2-9}{5-4x}$ is positive, i.e. $\dfrac{(x-2)^2}{5-4x} > 1$

for values of x in *both* of the ranges found above,

i.e. for $1\frac{1}{4} < x < 3$ *and* $x < -3$



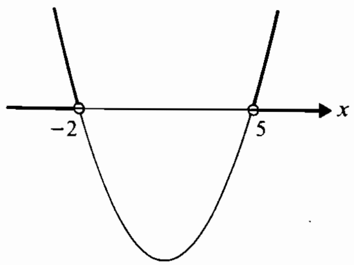
4. Find the range of values of x for which $3x+4 < x^2-6 < 9-2x$

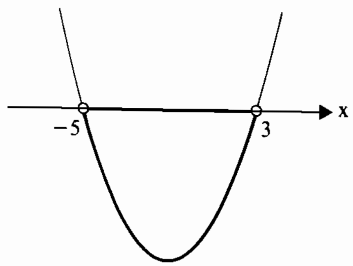
$3x+4 < x^2-6 < 9-2x$ is called a *double inequality* because it contains two inequalities, i.e. $3x+4 < x^2-6$ and $x^2-6 < 9-2x$

We are looking for the set of values of x for which *both* inequalities are satisfied so we first solve each of them separately.

$$3x+4 < x^2-6$$
$$\Rightarrow \quad 3x - x^2 + 10 < 0$$
$$\Rightarrow \quad x^2 - 3x - 10 > 0$$
$$\Rightarrow \quad (x-5)(x+2) > 0$$

$$x^2-6 < 9-2x$$
$$\Rightarrow \quad x^2 + 2x - 15 < 0$$
$$\Rightarrow \quad (x+5)(x-3) < 0$$

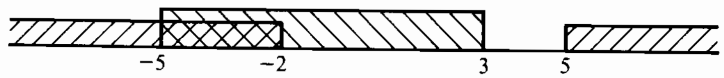




$\therefore$ $x < -2$ or $x > 5$ $\qquad$ $\therefore$ $-5 < x < 3$

The required set of values of x must satisfy *both* of these conditions,

i.e.



$\therefore$ the solution set is $-5 < x < -2$

EXERCISE 13c

For what range(s) of values of x are the following inequalities valid?

1. $\dfrac{x-1}{2+x} < 1$
2. $\dfrac{x-1}{2+x} > 1$
3. $\dfrac{3}{x-1} > 2$
4. $\dfrac{x}{2x-8} > 3$
5. $\dfrac{12}{x-3} < x+1$
6. $\dfrac{x}{x-2} < \dfrac{x}{x-1}$
7. $\dfrac{(x-2)}{(x-1)(x-3)} > 0$
8. $\dfrac{2x}{(x-4)^2} > 1$
9. $\dfrac{(x-1)(x-2)}{(x+1)(x-3)} < 0$
10. $(3x-2)(2x+1) < 6x-3$
11. $(x+1)(x+3)(x+5) > 0$
12. $2-x > 2x+4 > x$
13. $x-1 < 3x+1 < x+5$
14. $2x-1 < x^2-4 < 12$
15. $x-4 < x(x-4) < 5$
16. $x-3 > x^2-9 > -5$

PROBLEMS

The types of problem which involve inequalities are very varied. Their solutions depend not only on all the methods used so far in this chapter but also on other facts known to the reader, for example:

1) a perfect square can never be negative.

2) the nature of the roots of a quadratic equation $ax^2+bx+c=0$ depends upon whether $b^2-4ac=0$ or $b^2-4ac>0$ or $b^2-4ac<0$. As two of the above conditions are inequalities, many problems about the roots of a quadratic equation require the solution or interpretation, of inequalities.

The worked examples that follow are intended to give the reader some ideas to use in problem solving and make no claim to cover every situation.

Examples 13d

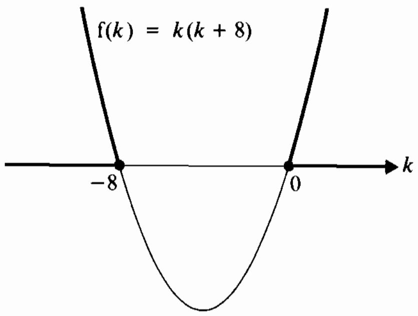
1. Find the range(s) of values of k for which the roots of the equation $kx^2+kx-2=0$ are real.

$$kx^2+kx-2=0$$

For real roots 'b^2-4ac' $\geqslant 0$

i.e. $\quad k^2-4(k)(-2) \geqslant 0$

$\Rightarrow \quad k(k+8) \geqslant 0$



From the sketch we see that $k(k+8) \geqslant 0$ for $k \leqslant -8$ and $k \geqslant 0$

Therefore the equation $kx^2+kx-2=0$ has real roots if the value of k lies in either of the ranges $k \leqslant -8$ or $k \geqslant 0$

Note. This type of question is sometimes expressed in another, less obvious, way, i.e. 'If x is real and $kx^2+kx-2=0$, find the values that k can take'. Once the reader appreciates that, because x is real the roots of the equation are real, the solution is identical to that above.

---

2. Prove that $x^2+2xy+2y^2$ cannot be negative.

Knowing that a perfect square cannot be negative, we rearrange the given expression in the form of perfect squares.

$$x^2+2xy+2y^2 = x^2+2xy+y^2+y^2$$
$$= (x+y)^2+y^2$$

Each of the two terms on the RHS is a square and so cannot be negative.

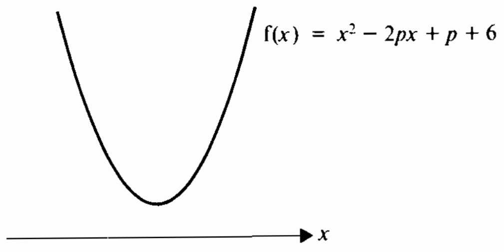
Therefore $x^2+2xy+2y^2$ cannot be negative.

---

3. Find the values of p for which $x^2 - 2px + p + 6$ is positive for all real values of x

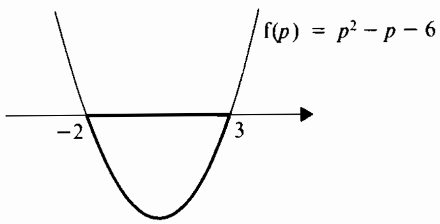
$$x^2 - 2px + p + 6 \text{ is positive for all values of } x$$

Therefore the graph of $f(x)$, where $f(x) = x^2 - 2px + p + 6$, is entirely above the x-axis, i.e.



The graph never crosses the x-axis so there are no values of x for which $f(x) = 0$, i.e. $x^2 - 2px + p + 6 = 0$ has no real roots.

$$\therefore \quad 'b^2 - 4ac' < 0 \Rightarrow (-2p)^2 - 4(1)(p+6) < 0$$
$$\Rightarrow 4p^2 - 4p - 24 < 0$$
$$\Rightarrow p^2 - p - 6 < 0$$
$$\Rightarrow (p+2)(p-3) < 0$$



From the graph of $f(p) = p^2 - p - 6$ we see that $f(p) < 0$ for values of p between -2 and 3

Therefore $x^2 - 2px + p + 6$ is positive for all real values of x provided that $-2 < p < 3$

---

4. If x is real find the set of possible values of the function $\dfrac{x^2}{x+1}$

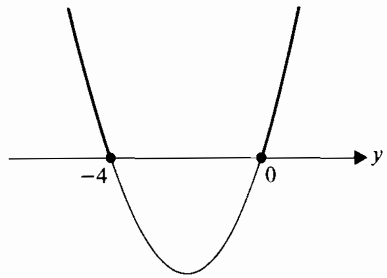
If we use $y = \dfrac{x^2}{x+1}$ we are looking for the range of values of y

To make use of the fact that x is real we need a quadratic equation in x

$$y = \frac{x^2}{x+1} \Rightarrow x^2 - yx - y = 0$$

Since x is real, the roots of this equation are real, so $'b^2 - 4ac' \geqslant 0$

i.e. $\quad (-y)^2 - 4(1)(-y) \geqslant 0 \Rightarrow y(y+4) \geqslant 0$



$\therefore \ y \leqslant -4$ or $y \geqslant 0$

Therefore, for real values of x,

$$\frac{x^2}{x+1} \leqslant -4 \quad \text{or} \quad \frac{x^2}{x+1} \geqslant 0$$

EXERCISE 13d

1. Find the values of p for which the given equation has real roots.
   (a) $x^2 + (p+3)x + 4p = 0$ (b) $x^2 + 3x + 1 = px$

2. Find the range of values of a for which the equation $x^2 - ax + (a+3) = 0$ has no real roots.

3. What is the set of values of p for which $p(x^2 + 2) < 2x^2 + 6x + 1$ for all real values of x?

In Questions 4 to 9 find the set of possible values of the given function.

4. $\dfrac{x+1}{2x^2+x+1}$ 5. $\dfrac{1+x^2}{x}$ 6. $\dfrac{x-2}{(x+2)(x-3)}$

7. $\dfrac{x}{1+x^2}$ 8. $\dfrac{x-1}{x(x+1)}$ 9. $x+1+\dfrac{1}{x+1}$

10. Find the set of values of k for which $x^2+3kx+k$ is positive for all real values of x

11. Show that, if x is real, the function $\dfrac{2-x}{x^2-4x+1}$ can take any real value.

12. If x is real, find the range of the function $\dfrac{(2x+1)}{(x^2+2)}$

13. Show that $x^2-4xy+5y^2 \geqslant 0$ for all real values of x and y

14. Prove that $(a+b)^2 \geqslant 4ab$ for all real values of a and b

MIXED EXERCISE 13

Solve each of the inequalities given in Questions 1 to 10.

1. $2x+1 < 4-x$
2. $x-5 > 1-3x$
3. $6x-5 > 1+2x$
4. $(x-3)(x+2) > 0$
5. $(2x-3)(3x+2) < 0$
6. $x^2-3 < 10$
7. $(x-3)^2 > 2$
8. $(3-x)(2-x) < 20$
9. $x(4x+3) > 2x-1$
10. $(x-6)(x+1) > 2x-12$

In Questions 11 to 16 find the set of values of x for which:

11. $-3 < 5-2x < 3$
12. $x^2+x+1 < x+2 < x^2-6x+12$
13. $\dfrac{2x-4}{x-1} < 1$
14. $\dfrac{x-1}{x+1} \leqslant x$
15. $2 \geqslant \dfrac{x-1}{x+1} \geqslant 0$
16. $\dfrac{(x-1)(x+1)}{(x+2)(x-2)} \leqslant 0$

17. Prove that $x^2+y^2-10y+25 \geqslant 0$ for all real values of x and y

18. For what values of k does the equation $4x^2+8x-8 = k(4x-3)$ have real roots?

19. If x is real find the set of possible values of the function
$$\frac{x^2+1}{x^2+x+1}$$

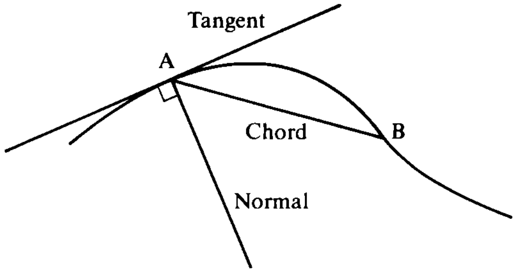
20. Provided that x is real, prove that the function $\dfrac{2(3x+1)}{3(x^2-9)}$ can take all real values.

CHAPTER 14

DIFFERENTIATION — BASIC

CHORDS, TANGENTS AND NORMALS

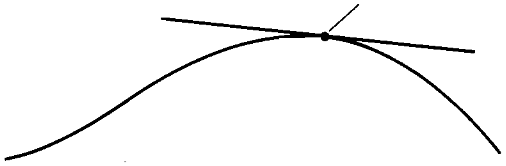
Consider any two points, A and B, on any curve.



The line joining A and B is called a chord.

The line that touches the curve at A is called the tangent at A.

Note that the word *touch* has a precise mathematical meaning, i.e. a line that meets a curve at a point and carries on without crossing to the other side of the curve at that point, is said to *touch* the curve at the *point of contact*.

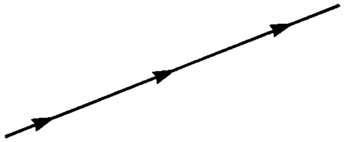


The line perpendicular to the tangent at A is called the normal at A.

THE GRADIENT OF A CURVE

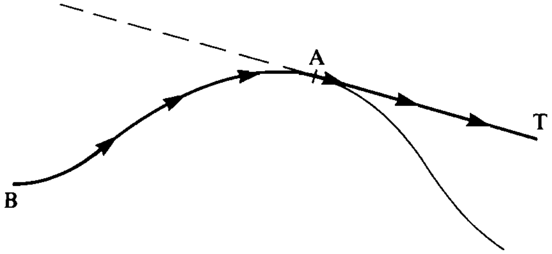
107
108

Gradient, or slope, defines the direction of a line (lines can be straight or curved).



If we imagine walking along a straight line, we walk in the same direction all the time, i.e. the gradient of a straight line is constant.

If, however, we imagine walking along a curve, our direction is continually changing. It follows that the gradient of a curve is not constant but has different values at different points on the curve.



Suppose that we move from B to A along the curve in the diagram above, then our direction is changing all the time. Now if at A we continued to move, but without any further change in direction, we would go along the straight line AT, i.e. along the tangent to the curve at A, so

the gradient of the curve at A is the same as the gradient of the tangent to the curve at A.

Before a numerical value can be given to a gradient, the line or curve must be drawn on a pair of x and y axes. Then, as the reader will already know, the gradient is the rate at which y increases with respect to x.

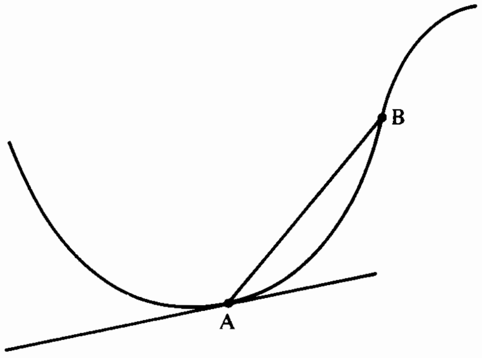
For a straight line this is found by taking the coordinates of two points and working out

$$\frac{\text{increase in } y}{\text{increase in } x}$$

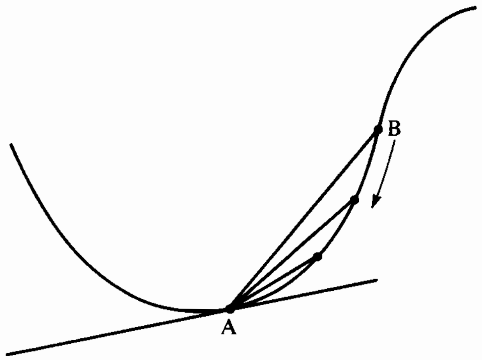
This can be used to find the gradient of a tangent to a curve but, if the tangent is just drawn by eye, the value obtained can only be an approximation.

A more precise method is needed for determining the gradient of a curve whose equation is known, so that further analysis can be made of the properties of such a curve.

Let us consider the problem of finding the gradient of the tangent at a point A on a curve.



If B is another point on the curve, fairly close to A, then the gradient of the chord AB gives an *approximate* value for the gradient of the tangent at A. As B gets nearer to A, the chord AB gets closer to the tangent at A, so the approximation becomes more accurate.



So, as B gets closer and closer to A, we can say,

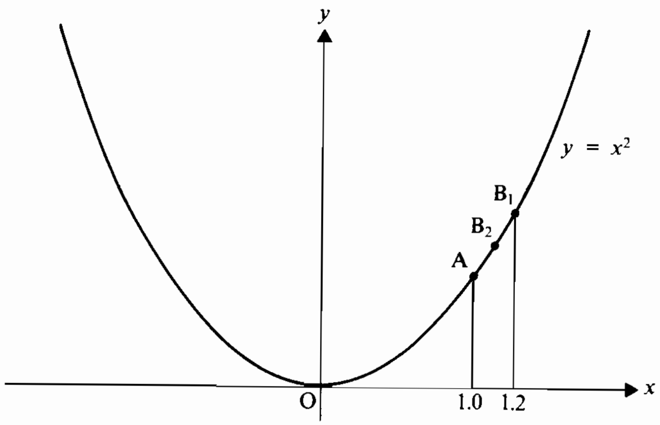
as B → A

the gradient of chord AB → the gradient of the tangent at A

This fact can also be expressed in the form

$$\lim_{\text{as } B \to A} (\text{gradient of chord AB}) = \text{gradient of tangent at A}$$

This definition can be applied to a particular point on a particular curve. Suppose, for instance, that we want the gradient of the curve $y = x^2$ at the point where $x = 1$



A is the point (1, 1) and we need a succession of points, B_1, B_2, ... getting closer and closer to A. Let us take the points where $x = 1.2,\ 1.1,\ 1.05,\ 1.01,\ 1.001$, then calculate the corresponding y coordinates, find the increases in x and y between A and B and hence find the gradient of AB.

x	1.2	1.1	1.05	1.01	1.001
$y \quad (= x^2)$	1.44	1.21	1.1025	1.0201	1.002 001
Increase in y	0.44	0.21	0.1025	0.0201	0.002 001
Increase in x	0.2	0.1	0.05	0.01	0.001
Gradient of chord AB	2.2	2.1	2.05	2.01	2.001

From the numbers in the last row of the table it is clear that, as B gets nearer to A, the gradient of the chord gets nearer to 2, i.e.

$$\lim_{\text{as } B \to A} (\text{gradient of chord AB}) = 2$$

It is equally clear that it is much too tedious to go through this process each time we want the gradient at just one point on just one curve and that we need a more general method. For this we use a general point $A(x, y)$ and a variable small change in the value of x between A and B.

A new symbol, δ, is used to denote this small change.

When δ appears as a prefix to any letter representing a variable quantity, it denotes a small increase in that quantity,

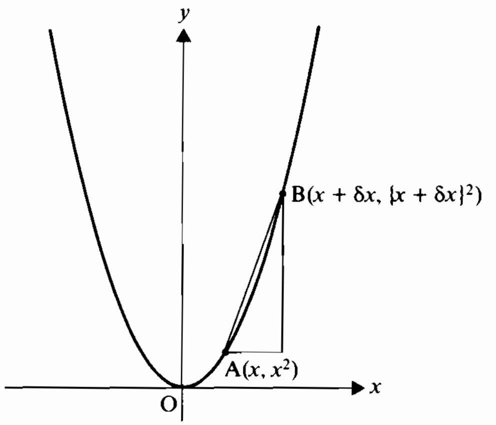
e.g. δx means a small increase in x

δy means a small increase in y

δt means a small increase in t

Note that δ is only a prefix. It does not have an independent value and cannot be treated as a factor.

Now consider again the gradient of the curve with equation $y = x^2$



This time we will look for the gradient at *any* point $A(x, y)$ on the curve and use a point B where the x-coordinate of B is $x + \delta x$.

For any point on the curve, $y = x^2$

So, at B, the y-coordinate is $(x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$

Therefore the gradient of chord AB, which is given by $\dfrac{\text{increase in } y}{\text{increase in } x}$, is

$$\frac{(x + \delta x)^2 - x^2}{(x + \delta x) - x} = \frac{2x\delta x + (\delta x)^2}{\delta x}$$

$$= 2x + \delta x$$

Now as $B \to A$, $\delta x \to 0$, therefore

$$\text{gradient of curve at A} = \lim_{\text{as } B \to A} (\text{gradient of chord AB})$$

$$= \lim_{\text{as } \delta x \to 0} (\text{gradient of chord AB})$$

$$= \lim_{\text{as } \delta x \to 0} (2x + \delta x)$$

$$= 2x$$

This result can now be used to give the gradient at any point on the curve with equation $y = x^2$, where the x-coordinate is given, e.g.

at the point where $x = 3$, the gradient is $2(3) = 6$
and at the point $(4, 16)$, the gradient is $2(4) = 8$

Looking back at the longer method we used on page 225 to find the gradient at the point where $x = 1$, we see that the value obtained there is confirmed by using the general result,
i.e. gradient $= 2x = 2(1) = 2$

DIFFERENTIATION

109 The process of finding a general expression for the gradient of a curve at any point is known as differentiation.

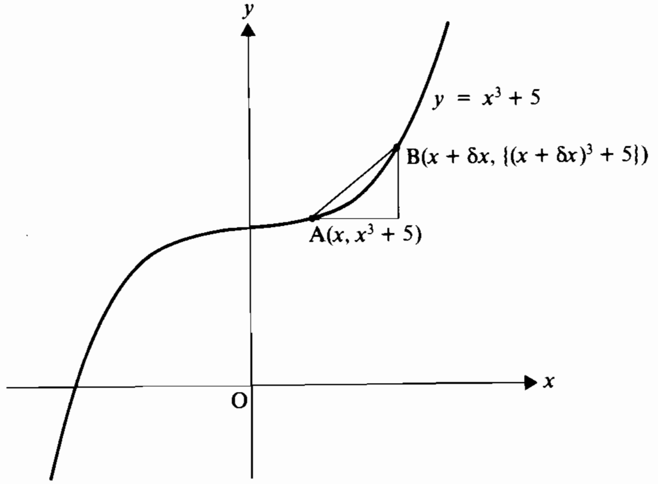
110 The general gradient expression for a curve $y = f(x)$ is itself a function so it is called the *gradient function*. For the curve $y = x^2$ for example, the gradient function is $2x$.

Because the gradient function is derived from the given function, it is more often called the *derived function* or the *derivative*.

The method used above, in which the limit of the gradient of a chord was used to find the derived function, is known as *differentiating from first principles*. It is the fundamental way in which the gradient of each new type of function is found and, although many short cuts can be developed, it is important to understand this basic method.

Example 14a

By differentiating from first principles, find the gradient function of the expression $x^3 + 5$. Find also the gradient at the point (2, 13) on the curve $y = x^3 + 5$



Let A and B be two neighbouring points on the curve, where A is the point (x, y) and the x-coordinate of B is $x + \delta x$

Therefore at A $\quad y = x^3 + 5$

and at B $\quad y = (x + \delta x)^3 + 5$

The gradient of chord AB is $\dfrac{\{(x + \delta x)^3 + 5\} - \{x^3 + 5\}}{\{x + \delta x\} - x}$

which simplifies to $\quad 3x^2 + 3x\delta x + (\delta x)^2$

The gradient of the curve at A $= \lim_{\text{as } \delta x \to 0} \{3x^2 + 3x\delta x + (\delta x)^2\}$

$= 3x^2$

Therefore the gradient function of $x^3 + 5$ is $3x^2$

At the point (2, 13), $x = 2$, so the gradient is $3(2)^2 = 12$

EXERCISE 14a

Find, by differentiating from first principles, the derivative (i.e. the gradient function) of each of the following expressions. Hence find the gradient of the curve $y = f(x)$ at the given point.

1. $4x$; (1, 4)
2. $x - 1$; (3, 2)
3. x^3; (1, 1)
4. $x^2 + 2$; (2, 6)
5. $x^2 - x$; (1, 0)
6. x^4; (2, 16) (Pascal's triangle will help)

NOTATION

We have seen that differentiating x^2 gives $2x$

One way to write this fact, based on the equation of the curve, is

for the curve $y = x^2$, $\dfrac{dy}{dx} = 2x$ (we say dy by dx)

Each of the results obtained in Exercise 14a can be written using this notation, e.g.

for $y = x^2 - x$, $\dfrac{dy}{dx} = 2x - 1$ and for $y = x^4$, $\dfrac{dy}{dx} = 4x^3$

Note carefully that d *has no independent meaning and must never be regarded as a factor.*

The complete symbol $\dfrac{d}{dx}$ means 'the derivative with respect to x of'

So, $\dfrac{dy}{dx}$ means 'the derivative with respect to x of y'

and $\dfrac{d}{dx}(x^2 - x)$ means 'the derivative with respect to x of $(x^2 - x)$'

An alternative notation concentrates on the function of x rather than the equation of the curve. An example is

for $f(x) = x^2$, $f'(x) = 2x$

In this form, f′ means 'the gradient function' or 'the derived function'.

Again we can illustrate this notation using results from Exercise 14a, e.g.

for $f(x) = x^3$, $f'(x) = 3x^2$ and for $f(x) = x^4$, $f'(x) = 4x^3$

Either of these notations can be used for variables other than x, e.g.

if $y = z^3$ then we differentiate y with respect to z

and write $\dfrac{dy}{dz} = 3z^2$

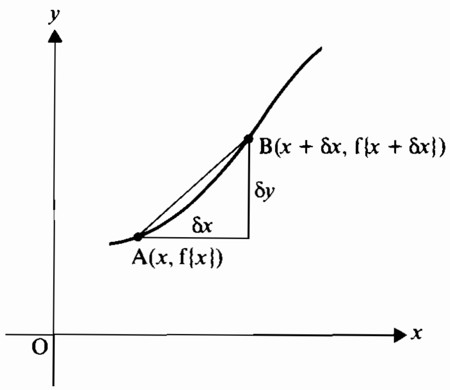
Similarly, if $s = t^2 - t$, we differentiate s with respect to t

and write $\dfrac{ds}{dt} = 2t - 1$

(Because the phrase 'with respect to' is used very frequently, it is often abbreviated to w.r.t.)

THE GENERAL GRADIENT FUNCTION

Consider any curve with equation $y = f(x)$



Taking two points on the curve, $A(x, y)$ and $B(x + \delta x, y + \delta y)$ we have,

at A $\quad y = f(x) \quad$ and at B $\quad y + \delta y = f(x + \delta x)$

Therefore the gradient of AB is given by

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Now as $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$ therefore, in general

$$\frac{dy}{dx} = f'(x) = \lim_{\text{as } \delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Example 14b

Find the derivative of the function $1/x$

$$f(x) = \frac{1}{x} \Rightarrow f(x + \delta x) = \frac{1}{x + \delta x}$$

$$f(x + \delta x) - f(x) = \frac{1}{x + \delta x} - \frac{1}{x} = \frac{x - (x + \delta x)}{x(x + \delta x)}$$

$$= \frac{-\delta x}{x(x + \delta x)}$$

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{-\delta x}{x(x + \delta x)(\delta x)} = \frac{-1}{x(x + \delta x)}$$

Hence

$$f'(x) = \lim_{\text{as } \delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\text{as } \delta x \to 0} \frac{-1}{x(x + \delta x)}$$

$$= \frac{-1}{x^2}$$

i.e. the derivative of $1/x$ is $-1/x^2$

EXERCISE 14b

Use the general formula for the derivative of $f(x)$ to differentiate

1. $1/x^2$ **2.** $1/x$ **3.** $2/x$

DIFFERENTIATING x^n WITH RESPECT TO x

Some of the results that have been produced so far can now be collected and tabulated.

y	x^2	x^3	x^4	x^{-1}
$\dfrac{dy}{dx}$	$2x$	$3x^2$	$4x^3$	$-x^{-2}$ (or $-1/x^2$)

From this table it *appears* that when we differentiate a power of x we multiply by the power and then reduce the power by 1, i.e. it looks as though

$$\text{if } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1}$$

This result, although deduced from just a few examples, is in fact valid for all powers, including those that are fractional or negative. It is not possible to give a proof at this stage and this is one example of a 'rule' which, for the moment, we must just take on trust. It is easy to apply and makes the task of differentiating a power of x very much simpler, e.g.

$$\frac{d}{dx}(x^7) = 7x^6 \qquad \frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3} \qquad \frac{d}{dx}(x^{3/2}) = (3/2)x^{1/2}$$

Example 14c

Differentiate with respect to x

(a) $x^{-1/3}$ (b) $\sqrt[4]{(x^3)}$

(a) Using $\frac{d}{dx}(x^n) = nx^{n-1}$, where $n = -1/3$, we have

$$\frac{dy}{dx} = -\tfrac{1}{3}x^{-1/3-1} = -\tfrac{1}{3}x^{-4/3} \text{ or } -\frac{1}{3x^{4/3}}$$

(b) $\sqrt[4]{(x^3)}$ can be written $x^{3/4}$, i.e. $n = 3/4$

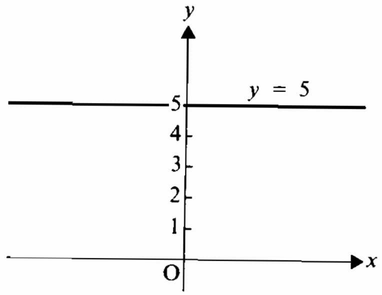
Therefore $\frac{d}{dx}(x^{3/4}) = \frac{3}{4}x^{3/4-1} = \frac{3}{4}x^{-1/4}$ or $\dfrac{3}{4\sqrt[4]{x}}$

EXERCISE 14c

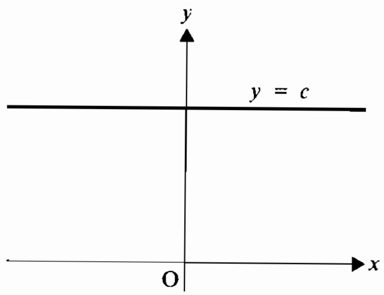
Differentiate with respect to x.

1. x^5	**2.** x^{-3}	**3.** $x^{4/3}$	**4.** $1/x$
5. x^{10}	**6.** $1/x^2$	**7.** $\sqrt{x^3}$	**8.** $x^{-1/2}$
9. $1/x^4$	**10.** $x^{1/3}$	**11.** $x^{-1/4}$	**12.** x
13. $\sqrt{x^7}$	**14.** $1/x^7$	**15.** $x^{1/7}$	**16.** $\sqrt{(x^2)^3}$

Differentiating a Constant



Consider the equation $y = 5$. Graphically this is a horizontal straight line and its gradient is zero, i.e. $\dfrac{dy}{dx} = 0$

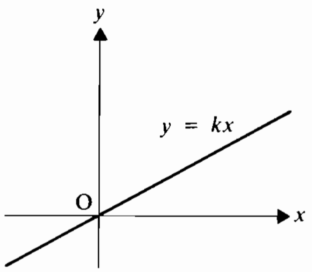


This argument applies to any equation of the form $y = c$ where c is a constant,

i.e. $$\text{if } y = c \text{ then } \frac{dy}{dx} = 0$$

Differentiating a Linear Function of x

The graph of the equation $y = kx$, where k is a constant, is a straight line with gradient k.



Hence $$\frac{d}{dx}(kx) = k$$

Now if we apply the general rule for differentiating x^n to $y = x$, i.e. to $y = x^1$

$$\frac{d}{dx}(x^1) = 1x^0 = 1$$

Combining these two facts shows that

$$\frac{d}{dx}(kx) = k \times \frac{d}{dx}(x)$$

This conclusion applies, in fact, to a constant multiple of *any* function of x, e.g.

if $y = 3x^5$, $\frac{dy}{dx} = 3 \times 5x^4 = 15x^4$

and if $y = 4x^{-2}$, $\frac{dy}{dx} = 4 \times -2x^{-3} = -8x^{-3}$

In general then, if a is a constant,

$$\frac{d}{dx}ax^n = anx^{n-1}$$

This rule, although not proved for the general case, can be used freely.

Another very useful property is

a function of x which contains a number of different terms can be differentiated term by term, applying the basic rule to each in turn.

For example

$$\text{if } y = x^4 + \frac{1}{x} - 6x \quad \text{then} \quad \frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(6x)$$
$$= 4x^3 - \frac{1}{x^2} - 6$$

(This property is justified, though not proved, by Question 5 in Exercise 14a.)

EXERCISE 14d

Differentiate each of the following functions w.r.t. x

1. $x^3 - x^2 + 5x - 6$
2. $3x^2 + 7 - 4/x$
3. $\sqrt{x} + 1/\sqrt{x}$
4. $2x^4 - 4x^2$
5. $x^3 - 2x^2 - 8x$
6. $x^2 + 5\sqrt{x}$
7. $x^{-3/4} - x^{3/4} + x$
8. $3x^3 - 4x^2 + 9x - 10$
9. $x^{3/2} - x^{1/2} + x^{-1/2}$
10. $\sqrt{x} + \sqrt{x^3}$
11. $\frac{1}{x^2} - \frac{1}{x^3}$
12. $\frac{1}{\sqrt{x}} - \frac{2}{x}$
13. $x^{-1/2} + 3x^{3/2}$
14. $x^{1/4} - x^{1/5}$
15. $\frac{4}{x^3} + \frac{x^3}{4}$
16. $\frac{4}{x} + \frac{5}{x^2} - \frac{6}{x^3}$
17. $3\sqrt{x} - 3x$
18. $x - 2x^{-1} - 3x^{-3}$
19. $x\sqrt{x} - x^2\sqrt{x}$
20. $\frac{\sqrt{x}}{x^2} + \frac{x^2}{\sqrt{x}}$

Differentiating Products and Fractions

All the rules given above can be applied to the differentiation of expressions containing products or quotients provided that, at this stage, they are multiplied out or divided into separate terms.

Examples 14e

1. If $y = (x-3)(x^2+7x-1)$, find $\dfrac{dy}{dx}$

$$y = (x-3)(x^2+7x-1) = x^3+4x^2-22x+3$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2+8x-22$$

2. Find $\dfrac{dt}{dz}$ given that $t = \dfrac{6z^2+z-4}{2z}$

$$t = \frac{6z^2+z-4}{2z} = \frac{6z^2}{2z}+\frac{z}{2z}-\frac{4}{2z}$$

$$= 3z+\tfrac{1}{2}-2/z$$

$$\Rightarrow \quad \frac{dt}{dz} = 3+0-2(-z^{-2})$$

$$= 3+\frac{2}{z^2}$$

EXERCISE 14e

Differentiate each of the following expressions with respect to the variable concerned.

1. $y = (x+1)^2$
2. $z = x^{-2}(2-x)$
3. $y = (3x-4)(x+5)$
4. $y = (4-z)^2$
5. $s = \dfrac{t^{-1}+3t^2}{2t^2}$
6. $s = \dfrac{t^2+t}{2t}$
7. $y = \left(\dfrac{1}{x}\right)(x^2+1)$
8. $y = \dfrac{z^3-z}{\sqrt{z}}$
9. $y = 2x(3x^2-4)$
10. $s = (t+2)(t-2)$
11. $s = \dfrac{t^3-2t^2+7t}{t^2}$
12. $y = \dfrac{\sqrt{x}+7}{x^2}$

GRADIENTS OF TANGENTS AND NORMALS

If the equation of a curve is known, and the gradient function can be found, then the gradient, m say, at a particular point A on that curve can be calculated. This is also the gradient of the tangent to the curve at A.

The normal at A is perpendicular to the tangent at A, therefore its gradient is $-1/m$

Examples 14f

1. The equation of a curve is $s = 6-3t-4t^2-t^3$. Find the gradient of the tangent and of the normal to the curve at the point $(-2, 4)$.

$$s = 6-3t-4t^2-t^3 \quad \Rightarrow \quad \frac{ds}{dt} = 0-3-8t-3t^2$$

At the point $(-2, 4)$, $\dfrac{ds}{dt} = -3-8(-2)-3(-2)^2 = 1$

Therefore the gradient of the tangent at $(-2, 4)$ is 1 and the gradient of the normal is $-1/1$, i.e. -1

2. Find the coordinates of the points on the curve $y = 2x^3-3x^2-8x+7$ where the gradient is 4

$$y = 2x^3-3x^2-8x+7 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2-6x-8$$

If the gradient is 4 then $\dfrac{dy}{dx} = 4$

i.e. $6x^2-6x-8 = 4 \quad \Rightarrow \quad 6x^2-6x-12 = 0$

$\Rightarrow \quad x^2-x-2 = 0$

$\therefore \quad (x-2)(x+1) = 0 \quad \Rightarrow \quad x = 2$ or -1

When $x = 2$, $y = 16-12-16+7 = -5$

when $x = -1$, $y = -2-3+8+7 = 10$

Therefore the gradient is 4 at the points $(2, -5)$ and $(-1, 10)$

EXERCISE 14f

Find the gradient of the tangent and the gradient of the normal at the given point on the given curve.

1. $y = x^2 + 4$ where $x = 1$
2. $y = 3/x$ where $x = -3$
3. $y = \sqrt{z}$ where $z = 4$
4. $s = 2t^3$ where $t = -1$
5. $v = 2 - 1/u$ where $u = 1$
6. $y = (x+3)(x-4)$ where $x = 3$
7. $y = z^3 - z$ where $z = 2$
8. $s = t + 3t^2$ where $t = -2$
9. $z = x^2 - 2/x$ where $x = 1$
10. $y = \sqrt{x} + 1/\sqrt{x}$ where $x = 9$
11. $s = \sqrt{t}(1 + \sqrt{t})$ where $t = 4$
12. $y = \dfrac{x^2 - 4}{x}$ where $x = -2$

Find the coordinates of the point(s) on the given curve where the gradient has the value specified.

13. $y = 3 - 2/x;\ \frac{1}{2}$
14. $z = x^2 - x^3;\ -1$
15. $s = t^3 - 12t + 9;\ 15$
16. $v = u + 1/u;\ 0$
17. $s = (t+3)(t-5);\ 0$
18. $y = 1/x^2;\ \frac{1}{4}$
19. $y = (2x-5)(x+1);\ -3$
20. $y = z^3 - 3z;\ 0$

MIXED EXERCISE 14

1. Differentiate $3x^2 + x$ with respect to x from first principles.
2. Find the derivative of
   (a) $x^{-3} - x^3 + 7$ (b) $x^{1/2} - x^{-1/2}$ (c) $1/x^2 + 2/x^3$
3. Differentiate w.r.t. x.
   (a) $y = x^{3/2} - x^{2/3} + x^{-1/3}$ (b) $y = \sqrt{x} - 1/x + 1/x^3$
   (c) $1/x^{3/4} - 1/x^{1/4}$

4. Find the gradient of the curve $y = 2x^3 - 3x^2 + 5x - 1$ at the point
   (a) $(0, -1)$ (b) $(1, 3)$ (c) $(-1, -11)$
5. Find the gradient of the given curve at the given point.
   (a) $y = x^2 + x - 9;\ x = 2$ (b) $y = x(x-4);\ x = 5$
6. The equation of a curve is $y = (x-3)(x+4)$. Find the gradient of the curve
   (a) at the point where the curve crosses the y-axis
   (b) at each of the points where the curve crosses the x-axis
7. If the equation of a curve is $y = 2x^2 - 3x - 2$ find
   (a) the gradient at the point where $x = 0$
   (b) the coordinates of the points where the curve crosses the x-axis
   (c) the gradient at each of the points found in (b).
8. Find the coordinates of the point(s) on the curve $y = 3x^3 - x + 8$ at which the gradient is
   (a) 8 (b) 0
9. Find $\dfrac{dy}{dx}$ if
   (a) $y = x^4 - x^2$ (b) $y = (3x+4)^2$ (c) $y = \dfrac{x-3}{\sqrt{x}}$
10. Find the gradient of the tangent at the point where $x = 2$ on the curve $y = (2 - \sqrt{x})^2$
11. Find the coordinates of the point on the curve $y = x^2$ where the gradient of the normal is $\frac{1}{4}$.
12. The equation of a curve is $s = 4t^2 + 5t$. Find the gradient of the normal at each of the points where the curve crosses the t-axis.
13. Find the coordinates of the points on the curve $y = x^3 - 6x^2 + 12x + 2$ at which the tangent is parallel to the line $y = 3x$
14. The curve $y = (x-2)(x-3)(x-4)$ cuts the x-axis at the points P(2, 0), Q(3, 0) and R(4, 0). Prove that the tangents at P and R are parallel and find the gradient of the normal at Q.

15. For a certain equation, $\dfrac{dy}{dx} = 2x + 1$ Which of the following could be the given equation

(a) $y = 2x^2 + x$ (b) $y = x^2 + x - 1$ (c) $y = x^2 + 1$
(d) $y = x^2 + x$

CHAPTER 15

TANGENTS, NORMALS AND STATIONARY POINTS

THE EQUATIONS OF TANGENTS AND NORMALS

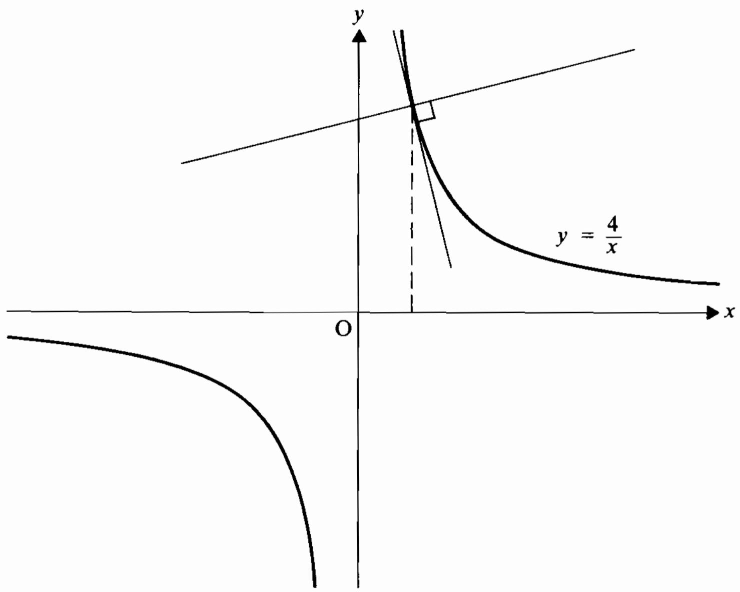
We have seen how to find the gradient of a tangent at a particular point, A, on a curve. We also know that the tangent passes through the point A. Therefore the tangent is a line passing through a known point and having a known gradient and its equation can be found using

$$y - y_1 = m(x - x_1)$$

The equation of a normal can be found in the same way.

Examples 15a

1. Find the equation of the normal to the curve $y = \dfrac{4}{x}$ at the point where $x = 1$



$$y = \frac{4}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{4}{x^2}$$

When $x = 1$, $y = 4$ and $\dfrac{dy}{dx} = -4$

The gradient of the tangent at $(1, 4)$ is -4, therefore the gradient of the normal at $(1, 4)$ is $-\frac{1}{-4}$ i.e. $\frac{1}{4}$

The equation of the normal is given by $y - y_1 = m(x - x_1)$

i.e. $\qquad y - 4 = \frac{1}{4}(x - 1)$

$\Rightarrow \qquad 4y = x + 15$

---

2. Find the equation of the tangent to the curve $y = x^2 - 6x + 5$ at each of the points where the curve crosses the x-axis. Find also the coordinates of the point where these tangents meet.

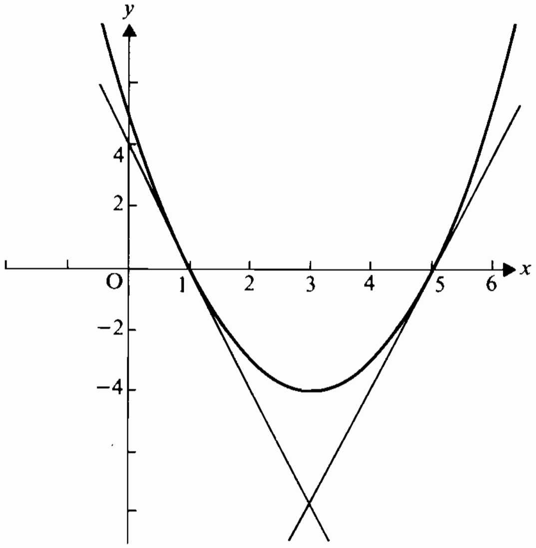
The curve crosses the x-axis where $y = 0$,

i.e. where $\quad x^2 - 6x + 5 = 0 \quad \Rightarrow \quad (x - 5)(x - 1) = 0$

$\Rightarrow \qquad x = 5$ and $x = 1$

Therefore the curve crosses the x-axis at $(5, 0)$ and $(1, 0)$

$$y = x^2 - 6x + 5 \quad \Rightarrow \quad \frac{dy}{dx} = 2x - 6$$



At $(5, 0)$, the gradient of the tangent is given by

$$\frac{dy}{dx} = 10 - 6 = 4$$

therefore the equation of this tangent is

$$y - 0 = 4(x - 5) \quad \Rightarrow \quad y = 4x - 20$$

At $(1, 0)$ the gradient of the tangent is given by

$$\frac{dy}{dx} = 2 - 6 = -4$$

Therefore the equation of the tangent is

$$y - 0 = -4(x - 1) \quad \Rightarrow \quad y + 4x = 4$$

If the two tangents meet at P then, at P,

$$y + 4x = 4 \qquad [1]$$

and $$y - 4x = -20 \qquad [2]$$

[1] + [2] gives $\quad 2y = -16 \quad \Rightarrow \quad y = -8$

Using $y = -8$ in [1] gives $\quad -8 + 4x = 4 \quad \Rightarrow \quad x = 3$

Therefore the tangents meet at $(3, -8)$

EXERCISE 15a

In each question from 1 to 6 find, at the given point,

(a) the equation of the tangent

(b) the equation of the normal.

1. $y = x^2 - 4$ where $x = 1$

2. $y = x^2 + 4x - 2$ where $x = 0$

3. $y = 1/x$ where $x = -1$

4. $y = x^2 + 5$ where $x = 0$

5. $y = x^2 - 5x + 7$ where $x = 2$

6. $y = (x - 2)(x^2 - 1)$ where $x = -2$

7. Find the equation of the normal to the curve $y = x^2 + 4x - 3$ at the point where the curve cuts the y-axis.

8. Find the equation of the tangent to the curve $y = x^2 - 3x - 4$ at the point where this curve cuts the line $x = 5$

9. Find the equation of the tangent to the curve $y = (2x - 3)(x - 1)$ at each of the points where this curve cuts the x-axis. Find the point of intersection of these tangents.

10. Find the equation of the normal to the curve $y = x^2 - 6x + 5$ at each of the points where the curve cuts the x-axis.

11. Find the equation of the tangent to the curve $y = 3x^2 + 5x - 1$ at each of the points of intersection of the curve and the line $y = x - 1$

12. Find the equations of the tangent to the curve $y = x^2 + 5x - 3$ at the points where the line $y = x + 2$ crosses the curve.

13. Find the coordinates of the point on the curve $y = 2x^2$ at which the gradient is 8 Hence find the equation of the tangent to $y = 2x^2$ whose gradient is 8

14. Find the coordinates of the point on the curve $y = 3x^2 - 1$ at which the gradient is 3

15. Find the equation of the tangent to the curve $y = 4x^2 + 3x$ which has a gradient of -1

16. Find the equation of the normal to the curve $y = 2x^2 - 2x + 1$ which has a gradient of $\frac{1}{2}$

17. Find the value of k for which $y = 2x + k$ is a tangent to the curve $y = 2x^2 - 3$

18. Find the equation of the tangent to the curve $y = (x - 5)(2x + 1)$ which is parallel to the x-axis.

19. Find the coordinates of the point(s) on the curve $y = x^2 - 5x + 3$ where the gradient of the normal is $\frac{1}{3}$

20. A curve has the equation $y = x^3 - px + q$. The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis. Find the values of p and q.
Find also the coordinates of the other point where the tangent is parallel to the x-axis.

STATIONARY VALUES

110

Consider a function $f(x)$. The derived function, $f'(x)$, expresses the rate at which $f(x)$ increases with respect to x

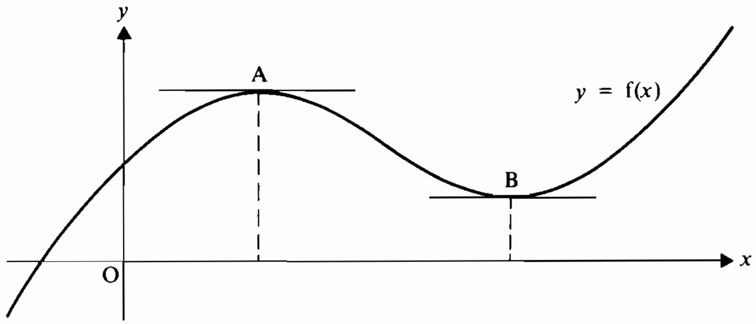
If, at a particular point, $f'(x)$ is positive then $f(x)$ is increasing as x increases, whereas if $f'(x)$ is negative then $f(x)$ is decreasing as x increases.

Now there may be points where $f'(x)$ is zero, i.e. $f(x)$ is momentarily neither increasing nor decreasing with respect to x

The value of $f(x)$ at such a point is called a *stationary value* of $f(x)$

i.e. $\quad f'(x) = 0 \quad \Rightarrow \quad f(x)$ has a stationary value.

To look at this situation graphically we consider the curve with equation $y = f(x)$.



At A and B, $f(x)$, and therefore y, is neither increasing nor decreasing with respect to x. So the values of y at A and B are stationary values.

i.e. $\quad \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad y$ has a stationary value.

The point on a curve where y has a stationary value is called a *stationary point* and we see that, at any stationary point, the gradient of the tangent to the curve is zero, i.e. the tangent is parallel to the x-axis.

To sum up:

at a stationary point $\begin{cases} y \quad [\text{or } f(x)] \quad \text{has a stationary value} \\ dy/dx \quad [\text{or } f'(x)] \quad \text{is zero} \\ \text{the tangent is parallel to the } x\text{-axis.} \end{cases}$

Example 15b

Find the stationary values of the function $x^3 - 4x^2 + 7$.

If $\qquad f(x) = x^3 - 4x^2 + 7$

then $\qquad f'(x) = 3x^2 - 8x$

At stationary points, $f'(x) = 0$, i.e. $3x^2 - 8x = 0$

$\Rightarrow \qquad x(3x - 8) = 0 \quad \Rightarrow \quad x = 0$ and $x = \frac{8}{3}$

Therefore there are stationary points where $x = 0$ and $x = \frac{8}{3}$

When $x = 0$, $\qquad f(x) = 0 - 0 + 7 = 7$

When $x = \frac{8}{3}$, $\qquad f(x) = (\frac{8}{3})^3 - 4(\frac{8}{3})^2 + 7 = -2\frac{13}{27}$

Therefore the stationary values of $x^3 - 4x^2 - 5$ are 7 and $-2\frac{13}{27}$

EXERCISE 15b

Find the value(s) of x at which the following functions have stationary values.

1. $x^2 + 7$
2. $2x^2 - 3x - 2$
3. $x^3 - 4x^2 + 6$
4. $4x^3 - 3x - 9$
5. $x^3 - 2x^2 + 11$
6. $x^3 - 3x - 5$

Find the value(s) of x for which y has a stationary value.

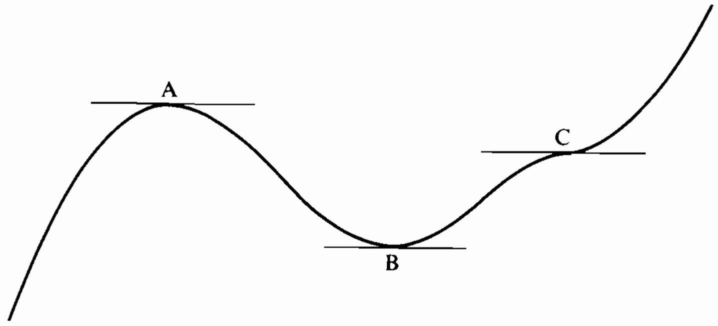
7. $y = x^2 - 8x + 1$
8. $y = x + 9/x$
9. $y = 2x^3 + x^2 - 8x + 1$
10. $y = 9x^3 - 25x$
11. $y = 2x^3 + 9x^2 - 24x + 7$
12. $y = 3x^3 - 12x + 19$

Find the coordinates of the stationary points on the following curves.

13. $y = \dfrac{x^2 + 9}{2x}$
14. $y = x^3 - 2x^2 + x - 7$
15. $y = (x - 3)(x + 2)$
16. $y = x^{3/2} - x^{1/2}$
17. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
18. $y = 8 + \dfrac{x}{4} + \dfrac{4}{x}$

TURNING POINTS

In the immediate neighbourhood of a stationary point a curve can have any one of the shapes shown in the following diagram.



Moving through A from left to right we see that the curve is rising, then turns at A and begins to fall, i.e. the gradient changes from positive to zero at A and then becomes negative.

At A there is a *turning point.*

The value of y at A is called a *maximum value* and A is called a *maximum point.*

Moving through B from left to right the curve is falling, then turns at B and begins to rise, i.e. the gradient changes from negative to zero at B and then becomes positive.

At B there is a *turning point.*

The value of y at B is called a *minimum value* and B is called a *minimum point.*

The tangent is always horizontal at a turning point.

Note that a maximum value of y is *not necessarily the greatest value of* y *overall.* The terms maximum and minimum apply only to the behaviour of the curve in the neighbourhood of a stationary point.

At C the curve does not turn. The gradient goes from positive, to zero at C and then becomes positive again, i.e. the gradient does not change sign at C.

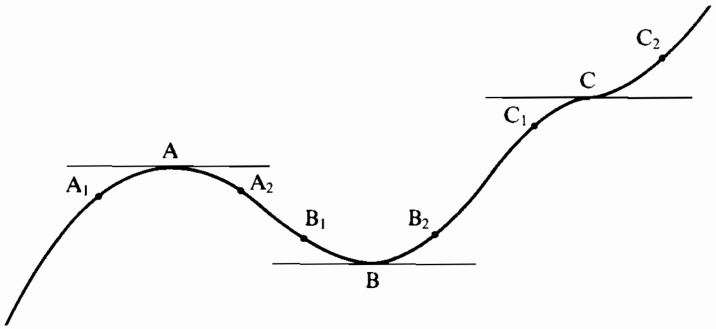
C is not a turning point but, because there is a change in the sense in which the curve is turning (from clockwise to anti-clockwise), C is called a *point of inflexion.*

Any point on a curve where the sense of turning changes, is a point of inflexion. In the diagram there are two points of inflexion other than C; one is between A and B and the other is between B and C. We see from these points that

the tangent is not necessarily horizontal at a point of inflexion.

INVESTIGATING THE NATURE OF STATIONARY POINTS

We already know how to locate stationary points on a curve and now examine several ways of distinguishing between the different types of stationary point.



METHOD (I)

This method compares the value of y at the stationary point with values of y at points on either side of, and near to, the stationary point.

For a maximum value, e.g. at A

$$y \text{ at } A_1 < y \text{ at } A$$

$$y \text{ at } A_2 < y \text{ at } A$$

For a minimum point, e.g. at B

$$y \text{ at } B_1 > y \text{ at } B$$

$$y \text{ at } B_2 > y \text{ at } B$$

For a point of inflexion, e.g. at C

$$y \text{ at } C_1 < y \text{ at } C$$

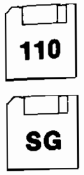
$$y \text{ at } C_2 > y \text{ at } C$$

Collecting these conclusions we have:

	Maximum	Minimum	Inflexion
y values on each side of the stationary point	both smaller	both larger	one larger and one smaller

Note that the points chosen on either side of the stationary point must be such that no *other* stationary point, nor any discontinuity on the graph, lies between them.

METHOD (II)



This method examines the sign of the gradient, again at points close to, and on either side of, the stationary point where the gradient is zero.

For a maximum point, A

$$\frac{dy}{dx} \text{ at } A_1 \text{ is +ve}$$

$$\frac{dy}{dx} \text{ at } A_2 \text{ is −ve}$$

For a minimum point, B

$$\frac{dy}{dx} \text{ at } B_1 \text{ is −ve}$$

$$\frac{dy}{dx} \text{ at } B_2 \text{ is +ve}$$

For a point of inflexion, C

$$\frac{dy}{dx} \text{ at } C_1 \text{ is +ve}$$

$$\frac{dy}{dx} \text{ at } C_2 \text{ is +ve}$$

Collecting these conclusions we have:

Sign of $\frac{dy}{dx}$	Passing through maximum + 0 −	Passing through minimum − 0 +	Passing through point of inflexion + 0 + or − 0 −
Gradient of tangent	/ — \	\ _ /	/ — / or \ — \

METHOD (III)

In this method we observe how dy/dx changes with respect to x as we pass through a stationary point. Now the rate at which dy/dx increases with respect to x could be written $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ but, because this notation is clumsy, it is condensed to $\dfrac{d^2y}{dx^2}$ (we say d 2 y by dx squared).

For example, using this notation the fact that $\dfrac{dy}{dx}$ is increasing as x increases can be expressed as $\dfrac{d^2y}{dx^2}$ is +ve.

$\left(\dfrac{d^2y}{dx^2}\right.$ is the *second derivative* with respect to x, of $\left.y\right)$

Now we can examine the behaviour of $\dfrac{dy}{dx}$ at each stationary point.

For the maximum point A,

at A_1 $\dfrac{dy}{dx}$ is +ve and at A_2 $\dfrac{dy}{dx}$ is −ve

so, passing through A, $\dfrac{dy}{dx}$ goes from + to −, i.e. $\dfrac{dy}{dx}$ decreases

$\Rightarrow$ at A, $\dfrac{d^2y}{dx^2}$ is negative.

For the minimum point B,

at B_1 $\dfrac{dy}{dx}$ is −ve and at B_2 $\dfrac{dy}{dx}$ is +ve

so, passing through B, $\dfrac{dy}{dx}$ goes from − to +, i.e. $\dfrac{dy}{dx}$ increases

$\Rightarrow$ at B, $\dfrac{d^2y}{dx^2}$ is positive.

Points of inflexion are not so easily dealt with by this method because, although $\dfrac{d^2y}{dx^2}$ is zero at a point of inflexion it is also possible for $\dfrac{d^2y}{dx^2}$ to be zero at a turning point. So, for any stationary point where $\dfrac{d^2y}{dx^2} = 0$, this method fails and one of the other two approaches must be used.

Summing up method (iii) we have:

	Maximum	Minimum
Sign of $\dfrac{d^2y}{dx^2}$	negative (or zero)	positive (or zero)

Note that, if $\dfrac{d^2y}{dx^2}$ is zero one of the other two methods must be used to determine the nature of a stationary point.

Examples 15c

1. Locate the stationary points on the curve $y = 4x^3 + 3x^2 - 6x - 1$ and determine the nature of each one.

$$y = 4x^3 + 3x^2 - 6x - 1 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^2 + 6x - 6$$

At stationary points, $\dfrac{dy}{dx} = 0$

i.e. $12x^2 + 6x - 6 = 0 \quad \Rightarrow \quad 6(2x-1)(x+1) = 0$

$\therefore$ there are stationary points where $x = \frac{1}{2}$ and $x = -1$

When $x = \frac{1}{2}$, $y = -2\frac{3}{4}$ and when $x = -1$, $y = 4$

i.e. the stationary points are $(\frac{1}{2}, -2\frac{3}{4})$ and $(-1, 4)$

Differentiating $\dfrac{dy}{dx}$ w.r.t. x gives

$$\frac{d^2y}{dx^2} = 24x + 6$$

When $x = \frac{1}{2}$, $\dfrac{d^2y}{dx^2} = 12 + 6$ which is positive

$\Rightarrow$ $(\frac{1}{2}, -2\frac{3}{4})$ is a minimum point.

When $x = -1$, $\dfrac{d^2y}{dx^2} = -24 + 6$ which is negative

$\Rightarrow$ $(-1, 4)$ is a maximum point.

2. Find the stationary values of the function $3x^4 - 8x^3 + 6x^2 - 3$ and investigate their nature.

$$f(x) = 3x^4 - 8x^3 + 6x^2 - 3 \quad \Rightarrow \quad f'(x) = 12x^3 - 24x^2 + 12x$$

At stationary values $f'(x) = 0$

i.e. $12x^3 - 24x^2 + 12x = 0 \quad \Rightarrow \quad 12x(x^2 - 2x + 1) = 0$

$\Rightarrow \quad 12x(x-1)(x-1) = 0$

there are stationary values when $x = 0$ and $x = 1$

$x = 0 \quad \Rightarrow \quad f(x) = -3$

$x = 1 \quad \Rightarrow \quad f(x) = 3 - 8 + 6 - 3 = -2$

i.e. the stationary values of $f(x)$ are -2 and -3

Differentiating $f'(x)$ w.r.t. x [the notation for $f'\{f'(x)\}$ is $f''(x)$] gives

$$f''(x) = 36x^2 - 48x + 12 = 12(3x^2 - 4x + 1)$$

When $x = 0$, $f''(x) = 12$ which is positive

$\Rightarrow \quad f(x) = -3$ is a minimum value

when $x = 1$, $f''(x) = 12(3 - 4 + 1)$ which is zero.

This is inconclusive so we will look at the signs of $f'(x)$ on either side of $x = 1$

x	$\frac{1}{2}$	1	$1\frac{1}{2}$
$f'(x)$	+	0	+
Gradient	/	–	/

From this table we see that the stationary value at $x = 1$, i.e. -2, is an inflexion.

EXERCISE 15c

Find the stationary points on the following curves and distinguish between them.

1. $y = 2x - x^2$ **2.** $y = 3x - x^3$

3. $y = 9/x + x$ **4.** $y = x^2(x - 5)$

5. $y = x^2$ **6.** $y = x + 1/2x^2$

7. $y = 2x^2 - x^4$ **8.** $y = x^4$

9. $y = (2x + 1)(x - 3)$ **10.** $y = x^5 - 5x$

11. $y = x^2(x^2 - 8)$ **12.** $y = x^2 + 16/x^2$

Find the stationary value(s) of each of the following functions and determine their character.

13. $x + 1/x$ **14.** $3 - x + x^2$ **15.** $4x^3 - x^4$

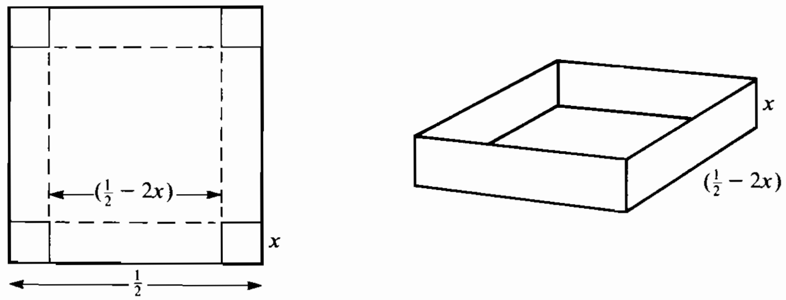
16. $8 - x^3$ **17.** $x^3 + 7$ **18.** $x^2(3x^2 - 2x - 3)$

PROBLEMS

Examples 15d

1. An open box is made from a square sheet of cardboard, with sides half a metre long, by cutting out a square from each corner, folding up the sides and joining the cut edges. Find the maximum capacity of the box.

The capacity of the box depends upon the unknown length of the side of the square cut from each corner so we denote this by x metres. The side of the cardboard sheet is $\frac{1}{2}$ m, so we know that $0 < x < \frac{1}{4}$



Using metres throughout,

the base of the box is a square of side $(\frac{1}{2} - 2x)$ and the height of the box is x

$\therefore$ the capacity, C, of the box is given by

$$C = x(\tfrac{1}{2} - 2x)^2 = \tfrac{1}{4}x - 2x^2 + 4x^3 \quad \text{for} \quad 0 < x < \tfrac{1}{4}$$

$$\Rightarrow \quad \frac{dC}{dx} = \tfrac{1}{4} - 4x + 12x^2$$

At a stationary value of C, $\dfrac{dC}{dx} = 0$

i.e. $\quad 12x^2 - 4x + \tfrac{1}{4} = 0 \quad \Rightarrow \quad 48x^2 - 16x + 1 = 0$

$(4x - 1)(12x - 1) = 0 \quad \Rightarrow \quad x = \tfrac{1}{4} \text{ or } x = \tfrac{1}{12}$

there are stationary values of C when $x = \tfrac{1}{4}$ and when $x = \tfrac{1}{12}$

It is obvious that it is not possible to make a box if $x = \tfrac{1}{4}$ so we need only check that $x = \tfrac{1}{12}$ gives a maximum capacity.

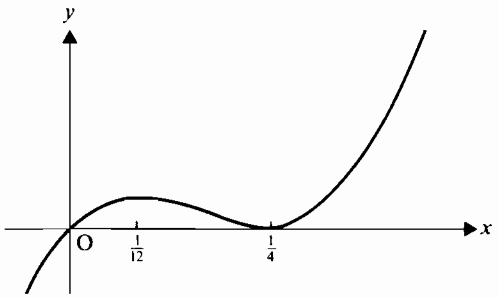
$\dfrac{d^2C}{dx^2} = -4 + 24x$ which is negative when $x = \tfrac{1}{12}$

Therefore C has a maximum value of $\tfrac{1}{12}(\tfrac{1}{2} - \tfrac{1}{6})^2$, i.e. $\tfrac{1}{108}$

i.e. the maximum capacity of the box is $\tfrac{1}{108}\ \text{m}^3$

or, correct to 3 s.f., $9260\ \text{cm}^3$

Alternatively the nature of the stationary point where $x = \tfrac{1}{12}$ can be investigated by using the sketch given by a graphics calculator for the curve $C = \tfrac{1}{4}x - 2x^2 + 4x^3$ and looking at the section for which $0 < x < \tfrac{1}{4}$, i.e.



The sketch shows that there is a maximum point between $x = 0$ and $x = \tfrac{1}{4}$ so $x = \tfrac{1}{12}$ must give the maximum value of C.

The sketch also shows that there is a minimum point *on the curve* where $x = \tfrac{1}{4}$ but this is *not* a minimum value of the *capacity*, as a box cannot be made if $x = \tfrac{1}{4}$

---

2. The function $ax^2 + bx + c$ has a gradient function $4x + 2$ and a stationary value of 1. Find the values of a, b and c

$$f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$$

But we know that $f'(x) = 4x + 2$

$\therefore \quad 2ax + b$ is identical to $4x + 2$

i.e. $\quad a = 2$ and $b = 2$

The stationary value of $f(x)$ occurs when $f'(x) = 0$

i.e. when $\quad 4x + 2 = 0 \quad \Rightarrow \quad x = -\tfrac{1}{2}$

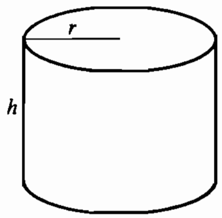
the stationary value of $f(x)$ is $2(-\tfrac{1}{2})^2 + 2(-\tfrac{1}{2}) + c = -\tfrac{1}{2} + c$

But the stationary value of $f(x)$ is also 1

$\therefore \quad -\tfrac{1}{2} + c = 1 \quad \Rightarrow \quad c = \tfrac{3}{2}$

---

3. A cylinder has a radius r metres and a height h metres. The sum of the radius and height is 2 m. Find an expression for the volume, V cubic metres, of the cylinder in terms of r only. Hence find the maximum volume.



$$V = \pi r^2 h \quad \text{and} \quad r + h = 2$$

$$\therefore \quad V = \pi r^2(2 - r) = \pi(2r^2 - r^3)$$

Now for maximum volume, $\dfrac{dV}{dr} = 0$,

i.e. $\quad \pi(4r - 3r^2) = 0 \quad \Rightarrow \quad \pi r(4 - 3r) = 0$

Therefore there are stationary values of V when $r = 0$ and $r = \tfrac{4}{3}$

It is obvious that, when $r = 0$, $V = 0$ and no cylinder exists, so we check the sign of $\dfrac{d^2V}{dr^2}$ only for $r = \tfrac{4}{3}$

$$\frac{d^2V}{dr^2} = \pi(4 - 6r) \quad \text{which is negative when} \quad r = \tfrac{4}{3}$$

Therefore the maximum value of V occurs when $r = \frac{4}{3}$
and is $\pi(\frac{4}{3})^2(2 - \frac{4}{3})$

i.e. the maximum volume is $\dfrac{32\pi}{27}\,\text{m}^3$

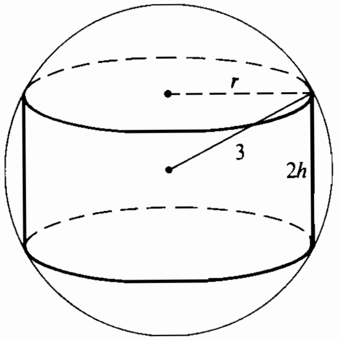
Note that the solution of this problem depends fundamentally on having an expression for V in terms of only one other variable. This is true of *all* problems on stationary points so, if three or more variables are involved initially, some of them must be replaced so that we have a basic relationship containing only two variables.

EXERCISE 15d

1. A farmer has an 80 m length of fencing. He wants to use it to form three sides of a rectangular enclosure against an existing fence which provides the fourth side. Find the maximum area that he can enclose and give its dimensions.

2. A large number of open cardboard boxes are to be made and each box must have a square base and a capacity of $4000\,\text{cm}^3$. Find the dimensions of the box which contains the minimum area of cardboard.

3.



The diagram shows a cylinder cut from a solid sphere of radius 3 cm. Given that the cylinder has a height of $2h$, find its radius in terms of h. Hence show that the volume, V cubic metres, of the cylinder is given by

$$V = 2\pi h(9 - h^2)$$

Find the maximum volume of the cylinder as h varies.

4. A variable rectangle has a constant perimeter of 20 cm. Find the lengths of the sides when the area is maximum.

5. A variable rectangle has a constant area of $35\,\text{cm}^2$. Find the lengths of the sides when the perimeter is minimum.

6. The curve $y = ax^2 + bx + c$ crosses the y-axis at the point $(0, 3)$ and has a stationary point at $(1, 2)$. Find the values of a, b and c

7. The gradient of the tangent to the curve $y = px^2 - qx - r$ at the point $(1, -2)$ is 1 If the curve crosses the x-axis where $x = 2$, find the values of p, q and r. Find the other point of intersection with the x-axis and sketch the curve.

8. y is a quadratic function of x. The line $y = 2x$ is a tangent to the curve at the point $(3, 6)$. The turning point on the curve occurs where $x = -2$ Find the equation of the curve.

MIXED EXERCISE 15

1. Find the gradient of the curve with equation $y = 6x^2 - x$ at the point where $x = 1$ Find the equation of the tangent at this point. Where does this tangent meet the line $y = 2x$?

2. Find the equation of the normal to the curve $y = 1 - x^2$ at the point where the curve crosses the positive x-axis. Find also the coordinates of the point where the normal meets the curve again.

3. Find the coordinates of the points on the curve $y = x^3 + 3x$ where the gradient is 15

4. Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$

5. Find the equation of the normal to the curve $y = x^2 - 6$ which is parallel to the line $x + 2y - 1 = 0$

6. Locate the turning points on the curve $y = x(x^2 - 12)$, determine their nature and draw a rough sketch of the curve.

7. Find the stationary values of the function $x + 1/x$ and sketch the function.

8. If the perimeter of a rectangle is fixed in length, show that the area of the rectangle is greatest when it is square.

9. A door is in the shape of a rectangle surmounted by a semicircle whose diameter is equal to the width of the rectangle. If the perimeter of the door is 7 m, and the radius of the semicircle is r metres, express the height of the rectangle in terms of r. Show that the area of the door has a maximum value when the width is $7/(4+\pi)$.

10. An open tank is constructed, with a square base and vertical sides, to hold 32 cubic metres of water. Find the dimensions of the tank if the area of sheet metal used to make it is to have a minimum value.

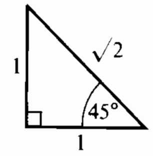
11. Triangle ABC has a right angle at C. The shape of the triangle can vary but the sides BC and CA have a fixed total length of 10 cm. Find the maximum area of the triangle.

CHAPTER 16

TRIGONOMETRIC FUNCTIONS

THE TRIG RATIOS OF 30°, 45°, 60°

The sine, cosine and tangent of 30°, 45°, and 60°, can be expressed exactly in surd form and are worth remembering.

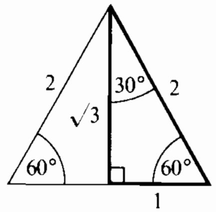


This triangle shows that

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = 1$$



And this triangle gives

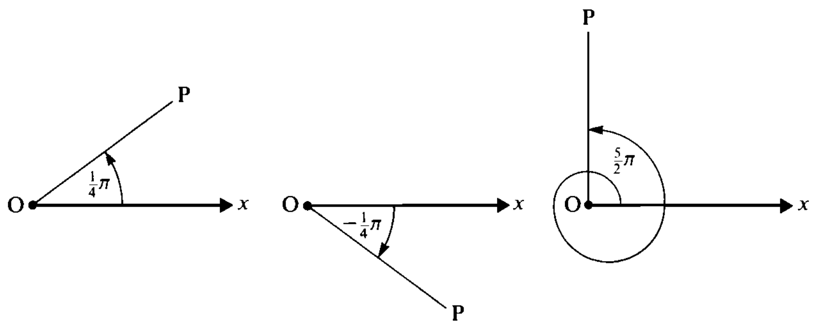
$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \sin 30^\circ = \frac{1}{2}$$

$$\cos 60^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \sqrt{3}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

THE TRIGONOMETRIC FUNCTIONS

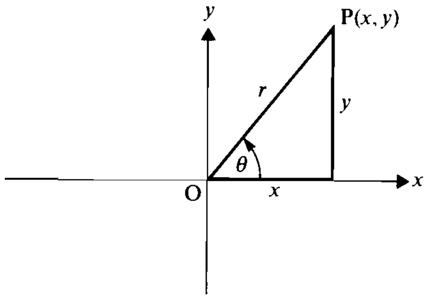
In Chapter 7, we gave the general definition of an angle as a measure of the rotation of a line OP about a fixed point O. Taking Ox as the initial direction of OP, anticlockwise rotation describes a positive angle and clockwise rotation describes a negative angle. The rotation of OP is not limited to one revolution, so an angle can be as big as we choose to make it.



If θ is any angle, then θ can be measured
either in degrees (one revolution $= 360°$)
or in radians (one revolution $= 2\pi$ radians)
and in either case we see that θ can take all real values.

Chapter 7 also contained the general definition of the sine, cosine and tangent of θ, but used those definitions only for values of θ in the range $0 \leqslant \theta \leqslant 180°$. They are valid for all values of θ however and, as a reminder, we repeat them here.

If OP is drawn on x and y axes as shown and if, for all values of θ, the length of OP is r and the coordinates of P are (x, y), then the sine, cosine and tangent functions are defined as follows.



$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

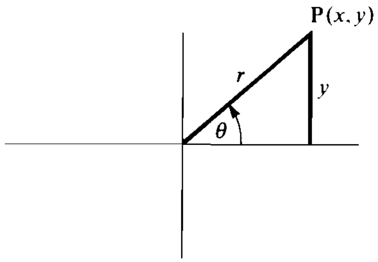
$$\tan\theta = \frac{y}{x}$$

We will now look at each of these functions in turn.

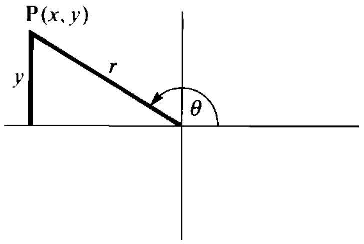
THE SINE FUNCTION

From the definition $f(\theta) = \sin\theta$, and measuring θ in radians, we can see that

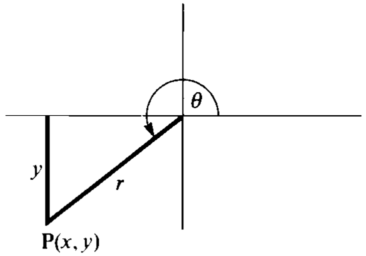
for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, OP is in the first quadrant; y is positive and increases in value from 0 to r as θ increases from 0 to $\frac{1}{2}\pi$. Now r is always positive, so $\sin\theta$ increases from 0 to 1



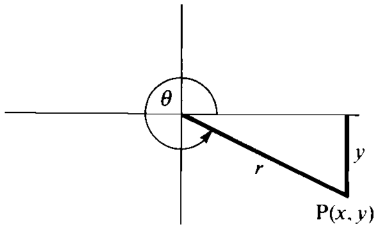
for $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$, OP is in the second quadrant; again y is positive but decreases in value from r to 0, so $\sin\theta$ decreases from 1 to 0



for $\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, OP is in the third quadrant; y is negative and decreases from 0 to $-r$, so $\sin\theta$ decreases from 0 to -1



for $\frac{3}{2}\pi \leqslant \theta \leqslant 2\pi$, OP is in the fourth quadrant; y is still negative but increases from $-r$ to 0, so $\sin\theta$ increases from -1 to 0

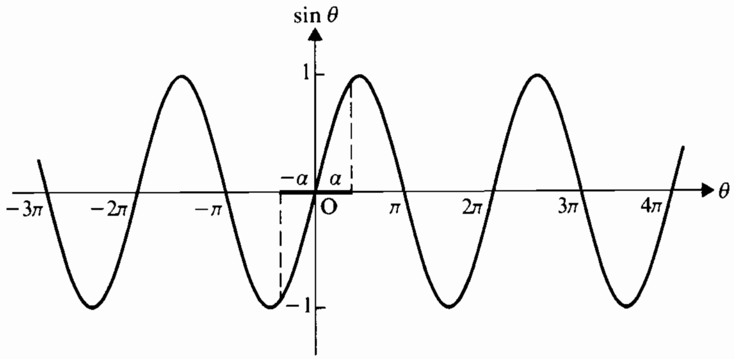


For $\theta \geqslant 2\pi$, the cycle repeats itself as OP travels round the quadrants again. For negative values of θ, OP rotates clockwise round the quadrants in the order 4th, 3rd, 2nd, 1st, etc. So $\sin\theta$ decreases from 0 to -1, then increases to 0 and on to 1 before decreasing to zero and repeating the pattern.

From this analysis we see that $\sin\theta$ is positive for $0 < \theta < \pi$ and negative when $\pi < \theta < 2\pi$

Further, $\sin\theta$ varies in value between -1 and 1 and the pattern repeats itself every revolution.

A plot of the graph of $f(\theta) = \sin\theta$ confirms these observations.



A graph of this shape is called, for obvious reasons, a *sine wave* and shows clearly the following characteristics of the sine function.

The curve is continuous (i.e. it has no breaks).

$$-1 \leqslant \sin\theta \leqslant 1$$

The shape of the curve from $\theta = 0$ to $\theta = 2\pi$ is repeated for each complete revolution. Any function with a repetitive pattern is called *periodic* or *cyclic*. The width of the repeating pattern, as measured on the horizontal scale, is called the *period*.

The period of the sine function is 2π

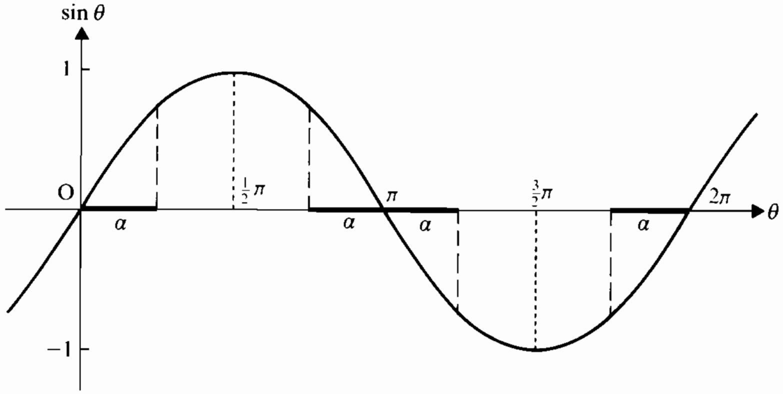
Other properties of the sine function shown by the graph are as follows.

$$\sin\theta = 0 \quad \text{when} \quad \theta = n\pi \quad \text{where } n \text{ is an integer.}$$

The curve has rotational symmetry about the origin so, for any angle α

$$\sin(-\alpha) = -\sin\alpha, \quad \text{e.g.} \quad \sin(-30°) = -\sin 30° = -\tfrac{1}{2}$$

Taking an enlarged section of the graph for $0 \leqslant \theta \leqslant 2\pi$, we find further relationships.



The curve is symmetrical about the line $\theta = \frac{1}{2}\pi$, so

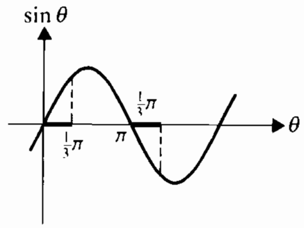
$$\sin(\pi - \alpha) = \sin\alpha, \quad \text{e.g.} \quad \sin 130° = \sin(180° - 130°) = \sin 50°$$

The curve has rotational symmetry about $\theta = \pi$, so

$$\sin(\pi + \alpha) = -\sin\alpha \quad \text{and} \quad \sin(2\pi - \alpha) = -\sin\alpha$$

Examples 16a

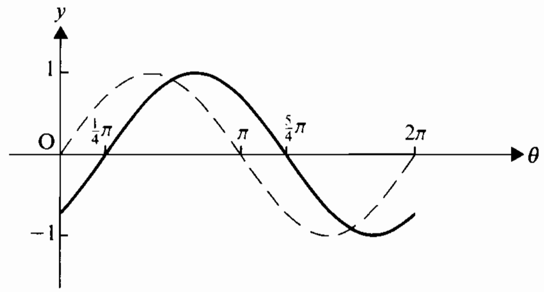
1. Find the exact value of $\sin\frac{4}{3}\pi$.



$$\sin\tfrac{4}{3}\pi = \sin(\pi + \tfrac{1}{3}\pi) = -\sin\tfrac{1}{3}\pi = -\frac{\sqrt{3}}{2}$$

2. Sketch the graph of $y = \sin(\theta - \frac{1}{4}\pi)$ for values of θ between 0 and 2π

Remember that the curve $y = f(x - a)$ is a translation of the curve $y = f(x)$ by a units in the positive direction of the x-axis.



EXERCISE 16a

Find the exact value of

1. sin 120° 2. sin -2π 3. sin 300° 4. sin −210°

5. Write down all the values of θ between 0 and 6π for which $\sin\theta = 1$

6. Write down all the values of θ between 0 and -4π for which $\sin\theta = -1$

Express in terms of the sine of an acute angle

7. sin 125° 8. sin 290° 9. sin −120° 10. $\sin\frac{7}{6}\pi$

Sketch each of the following curves for values of θ in the range $0 \leqslant \theta \leqslant 3\pi$

11. $y = \sin(\theta + \frac{1}{3}\pi)$ 12. $y = -\sin\theta$ 13. $y = \sin(-\theta)$

14. $y = 1 - \sin\theta$ 15. $y = \sin(\pi - \theta)$ 16. $y = \sin(\frac{1}{2}\pi - \theta)$

Use a graphics calculator or computer for Questions 17 to 19 and set the range for θ as -2π to 4π.

17. On the same set of axes draw the graphs of $y = \sin\theta$, $y = 2\sin\theta$, and $y = 3\sin\theta$. What can you deduce about the relationship between the curves $y = \sin\theta$ and $y = a\sin\theta$?

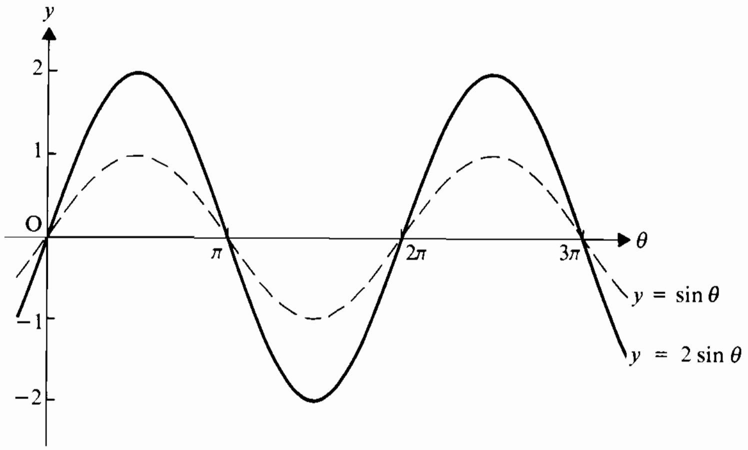
18. On the same set of axes draw the curves $y = \sin\theta$ and $y = \sin 2\theta$

19. On the same set of axes draw the curves $y = \sin\theta$ and $y = \sin 3\theta$
What can you deduce about the relationship between the two curves?

20. *Sketch* the curves (a) $y = \sin 4\theta$ (b) $y = 4\sin\theta$

One-way Stretches

Questions 17 to 20 in the last exercise show examples of one-way stretches. For example, the curve $y = 2\sin\theta$ is seen to be a one-way stretch of the curve $y = \sin\theta$ by a factor 2 parallel to the y-axis.



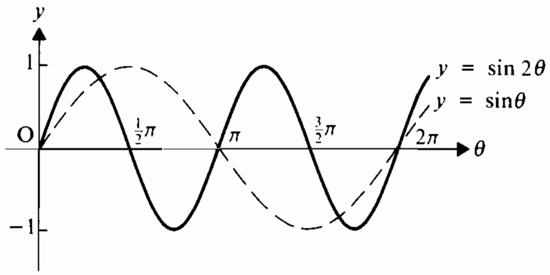
In general, if we compare points on the curves $y = f(x)$ and $y = af(x)$ with the same x-coordinate, then the y-coordinate of the point on $y = af(x)$ is a times the y-coordinate of the point on $y = f(x)$. Therefore

the curve $y = af(x)$ is a one-way stretch of the curve $y = f(x)$ by a factor a parallel to the y-axis.

Also, the curve $y = \sin 2\theta$ was seen to be a one-way stretch of the curve $y = \sin\theta$ by a factor $\frac{1}{2}$ parallel to the x-axis (or a one-way shrinkage by a factor 2).

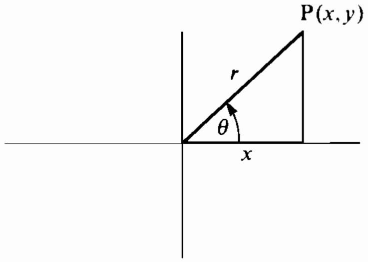
Now consider points on the curves $y = f(x)$ and $y = f(ax)$ with the same y-coordinate. The x-coordinate on $y = f(ax)$ must be $\frac{1}{a}$ times the x-coordinate on $y = f(x)$. Therefore, in general,

the curve $y = f(ax)$ is a one-way stretch of the curve $y = f(x)$ by a factor $\frac{1}{a}$ parallel to the x-axis.

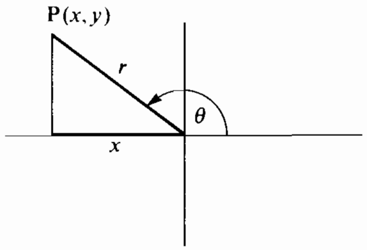


THE COSINE FUNCTION

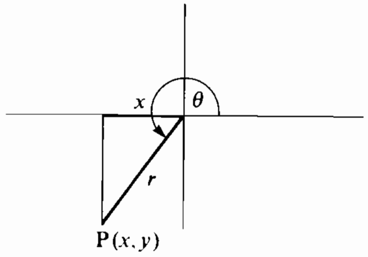
For any position of P, $\cos\theta = \dfrac{x}{r}$



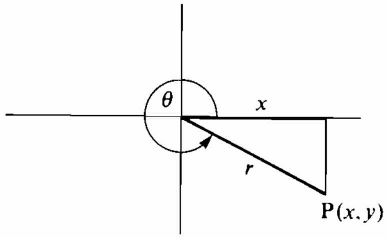
When P is in the first quadrant, x decreases from r to 0 as θ increases, so $\cos\theta$ decreases from 1 to 0



When P is in the second quadrant, x decreases from 0 to $-r$, so $\cos\theta$ decreases from 0 to -1

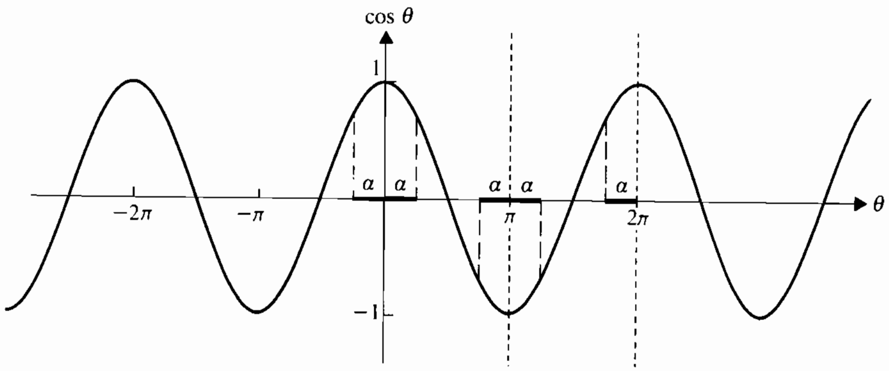


Similarly, when P is in the third quadrant, $\cos\theta$ increases from -1 to 0,



and when P is in the fourth quadrant, $\cos\theta$ increases from 0 to 1

The cycle then repeats itself, and we get this graph of $f(\theta) = \cos\theta$



The characteristics of this graph are as follows.

The curve is continuous

$$-1 \leqslant \cos\theta \leqslant 1$$

It is periodic with a period of 2π

It is the same shape as the sine wave but is translated a distance $\frac{1}{2}\pi$ to the left. Such a translation of a sine wave is called a *phase shift.*

$$\cos\theta = 0 \quad \text{when} \quad \theta = \ldots -\tfrac{1}{2}\pi, \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \ldots$$

The curve is symmetric about $\theta = 0$, so $\cos -\alpha = \cos\alpha$

The curve has rotational symmetry about $\theta = \frac{1}{2}\pi$, so
$\cos(\pi - \alpha) = -\cos\alpha$

Further considerations of symmetry show that

$$\cos(\pi + \alpha) = -\cos\alpha \quad \text{and} \quad \cos(2\pi - \alpha) = \cos\alpha$$

EXERCISE 16b

1. Write in terms of the cosine of an acute angle
   (a) $\cos 123°$ (b) $\cos 250°$ (c) $\cos(-20°)$ (d) $\cos(-154°)$

2. Find the exact value of
   (a) $\cos 150°$ (b) $\cos\frac{3}{2}\pi$ (c) $\cos\frac{5}{4}\pi$ (d) $\cos 6\pi$

3. *Sketch* each of the following curves.
   (a) $y = \cos(\theta + \pi)$ (b) $y = \cos(\theta - \frac{1}{3}\pi)$ (c) $y = \cos(-\theta)$

4. Sketch the graph of $y = \cos(\theta - \frac{1}{2}\pi)$
   What relationship does this suggest between $\sin\theta$ and $\cos(\theta - \frac{1}{2}\pi)$?
   Is there a similar relationship between $\cos\theta$ and $\sin(\theta - \frac{1}{2}\pi)$?

5. Sketch the graph of $y = \cos(\theta - \frac{1}{4}\pi)$ for values of θ between $-\pi$ and π. Use the graph to find the values of θ in this range for which
   (a) $\cos(\theta - \frac{1}{4}\pi) = 1$ (b) $\cos(\theta - \frac{1}{4}\pi) = -1$
   (c) $\cos(\theta - \frac{1}{4}\pi) = 0$

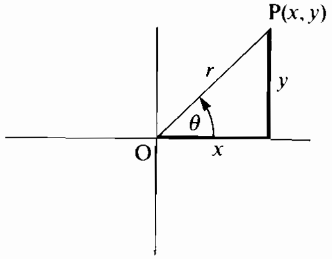
6. On the same set of axes, sketch the graphs $y = \cos\theta$ and $y = 3\cos\theta$

7. On the same set of axes, sketch the graphs $y = \cos\theta$ and $y = \cos 3\theta$

8. Sketch the graph of $f(\theta) = \cos 4\theta$ for $0 \leqslant \theta \leqslant \pi$
   Hence find the values of θ in this range for which $f(\theta) = 0$

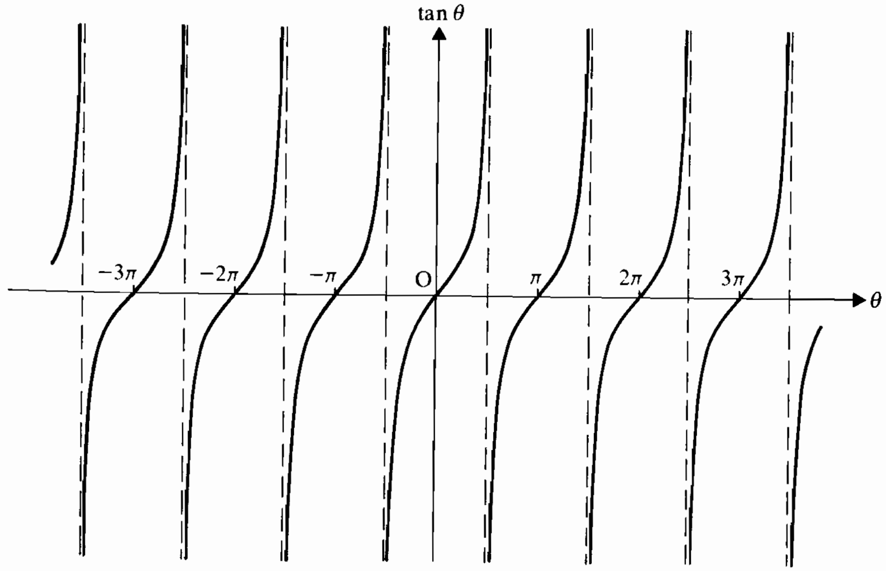
THE TANGENT FUNCTION

For any position of P, $\tan\theta = \dfrac{y}{x}$



As OP rotates through the first quadrant, x decreases from r to 0 while y increases from 0 to r. This means that the fraction y/x increases from 0 to very large values indeed. In fact, as $\theta \to \frac{1}{2}\pi$, $\tan\theta \to \infty$

Similar analysis of the behaviour of y/x in the other quadrants shows that in the second quadrant, $\tan\theta$ is negative and increases from $-\infty$ to 0, in the third quadrant, $\tan\theta$ is positive and increases from 0 to ∞, and in the fourth quadrant, $\tan\theta$ is negative and increases from $-\infty$ to 0. The cycle then repeats itself and we can draw the graph of $f(\theta) = \tan\theta$



From the graph we can see that the characteristics of the tangent function are different from those of the sine and cosine functions in several respects.

It is not continuous, being *undefined* when $\theta = \dots -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \dots$

The range of values of $\tan\theta$ is unlimited.

It is periodic with a period of π (not 2π as in the other cases).

The graph has rotational symmetry about $\theta = 0$, so

$$\tan(-\alpha) = -\tan\alpha$$

The graph has rotational symmetry about $\theta = \frac{1}{2}\pi$, giving

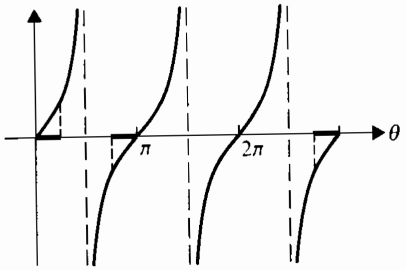
$$\tan(\pi - \alpha) = -\tan\alpha$$

As the cycle repeats itself from $\theta = \pi$ to 2π, we have

$$\tan(\pi + \alpha) = \tan\alpha \quad \text{and} \quad \tan(2\pi - \alpha) = -\tan\alpha$$

Example 16c

Express $\tan\frac{11}{4}\pi$ as the tangent of an acute angle.



$$\tan(\tfrac{11}{4}\pi) = \tan(2\pi + \tfrac{3}{4}\pi) = \tan\tfrac{3}{4}\pi$$
$$= \tan(\pi - \tfrac{1}{4}\pi)$$
$$= -\tan\tfrac{1}{4}\pi$$

EXERCISE 16c

1. Find the exact value of
   (a) $\tan\frac{9}{4}\pi$ (b) $\tan 120°$ (c) $\tan -\frac{2}{3}\pi$ (d) $\tan\frac{7}{4}\pi$

2. Write in terms of the tangent of an acute angle
   (a) $\tan 220°$ (b) $\tan\frac{12}{7}\pi$ (c) $\tan 310°$ (d) $\tan -\frac{7}{5}\pi$

3. Sketch the graph of $y = \tan\theta$ for values of θ in the range 0 to 2π. From this sketch find the values of θ in this range for which
   (a) $\tan\theta = 1$ (b) $\tan\theta = -1$ (c) $\tan\theta = 0$ (d) $\tan\theta = \infty$

4. Using the basic definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$, show that
   $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
   for all values of θ

RELATIONSHIPS BETWEEN SIN θ, COS θ AND TAN θ

Because each trig ratio is a ratio of two of the three quantities x, y and r, we would expect to find several relationships between $\sin\theta$, $\cos\theta$ and $\tan\theta$. Most of these relationships will be investigated in later chapters, but here is a summary of the results from the various exercises so far.

If the graph of $\cos\theta$ is shifted by $\frac{1}{2}\pi$ to the right we get the graph of $\sin\theta$. So $\cos(\theta - \frac{1}{2}\pi) = \sin\theta$

But $\quad \cos(\theta - \frac{1}{2}\pi) = \cos(\frac{1}{2}\pi - \theta)$

Therefore $\quad \cos(\frac{1}{2}\pi - \theta) = \sin\theta$

Two angles which add up to $\frac{1}{2}\pi$ (90°) are called *complementary* angles.

i.e. **the sine of an angle is equal to the cosine of the complementary angle and vice-versa.**

Now $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$ and $\tan\theta = \dfrac{y}{x}$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta$$

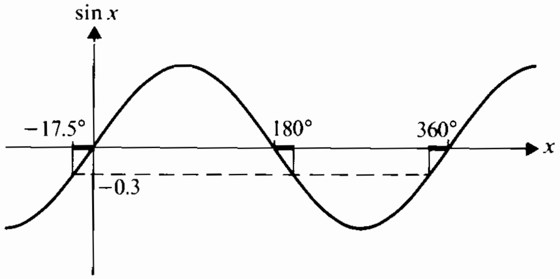
i.e. for all values of θ, $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

We have also seen that the sign of each trig ratio depends on the size of the angle, i.e. the quadrant in which P is. So we can summarise the sign of each ratio in a quadrant diagram:

sin +ve	All +ve
tan +ve	cos +ve

Examples 16d

1. Give all the values of x between 0 and 360° for which $\sin x = -0.3$



The value given for x by a calculator is $-17.5°$

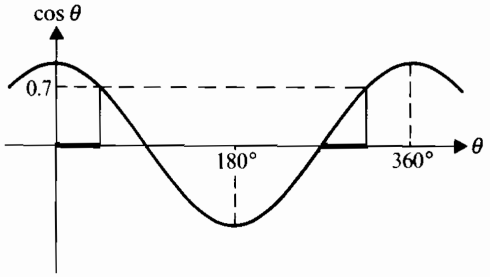
From the graph, we see that, when $\sin x = -0.3$, the values of x in the specified range are $180° + 17.5°$ and $360° - 17.5°$

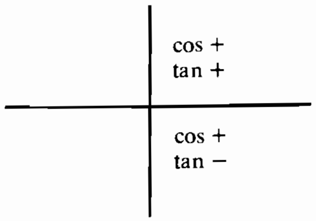
When $\sin x = -0.3$, $x = 197.5°$ and $342.5°$

(Note that when the range of values is given in degrees, the answer should also be given in degrees and the same applies for radians.)

---

2. Find the smallest positive value of θ for which $\cos\theta = 0.7$ and $\tan\theta$ is negative.





If $\cos\theta = 0.7$, the possible values of θ are 45.6°, 314.4°, . . .
Now $\tan\theta$ is positive if θ is in the first quadrant and negative if θ is in the fourth quadrant.

Therefore the required value of θ is 314.4°

EXERCISE 16d

1. Within the range $-2\pi \leqslant \theta \leqslant 2\pi$, give all the values of θ for which
(a) $\sin\theta = 0.4$ (b) $\cos\theta = -0.5$ (c) $\tan\theta = 1.2$

2. Within the range $0 \leqslant \theta \leqslant 720°$, give all the values of θ for which
(a) $\tan\theta = -0.8$ (b) $\sin\theta = -0.2$ (c) $\cos\theta = 0.1$

3. Find the smallest angle (positive or negative) for which
(a) $\cos\theta = 0.8$ and $\sin\theta \geqslant 0$
(b) $\sin\theta = -0.6$ and $\tan\theta \leqslant 0$
(c) $\tan\theta = \sin\frac{1}{6}\pi$

4. Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, show that the equation $\tan\theta = \sin\theta$ can be written as $\sin\theta(\cos\theta - 1) = 0$, provided that $\cos\theta \neq 0$
Hence find the values of θ between 0 and 2π for which $\tan\theta = \sin\theta$.

5. Sketch the graph of $y = \sin 2\theta$. Use your sketch to help find the values of θ in the range $0 \leqslant \theta \leqslant 360°$ for which $\sin 2\theta = 0.4$

6. Sketch the graph of $y = \cos 3\theta$. Hence find the values of θ in the range $0 \leqslant \theta \leqslant 2\pi$ for which $\cos 3\theta = -1$

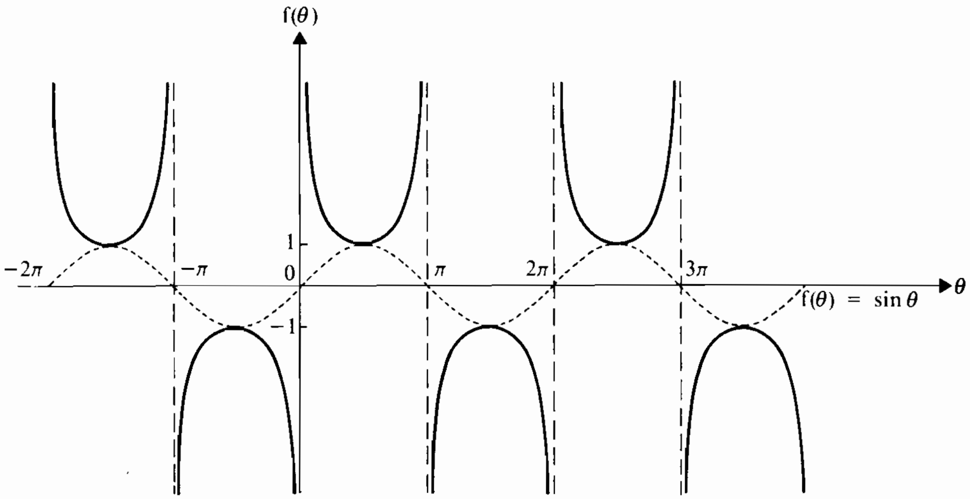
THE RECIPROCAL TRIGONOMETRIC FUNCTIONS

The reciprocals of the three main trig functions have their own names and are sometimes referred to as the *minor* trig ratios.

$$\frac{1}{\sin\theta} = \operatorname{cosec}\theta \qquad \frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\tan\theta} = \cot\theta$$

The names given above are abbreviations of cosecant, secant and cotangent respectively.

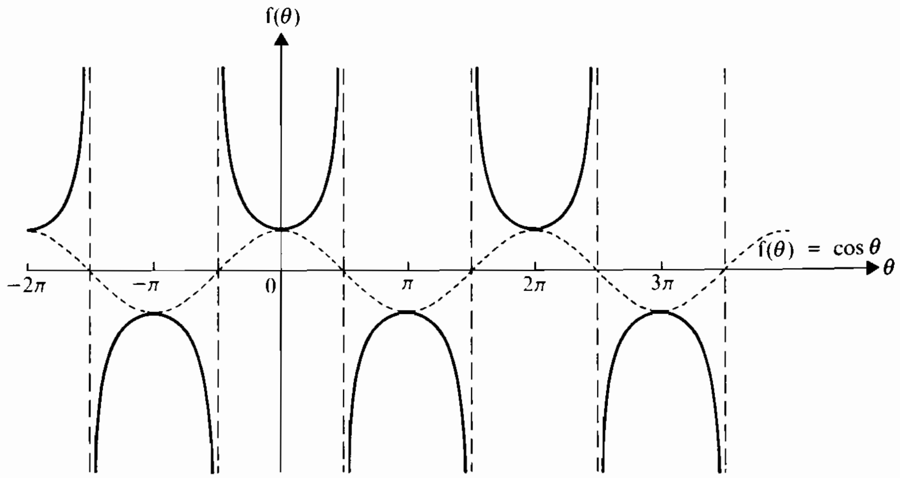
The graph of $f(\theta) = \operatorname{cosec}\theta$ is given below.



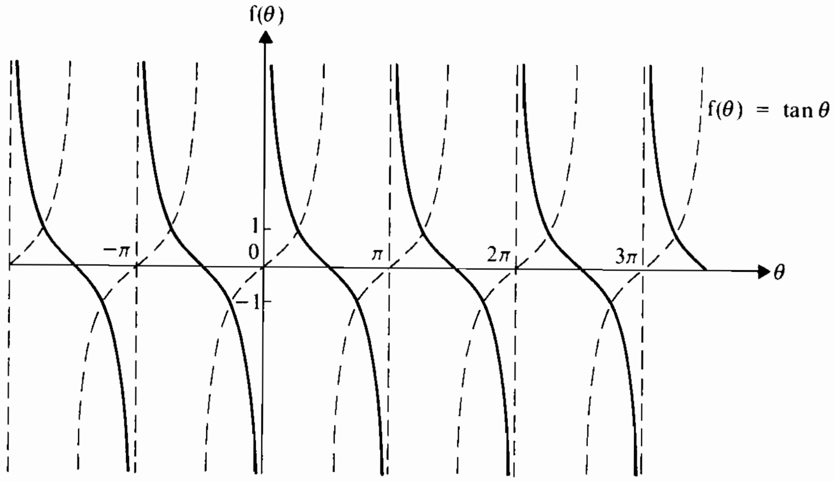
From this graph we can see that

> the cosec function is not continuous, being undefined when θ is any integral multiple of π (we would expect this because these are values of θ where $\sin\theta = 0$ and the reciprocal of 0 is $\pm\infty$).

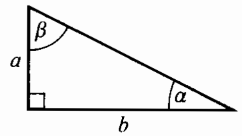
The pattern of the graph of $f(\theta) = \sec\theta$ is similar to that of the cosec graph, as would be expected.



The graph of $f(\theta) = \cot\theta$ is given below.



In this right-angled triangle,



$\tan\alpha = \dfrac{a}{b}$ and $\cot\beta = \dfrac{a}{b}$ $\qquad \left(\cot\beta = \dfrac{1}{\tan\beta}\right)$

Now $\alpha + \beta = 90°$, i.e. α and β are complementary angles.

Hence

> the cotangent of an angle is equal to the tangent of its complement.

In fact, for *any* angle θ, $\cot\theta = \tan(\frac{1}{2}\pi - \theta)$

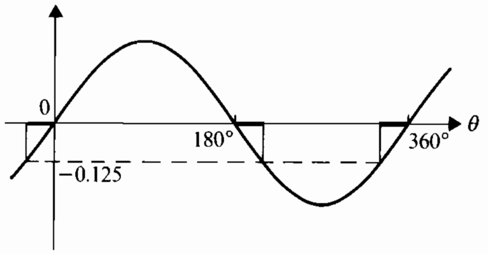
This can be seen from the graph.

Reflecting the curve $y = \tan\theta$ in the vertical axis gives $y = \tan(-\theta)$

Then translating this curve $\frac{1}{2}\pi$ to the left gives $y = \tan(\frac{1}{2}\pi - \theta)$, which is the curve $y = \cot\theta$

Example 16e

For $0 \leqslant \theta \leqslant 360°$, find the values of θ for which $\operatorname{cosec}\theta = -8$



$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = -\frac{1}{8} = -0.125$$

$\therefore$ from a calculator $\theta = -7.2°$

From the sketch, the required values of θ are $187.2°$ and $352.8°$

EXERCISE 16e

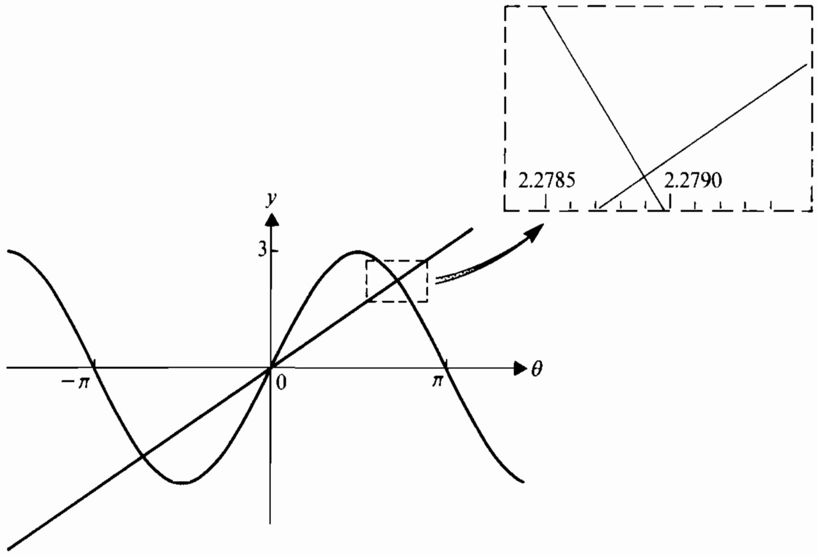
1. Find, for values of θ in the range $0 \leqslant \theta \leqslant 360°$, the values of θ for which
   (a) $\sec\theta = 2$ (b) $\cot\theta = 0.6$ (c) $\operatorname{cosec}\theta = 1.5$
2. Within the range $-180° \leqslant \theta \leqslant 180°$ find the values of θ for which
   (a) $\cot\theta = 1.2$ (b) $\sec\theta = -1.5$ (c) $\operatorname{cosec}\theta = -2$
3. Given that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, write $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$. Hence show that $\cot\theta - \cos\theta = 0$ can be written in the form $\cos\theta(1 - \sin\theta) = 0$, provided that $\sin\theta \neq 0$
   Thus find the values in the range $-\pi \leqslant \theta \leqslant \pi$ for which $\cot\theta - \cos\theta = 0$
4. Find, in surd form, the values of
   (a) $\cot\frac{1}{4}\pi$ (b) $\sec\frac{5}{4}\pi$ (c) $\operatorname{cosec}\frac{11}{6}\pi$
5. Sketch the graph of $f(\theta) = \sec(\theta - \frac{1}{4}\pi)$ for $0 \leqslant \theta \leqslant 2\pi$ and give the values of θ for which $f(\theta) = 1$
6. Sketch the graph of $f(\theta) = \cot(\theta + \frac{1}{3}\pi)$ for $-\pi \leqslant \theta \leqslant \pi$. Hence give the values of θ in this range for which $f(\theta) = 1$

GRAPHICAL SOLUTIONS OF TRIG EQUATIONS

If you have worked through all the exercises in this chapter, you will already have solved several trig equations with the help of sketch graphs. In this section we are going to look at more complicated equations which require accurate plots of the graphs to solve them.

Consider the equation $\theta = 3\sin\theta$

The values of θ for which $\theta = 3\sin\theta$ can be found by plotting the graphs of $y = \theta$ and $y = 3\sin\theta$ on the same axes and hence finding the values of θ at points of intersection.



From the enlarged section of the graph $\theta = 2.2789$ rad

Therefore the three points of intersection occur where

$$\theta = -2.2789 \text{ rad},\ 0,\ 2.2789 \text{ rad} \quad \text{correct to 4 d.p.}$$

If these graphs are produced on a graphics calculator or on a computer using suitable software, then it is possible to zoom in on the points of intersection and get very accurate values for θ. If the graphs are hand drawn, the accuracy of the results will depend on the patience and accuracy of the drawer! Plotting accurate graphs manually is tedious, so if you do not have either of the tools mentioned above, try just Question 1 in the following exercise and do not attempt to get answers correct to more than 2 s.f.

EXERCISE 16f

1. Plot the graphs of $y = \theta$ and $y = 2\cos\theta$ for values of θ in the range $-\pi \leqslant \theta \leqslant \pi$. Hence find the values of θ for which $\theta = 2\cos\theta$

   Repeat this question using *sketch* graphs and measuring θ in degrees. If this was plotted accurately, would it give the same solutions as when the angle is measured in radians?

2. Measuring the angle in radians throughout, find graphically the values of θ for which

   (a) $2\theta = 4\sin\theta$ (b) $\sin\theta = \theta^2$ (c) $\cos\theta = \theta - 1$

CHAPTER 17

TRIGONOMETRIC IDENTITIES AND EQUATIONS

IDENTITIES

At this stage it is important to know the difference between identities and equations.

This is an equation: $(x-1)^2 = 4$
The equality is true only when $x = 3$ or when $x = -1$
In any equation, the equality is valid only for a restricted set of values.

This is an identity: $(x-1)^2 = x^2 - 2x + 1$
The RHS is a different way of expressing the LHS, and the equality is true for all values of x.

In an identity, the equality is true for *any* value of the variable, and we use the symbol $\equiv$ to mean 'is identical to',

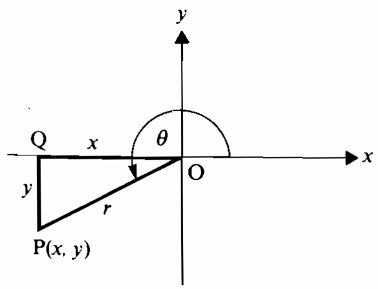
i.e. we would write $(x-1)^2 \equiv x^2 - 2x + 1$

In this chapter we concentrate on some trigonometric identities and some of their uses.

One such identity was introduced in Chapter 16, namely

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

THE PYTHAGOREAN IDENTITIES



For any angle θ,

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \quad \text{and}$$

$$\tan\theta = \frac{y}{x}$$

Also, in right-angled triangle OPQ,

$$x^2 + y^2 = r^2 \qquad \text{(Pythagoras)}$$

Therefore, $(\cos\theta)^2 + (\sin\theta)^2 = \left(\dfrac{x}{r}\right)^2 + \left(\dfrac{y}{r}\right)^2 = \dfrac{x^2+y^2}{r^2} = 1$

Using the notation $\cos^2\theta$ to mean $(\cos\theta)^2$, etc., we have

$$\cos^2\theta + \sin^2\theta \equiv 1 \qquad [1]$$

Using the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ we can write [1] in two other forms.

$[1] \div \cos^2\theta \Rightarrow \qquad 1 + \dfrac{\sin^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$

$\Rightarrow \qquad 1 + \tan^2\theta \equiv \sec^2\theta$

$[1] \div \sin^2\theta \Rightarrow \qquad \dfrac{\cos^2\theta}{\sin^2\theta} + 1 \equiv \dfrac{1}{\sin^2\theta}$

$\Rightarrow \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$

These identities can be used to

- simplify trig expressions,
- eliminate trig terms from pairs of equations,
- derive a variety of further trig relationships,
- calculate other trig ratios of any angle for which one trig ratio is known.

These identities are also very useful in the solution of certain types of trig equations and we will look at this application later in this chapter.

Examples 17a

1. Simplify $\dfrac{\sin\theta}{1+\cot^2\theta}$

$$\frac{\sin\theta}{1+\cot^2\theta} \equiv \frac{\sin\theta}{\operatorname{cosec}^2\theta} \equiv \sin^3\theta$$

Using $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$ and $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$

---

2. Eliminate θ from the equations $x = 2\cos\theta$ and $y = 3\sin\theta$

$$\cos\theta = \tfrac{1}{2}x \quad\text{and}\quad \sin\theta = \tfrac{1}{3}y$$

Using $\cos^2\theta + \sin^2\theta \equiv 1$ gives

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\Rightarrow \qquad 9x^2 + 4y^2 = 36$$

In Example 2, both x and y initially depend on θ, a variable angle. Used in this way, θ is called a *parameter*, and is a type of variable that plays an important part in the analysis of curves and functions.

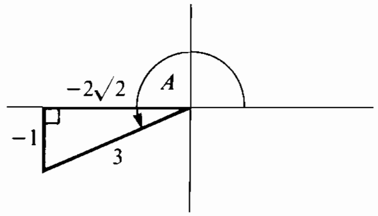
---

3. If $\sin A = -\frac{1}{3}$ and A is in the third quadrant, find $\cos A$ without using a calculator.

There are two ways of doing this problem. The first method involves drawing a quadrant diagram and working out the remaining side of the triangle, using Pythagoras theorem.

From the diagram, $x = -2\sqrt{2}$

$\therefore \qquad \cos A = \dfrac{x}{r} = -\dfrac{2\sqrt{2}}{3}$



The second method uses the identity $\cos^2 A + \sin^2 A \equiv 1$ giving

$$\cos^2 A + \frac{1}{3} = 1 \quad\Rightarrow\quad \cos A = \pm\sqrt{\frac{8}{3}} = \pm\frac{2\sqrt{2}}{3}$$

As A is between π and $\frac{3}{2}\pi$, $\cos A$ is negative, i.e.

$$\cos A = -\frac{2\sqrt{2}}{3}$$

---

4. Prove that $(1 - \cos A)(1 + \sec A) \equiv \sin A \tan A$

Because the relationship has yet to be proved, we must not assume its truth by using the complete identity in our working. The left and right hand sides must be isolated throughout the proof, preferably by working on only one of these sides.

Consider the LHS:

$$(1 - \cos A)(1 + \sec A) \equiv 1 + \sec A - \cos A - \cos A \sec A$$

$$\equiv 1 + \sec A - \cos A - \cos A\left(\frac{1}{\cos A}\right)$$

$$\equiv \sec A - \cos A$$

$$\equiv \frac{1 - \cos^2 A}{\cos A}$$

$$\equiv \frac{\sin^2 A}{\cos A} \qquad (\cos^2 A + \sin^2 A \equiv 1)$$

$$\equiv \sin A\left[\frac{\sin A}{\cos A}\right]$$

$$\equiv \sin A \tan A \equiv \text{RHS}$$

---

5. Prove that $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) \equiv \dfrac{1}{\tan A + \cot A}$

LHS is $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$$\equiv \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$$

$$\equiv \left(\frac{1 - \sin^2 A}{\sin A}\right)\left(\frac{1 - \cos^2 A}{\cos A}\right)$$

$$\equiv \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)$$

$$\equiv \cos A \sin A$$

Now this is already a very simple form but as it is not yet in the form of the given RHS, we begin working independently on the RHS.

RHS is $\dfrac{1}{\tan A + \cot A} \equiv 1 \div \left(\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}\right)$

$$\equiv 1 \div \left(\frac{\sin^2 A + \cos^2 A}{\cos A \sin A}\right)$$

$$\equiv \cos A \sin A$$

Since both LHS and RHS reduce to $\cos A \sin A$ they are identical.

EXERCISE 17a

1. Without using a calculator, complete the following table.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	type of angle
(a)		$-\frac{5}{13}$		reflex
(b)	$\frac{3}{5}$			obtuse
(c)			$\frac{7}{24}$	acute
(d)				straight line

Simplify the following expressions.

2. $\dfrac{1 - \sec^2 A}{1 - \operatorname{cosec}^2 A}$

3. $\dfrac{\sin\theta}{\sqrt{(1 - \cos^2\theta)}}$

4. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

5. $\dfrac{\sqrt{(1 + \tan^2\theta)}}{\sqrt{(1 - \sin^2\theta)}}$

6. $\dfrac{1}{\cos\theta\sqrt{(1 + \cot^2\theta)}}$

7. $\dfrac{\sin\theta}{1 + \cot^2\theta}$

Eliminate θ from the following pairs of equations.

8. $x = 4\sec\theta$, $y = 4\tan\theta$

9. $x = a\operatorname{cosec}\theta$, $y = b\cot\theta$

10. $x = 2\tan\theta$, $y = 3\cos\theta$

11. $x = 1 - \sin\theta$, $y = 1 + \cos\theta$

12. $x = 2 + \tan\theta$, $y = 2\cos\theta$

13. $x = a\sec\theta$, $y = b\sin\theta$

Prove the following identities.

14. $\cot\theta + \tan\theta \equiv \sec\theta \operatorname{cosec}\theta$

15. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} \equiv \sin A + \cos A$

16. $\tan^2\theta + \cot^2\theta \equiv \sec^2\theta + \operatorname{cosec}^2\theta - 2$

17. $\dfrac{\sin A}{1+\cos A} \equiv \dfrac{1-\cos A}{\sin A}$ (Hint. Multiply top and bottom of LHS by $(1-\cos A)$.)

18. $(\sec^2\theta + \tan^2\theta)(\operatorname{cosec}^2\theta + \cot^2\theta) \equiv 1 + 2\sec^2\theta \operatorname{cosec}^2\theta$

19. $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} \equiv \dfrac{2}{\sin A}$

20. $\sec^2 A \equiv \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - \sin A}$

21. $(1+\sin A+\cos A)^2 \equiv 2(1+\sin A)(1+\cos A)$

22. $\dfrac{\tan^2 A + \cos^2 A}{\sin A + \sec A} \equiv \sec A - \sin A$

SOLVING EQUATIONS

We have already solved some simple trig equations in Chapter 16. We can now solve slightly more complicated equations using the Pythagorean identities.

Examples 17b

1. Solve the equation $2\cos^2\theta - \sin\theta = 1$ for values of θ in the range 0 to 2π

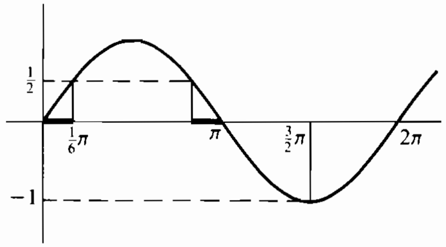
The given equation is quadratic, but it involves the sine and the cosine of θ, so we use $\cos^2\theta + \sin^2\theta \equiv 1$ to express the equation in terms of $\sin\theta$ only.

$$2\cos^2\theta - \sin\theta = 1$$

$$\Rightarrow \quad 2(1-\sin^2\theta) - \sin\theta = 1$$

$$\Rightarrow \quad 2\sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta + 1) = 0 \quad \Rightarrow \quad \sin\theta = \tfrac{1}{2} \text{ or } -1$$



If $\sin\theta = \frac{1}{2}$, $\theta = \frac{1}{6}\pi, \frac{5}{6}\pi$

If $\sin\theta = -1$, $\theta = \frac{3}{2}\pi$

Therefore the solution of the equation is $\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{3}{2}\pi$

---

2. Solve the equation $\cot x = \sin x$ for values of x from 0 to 360°

Using $\cot x \equiv \dfrac{\cos x}{\sin x}$ gives

$$\frac{\cos x}{\sin x} = \sin x$$

We can now multiply the equation by $\sin x$ provided that $\sin x \neq 0$ Thus we must exclude any values of x for which $\sin x = 0$ from the solution set.

$$\Rightarrow \quad \cos x = \sin^2 x$$

$$\Rightarrow \quad \cos^2 x + \cos x - 1 = 0$$

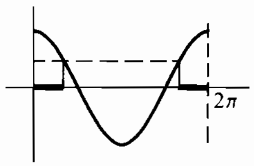
This equation does not factorise, so we use the formula, giving

$$\cos x = \tfrac{1}{2}(-1 \pm \sqrt{5})$$

$\therefore$ $\cos x = -1.618$ and there is no value of x for which this is true.

or $\cos x = 0.618$

$\Rightarrow$ $x = 51.8°$ or $308.2°$



EXERCISE 17b

Solve the following equations for angles in the range $0 \leqslant \theta \leqslant 360°$

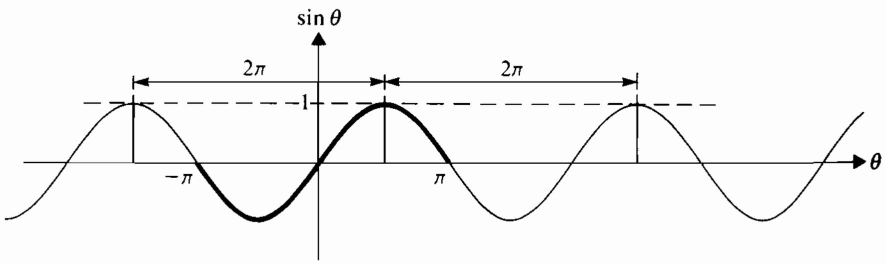
1. $\sec^2\theta + \tan^2\theta = 6$
2. $4\cos^2\theta + 5\sin\theta = 3$
3. $\cot^2\theta = \operatorname{cosec}\theta$
4. $\tan\theta + \cot\theta = 2\sec\theta$
5. $\tan\theta + 3\cot\theta = 5\sec\theta$
6. $\sec\theta = 1 - 2\tan^2\theta$

Solve the following equations for angles in the range $-\pi \leqslant \theta \leqslant \pi$

7. $5\cos\theta - 4\sin^2\theta = 2$
8. $4\cot^2\theta + 12\operatorname{cosec}\theta + 1 = 0$
9. $4\sec^2\theta - 3\tan\theta = 5$
10. $2\cos\theta - 4\sin^2\theta + 2 = 0$

GENERAL SOLUTIONS OF TRIG EQUATIONS

Consider the equation $\sin\theta = 1$

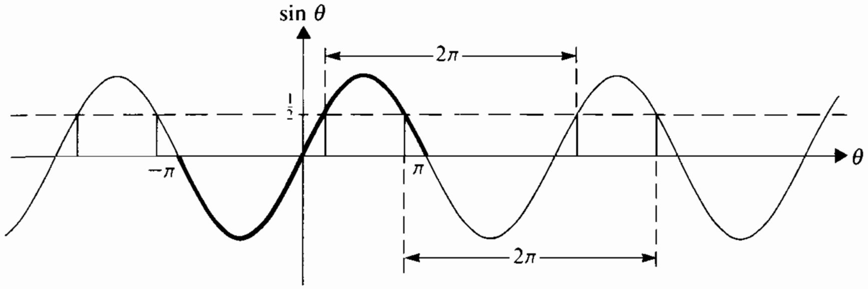


We see that, in the range $-\pi \leqslant \theta \leqslant \pi$, there is one solution, $\theta = \frac{1}{2}\pi$
We can also see that there are many more values of θ that satisfy $\sin\theta = 1$

The general solution is an expression which represents all angles which satisfy the given equation. From the graph we can see that the general solution is an infinite set of angles.

In looking for a general solution, use can be made of the graphs of trig functions and their periodic nature.

Now consider the equation $\sin\theta = \frac{1}{2}$



The period of the sine function is covered by the interval $[-\pi, \pi]$ and repeats every 2π

In the interval $[-\pi, \pi]$, $\theta = \frac{1}{6}\pi$ and $\theta = \frac{5}{6}\pi$ and we can get further solutions by adding (or subtracting) any multiple of 2π to $\frac{1}{6}\pi$ or to $\frac{5}{6}\pi$

Therefore the general solution of $\sin\theta = \frac{1}{2}$ can be expressed as

$$\theta = \begin{cases} \frac{1}{6}\pi + 2n\pi \\ \frac{5}{6}\pi + 2n\pi \end{cases} \quad \text{where } n \in \mathbb{Z}$$

The same argument can be applied to any equation $\sin\theta = s$, where $-1 \leqslant s \leqslant 1$, which usually has two solutions in the interval $[-\pi, \pi]$, showing that

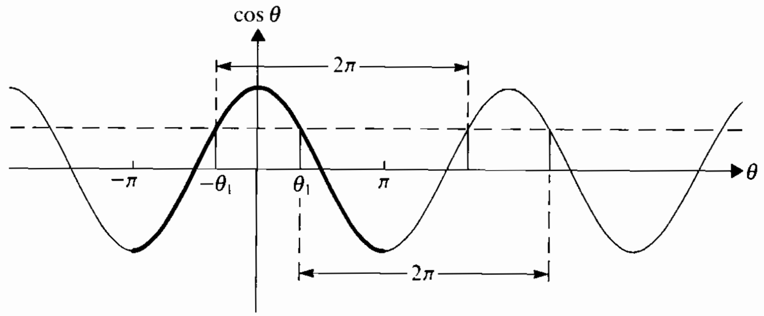
the general solution of $\sin\theta = s$, where $-1 \leqslant s \leqslant 1$, is

$$\theta = \begin{cases} \theta_1 + 2n\pi \\ \theta_2 + 2n\pi \end{cases} \quad \text{or} \quad \theta = \begin{cases} \theta_1 + 360n° \\ \theta_2 + 360n° \end{cases}$$

where $n \in \mathbb{Z}$, and θ_1 and θ_2 are the solutions in the range $-\pi \leqslant \theta \leqslant \pi$

A similar situation occurs when the equation $\cos\theta = c$ is considered. The cosine function is periodic with a period of 2π, and one period is covered by the interval $[-\pi, \pi]$

Within this interval there are usually two solutions, θ_1 and θ_2, and adding or subtracting any multiple of 2π gives another angle with the same cosine ratio. But, because the graph is symmetrical about $\theta = 0$, $\theta_2 = -\theta_1$, the general solution can be expressed in terms of θ_1 alone.

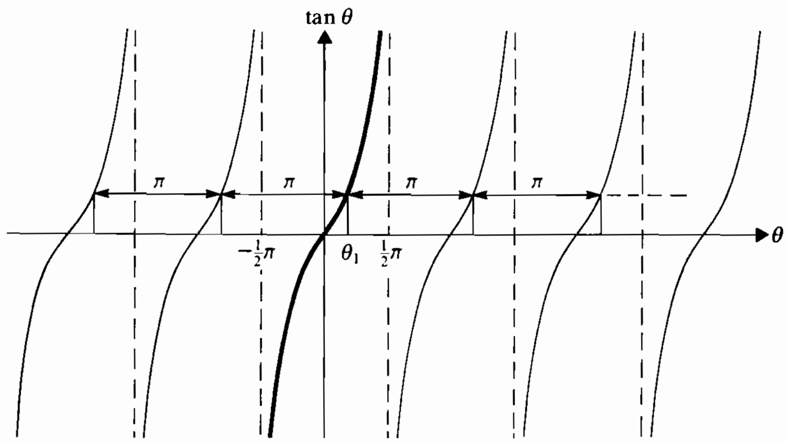


The general solution of the equation $\cos\theta = c$, where $-1 \leqslant c \leqslant 1$ is

$$\theta = \pm\theta_1 + 2n\pi$$

where $n \in \mathbf{Z}$ and θ_1 is a solution in the range $-\pi \leqslant \theta \leqslant \pi$

The situation with the equation $\tan\theta = t$ is different because the tangent function has a period of π which is covered by the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Within this interval there is only one solution of the equation $\tan\theta = t$ and adding or subtracting any multiple of π gives another angle with the same tangent ratio.



The general solution of the equation $\tan\theta = t$ is

$$\theta = \theta_1 + n\pi$$

where $n \in \mathbf{Z}$ and θ_1 is the solution in the range $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$

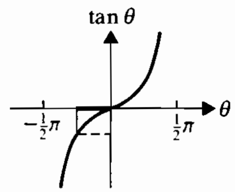
Examples 17c

1. Find the general solution of the equation $\tan\theta = -\sqrt{3}$

In the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ the solution of $\tan\theta = -\sqrt{3}$ is $\theta = -\frac{1}{3}\pi$

Therefore the general solution is

$\theta = -\frac{1}{3}\pi + n\pi$ where $n \in \mathbf{Z}$

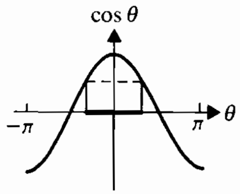


2. Find the general solution of the equation $\cos\theta = \dfrac{1}{\sqrt{2}}$

In the interval $[-\pi, \pi]$, the solutions of $\cos\theta = \dfrac{1}{\sqrt{2}}$ are $\pm\frac{1}{4}\pi$

So the general solution is

$\theta = \pm\frac{1}{4}\pi + 2n\pi$ where $n \in \mathbf{Z}$

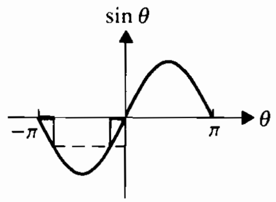


3. Find the general solution of the equation $\sin\theta = -\frac{1}{2}$

In the interval $[-\pi, \pi]$, the solutions are $\theta = -\frac{1}{6}\pi$ and $\theta = -\frac{5}{6}\pi$

So the general solution is

$$\left.\begin{aligned} \theta &= -\tfrac{1}{6}\pi + 2n\pi \\ \text{and}\quad \theta &= -\tfrac{5}{6}\pi + 2n\pi \end{aligned}\right\} \text{ where } n \in \mathbf{Z}$$



4. Find the general solution of the equation $3\sec^2\theta - 5\tan\theta - 4 = 0$ giving answers in degrees correct to 1 d.p.

Using $1 + \tan^2\theta \equiv \sec^2\theta$ we have

$$3(1+\tan^2\theta) - 5\tan\theta - 4 = 0$$

$$\Rightarrow \quad 3\tan^2\theta - 5\tan\theta - 1 = 0$$

This equation does not have any simple factors so we solve it by the formula.

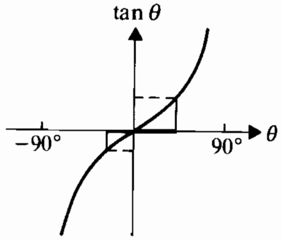
$$\Rightarrow \quad \tan\theta = \frac{5 \pm \sqrt{(25+12)}}{6}$$

$$= 1.8471 \text{ or } -0.1805$$

In the interval $[-90°, 90°]$

$\tan\theta = 1.8471 \quad \Rightarrow \quad \theta = 61.6°$

$\tan\theta = -0.1805 \quad \Rightarrow \quad \theta = -10.2°$



The general solution is therefore $\theta = \begin{cases} 61.6° + 180n° \\ -10.2° + 180n° \end{cases}$ where $n \in \mathbb{Z}$

EXERCISE 17c

Find the general solutions of each of the following equations.
Give answers in radians when they are exact; otherwise give answers in degrees to 1 d.p.

1. $\sin\theta = \frac{\sqrt{3}}{2}$
2. $\cos\theta = 0$
3. $\tan\theta = -\sqrt{3}$
4. $\sin\theta = -\frac{1}{4}$
5. $\cos\theta = -\frac{1}{2}$
6. $\tan\theta = 1$
7. $\sec\theta = 1$
8. $\operatorname{cosec}\theta = 2$
9. $\sin^2\theta = \frac{1}{4}$
10. $5\cos\theta - 4\sin^2\theta = 2$
11. $4\cot^2\theta + 12\operatorname{cosec}\theta + 1 = 0$
12. $4\sec^2\theta - 3\tan\theta = 5$
13. $2\cos\theta - 4\sin^2\theta + 2 = 0$
14. $2\sin\theta\cos\theta + \sin\theta = 0$
15. $4\cos\theta = \cos\theta\operatorname{cosec}\theta$
16. $\sqrt{3}\tan\theta = 2\sin\theta$
17. $\cot\theta = \sin\theta$

EQUATIONS INVOLVING MULTIPLE ANGLES

Many trig equations involve ratios of a multiple of θ, for example

$$\cos 2\theta = \tfrac{1}{2} \qquad \tan 3\theta = -2$$

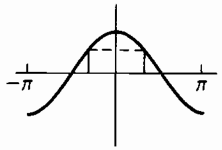
Simple equations of this type can be solved by finding first the values of the multiple angle and then, by division, the corresponding values of θ

Examples 17d

1. Find the general solution of the equation $\cos 2\theta = \frac{1}{2}$

Using $2\theta = \phi$ gives $\cos\phi = \frac{1}{2}$

In the interval $[-\pi, \pi]$, the solutions of $\cos\phi = \frac{1}{2}$ are $\phi = \pm\frac{1}{3}\pi$



So the general solution for ϕ is $\quad \phi = \pm\frac{1}{3}\pi + 2n\pi$

but $\phi = 2\theta$, therefore $\quad 2\theta = \pm\frac{1}{3}\pi + 2n\pi$

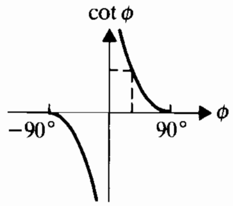
Hence $\quad \theta = \pm\frac{1}{6}\pi + n\pi$

---

2. Find the general solution of the equation $\cot(\frac{1}{3}\theta - 90°) = 1$, giving the answer in degrees.

Using $\frac{1}{3}\theta - 90° = \phi$ gives

$$\cot(\tfrac{1}{3}\theta - 90°) = \cot\phi$$



The general solution of the equation $\cot\phi = 1$ is

$$\phi = 45° + 180n°$$

But $\phi = \frac{1}{3}\theta - 90°$, so $\quad \frac{1}{3}\theta - 90° = 45° + 180n°$

$\Rightarrow \quad \theta = 135° + 270° + 540n°$

i.e. $\quad \theta = 405° + 540n°$

EXERCISE 17d

Find the general solutions of the following equations, giving your answers in degrees.

1. $\tan 2\theta = 1$
2. $\cos 3\theta = -0.5$
3. $\sin\frac{1}{2}\theta = -\frac{\sqrt{2}}{2}$
4. $\sec 5\theta = 2$
5. $\cot\frac{1}{3}\theta = -4$
6. $\cos 2\theta = 0.63$
7. $\cos(2\theta - 45°) = 0$
8. $\sin(\frac{1}{4}\theta + 30°) = -1$
9. $\tan(\theta - 60°) = 0$

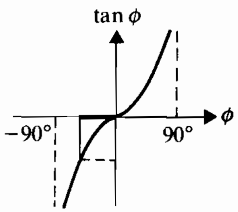
Find the general solutions of the following equations, giving your answers in radians.

10. $\cos(\theta + \frac{1}{4}\pi) = \frac{1}{2}$
11. $\tan(2\theta - \frac{1}{3}\pi) = -1$
12. $\sin(2\theta + \frac{1}{6}\pi) = \frac{1}{2}$

Sometimes an equation involving multiple angles requires a solution in a specified range and this can be found from the general solution.

Example 17e

Find the angles in the interval $[-180°, 180°]$ which satisfy the equation $\tan 3\theta = -2$



Using $\phi = 3\theta$ gives $\tan\phi = -2$

In the interval $[-90°, 90°]$, $\phi = -63.4°$

$\therefore$ the general solution is $3\theta = -63.4° + 180n°$

$\Rightarrow$ $\theta = -21.1° + 60n°$

Now to get values of θ from $-180°$ to $180°$, we see that we need values of n from -2 to 3

When $n = -2, -1, 0, 1, 2, 3$

$\theta = -141.1°, -81.1°, -21.1°, 38.9°, 98.9°, 158.9°$

Note that **when a multiple angle is involved, it is the general solution of the multiple angle that must be found.**

EXERCISE 17e

Solve the equations for values of θ in the range $-180° \leqslant \theta \leqslant 180°$

1. $\tan 2\theta = 1.8$
2. $\sin 3\theta = 0.7$
3. $\cos\frac{1}{2}\theta = 0.85$

Solve the equations for values of θ in the range $0 \leqslant \theta \leqslant 2\pi$

4. $\tan 4\theta = -\sqrt{3}$
5. $\sec 5\theta = 2$
6. $\cot\frac{1}{2}\theta = -1$

MIXED EXERCISE 17

1. Eliminate α from the equations $x = \cos\alpha$, $y = \operatorname{cosec}\alpha$
2. If $\cos\beta = 0.5$, find possible values for $\sin\beta$ and $\tan\beta$, giving your answers in exact form.
3. Simplify the expression $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta}$. Hence solve the equation $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta} = 4$ for values of θ in the range $0 \leqslant \theta \leqslant 2\pi$
4. Find the general solution of the equation $\sec\theta + \tan^2\theta = 5$ Give the answer in degrees.
5. Prove that $(\cot\theta + \operatorname{cosec}\theta)^2 \equiv \dfrac{1+\cos\theta}{1-\cos\theta}$
6. Find the values of θ for which $\tan(3\theta - \frac{1}{3}\pi) = 1$ in the interval $[-\pi, \pi]$
7. Eliminate θ from the equations
   (a) $x - 2 = \sin\theta$, $y + 1 = \cos\theta$
   (b) $x = \sec\theta - 3$, $y = 2 - \tan\theta$
8. Find the general solution of the equation $\tan 2\alpha = \cot 2\alpha$
9. Prove that $(\cos A + \sin A)^2 + (\cos A - \sin A)^2 \equiv 2$
10. Simplify $(1 + \cos A)(1 - \cos A)$
11. Find in degrees the general solution of the equation $\tan\theta = 3\sin\theta$
12. Simplify $\sec^4\theta - \sec^2\theta$

CHAPTER 18

COMPOUND ANGLE IDENTITIES

COMPOUND ANGLES

It is often useful to be able to express a trig ratio of an angle A + B in terms of trig ratios of A and of B.

It is dangerously easy to think, for instance, that $\sin(A+B)$ is $\sin A + \sin B$. However, this is *false* as can be seen by considering

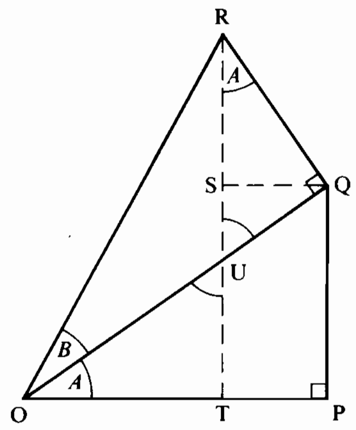
$$\sin(45° + 45°) = \sin 90° = 1$$

whereas $$\sin 45° + \sin 45° = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2} \neq 1$$

Thus the sine function is *not distributive* and neither are the other trig functions.

The correct identity is $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

This is proved geometrically when A and B are both acute, from the diagram below.



The right-angled triangles OPQ and OQR contain angles A and B as shown.
From the diagram, $\angle URQ = A$

$$\sin(A+B) = \frac{TR}{OR} = \frac{TS+SR}{OR} = \frac{PQ+SR}{OR}$$

$$= \frac{PQ}{OQ}\times\frac{OQ}{OR} + \frac{SR}{QR}\times\frac{QR}{OR}$$

$$\therefore \quad \sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

This identity is in fact valid for all angles and it can be adapted to give the full set of compound angle formulae. The reader is left to do this in the following exercise.

EXERCISE 18a

1. In the identity $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$, replace B by $-B$ to show that $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

2. In the identity $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$, replace A by $(\frac{1}{2}\pi - A)$ to show that $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$

3. In the identity $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$, replace B by $-B$ to show that $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$

4. Use $\dfrac{\sin(A+B)}{\cos(A+B)}$ to show that $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

5. Replace B by $-B$ in the formula for $\tan(A+B)$ to show that
$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Collecting these results we have:

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Examples 18b

1. Find exact values for (a) $\sin 75°$ (b) $\cos 105°$

To find exact values, we need to express the given angle in terms of angles whose trig ratios are known as exact values, e.g. 30°, 60°, 45°, 90°, 120°, ...
Now $75° = 45° + 30°$ (or $120° - 45°$ or other alternative compound angles).

(a) $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$$

(b) $\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

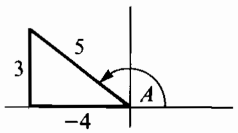
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4}(1 - \sqrt{3})$$

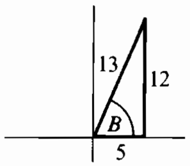
2. A is obtuse and $\sin A = \frac{3}{5}$, B is acute and $\sin B = \frac{12}{13}$. Find the exact value of $\cos(A + B)$.

$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$

In order to use this formula, we need values for $\cos A$ and $\cos B$. These can be found using Pythagoras' theorem in the appropriate right-angled triangle.



$\cos A = -\frac{4}{5}$



$\cos B = \frac{5}{13}$

$$\therefore \quad \cos(A + B) = \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = -\frac{56}{65}$$

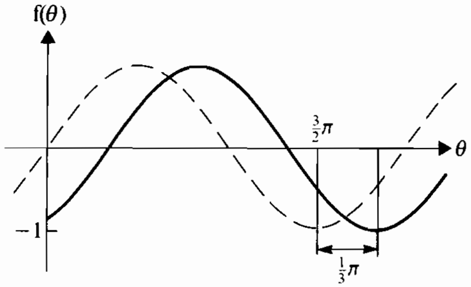
3. Simplify $\sin\theta\cos\frac{1}{3}\pi - \cos\theta\sin\frac{1}{3}\pi$ and hence find the smallest positive value of θ for which the expression has a minimum value.

$\sin\theta\cos\frac{1}{3}\pi - \cos\theta\sin\frac{1}{3}\pi$ is the expansion of $\sin(A - B)$ with $A = \theta$ and $B = \frac{1}{3}\pi$

$$\sin\theta\cos\tfrac{1}{3}\pi - \cos\theta\sin\tfrac{1}{3}\pi = \sin(\theta - \tfrac{1}{3}\pi)$$

Let $f(\theta) = \sin(\theta - \frac{1}{3}\pi)$

Now the graph $f(\theta)$ is a sine wave, but translated $\frac{1}{3}\pi$ in the direction of the positive θ-axis.



Therefore $f(\theta)$ has a minimum value of -1 and the smallest +ve value of θ at which this occurs is $\frac{3}{2}\pi + \frac{1}{3}\pi = \frac{11}{6}\pi$

4. Prove that $\dfrac{\sin(A - B)}{\cos A \cos B} + \dfrac{\sin(B - C)}{\cos B \cos C} \equiv \tan A - \tan C$

Expanding each of the numerators, the LHS becomes

$$\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C}$$

$$\equiv \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C}$$

$\equiv \tan A - \tan B + \tan B - \tan C$

$\equiv \tan A - \tan C \equiv \text{RHS}$

5. Find the general solution of the equation $2\cos\theta = \sin(\theta + \frac{1}{6}\pi)$

$$2\cos\theta = \sin(\theta + \tfrac{1}{6}\pi)$$
$$= \sin\theta\cos\tfrac{1}{6}\pi + \cos\theta\sin\tfrac{1}{6}\pi = \tfrac{\sqrt{3}}{2}\sin\theta + \tfrac{1}{2}\cos\theta$$
$$\therefore \quad \tfrac{3}{2}\cos\theta = \tfrac{\sqrt{3}}{2}\sin\theta$$
$$\Rightarrow \quad \frac{3}{\sqrt{3}} = \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$$

$\tan\frac{1}{3}\pi = \sqrt{3}$

Therefore the general solution is $\theta = \frac{1}{3}\pi + n\pi$

EXERCISE 18b

Find the exact value of each expression, leaving your answer in surd form where necessary.

1. $\cos 40° \cos 50° - \sin 40° \sin 50°$
2. $\sin 37° \cos 7° - \cos 37° \sin 7°$
3. $\cos 75°$
4. $\tan 105°$
5. $\sin 165°$
6. $\cos 15°$

Simplify each of the following expressions.

7. $\sin\theta\cos 2\theta + \cos\theta\sin 2\theta$
8. $\cos\alpha\cos(90° - \alpha) - \sin\alpha\sin(90° - \alpha)$
9. $\dfrac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$
10. $\dfrac{\tan 3\beta - \tan 2\beta}{1 + \tan 3\beta \tan 2\beta}$
11. A is acute and $\sin A = \frac{7}{25}$, B is obtuse and $\sin B = \frac{4}{5}$. Find an exact expression for
    (a) $\sin(A + B)$ (b) $\cos(A + B)$ (c) $\tan(A + B)$
12. Find the greatest value of each expression and the value of θ between 0 and 360° at which it occurs.
    (a) $\sin\theta\cos 25° - \cos\theta\sin 25°$ (b) $\sin\theta\sin 30° + \cos\theta\cos 30°$
    (c) $\cos\theta\cos 50° - \sin\theta\sin 50°$ (d) $\sin 60°\cos\theta - \cos 60°\sin\theta$

Prove the following identities.

13. $\cot(A + B) \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$
14. $(\sin A + \cos A)(\sin B + \cos B) \equiv \sin(A + B) + \cos(A - B)$
15. $\sin(A + B) + \sin(A - B) \equiv 2\sin A\cos B$
16. $\sin(\frac{1}{4}\pi + A) + \sin(\frac{1}{4}\pi - A) \equiv \sqrt{2}\cos A$
17. $\cos(A + B) + \cos(A - B) \equiv 2\cos A\cos B$
18. $\dfrac{\sin(A + B)}{\cos A\cos B} \equiv \tan A + \tan B$
19. $\sin(\theta + 60°) \equiv \sin(120° - \theta)$
20. $\tan(x + y) - \tan x \equiv \dfrac{\sin y}{\cos x\cos(x + y)}$

Solve the following equations for values of θ in the range $0 \leqslant \theta \leqslant 360°$

21. $\cos(45° - \theta) = \sin\theta$
22. $3\sin\theta = \cos(\theta + 60°)$
23. $\tan(A - \theta) = \frac{2}{3}$ and $\tan A = 3$
24. $\sin(\theta + 60°) = \cos\theta$

Find the general solution of the following equations.

25. $\sin(x + \frac{1}{3}\pi) = \cos x$
26. $\sin x = \cos(x - \frac{2}{3}\pi)$

THE DOUBLE ANGLE IDENTITIES

The compound angle formulae deal with any two angles A and B and can therefore be used for two equal angles, i.e. when $B = A$

Replacing B by A in the trig identities for $(A + B)$ gives the following set of double angle identities.

$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

The second of these identities can be expressed in several forms because

$$\cos^2 A - \sin^2 A \equiv \begin{cases}(1-\sin^2 A) - \sin^2 A \equiv 1 - 2\sin^2 A \\ \cos^2 A - (1-\cos^2 A) \equiv 2\cos^2 A - 1\end{cases}$$

i.e.

$$\cos 2A \equiv \begin{cases}\cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1\end{cases}$$

Examples 18c

1. If $\tan\theta = \frac{3}{4}$, find the values of $\tan 2\theta$ and $\tan 4\theta$

Using $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$ with $A = \theta$ and $\tan\theta = \frac{3}{4}$ gives

$$\tan 2\theta = \frac{2(\frac{3}{4})}{1-(\frac{3}{4})^2} = \frac{24}{7}$$

Using the identity for $\tan 2A$ again, but this time with $A = 2\theta$, gives

$$\tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2(\frac{24}{7})}{1-(\frac{24}{7})^2} = -\frac{336}{527}$$

2. Eliminate θ from the equations $x = \cos 2\theta$, $y = \sec\theta$

Using $\cos 2\theta \equiv 2\cos^2\theta - 1$ gives

$$x = 2\cos^2\theta - 1 \quad\text{and}\quad y = \frac{1}{\cos\theta}$$

$$\therefore \qquad x = 2\left(\frac{1}{y}\right)^2 - 1$$

$$\Rightarrow \qquad (x+1)y^2 = 2$$

Note that this is a Cartesian equation which has been obtained by *eliminating the parameter* θ from a *pair of parametric equations.*

3. Prove that $\sin 3A \equiv 3\sin A - 4\sin^3 A$

$$\begin{aligned}\sin 3A &\equiv \sin(2A + A)\\ &\equiv \sin 2A\cos A + \cos 2A\sin A\\ &\equiv (2\sin A\cos A)\cos A + (1-2\sin^2 A)\sin A\\ &\equiv 2\sin A\cos^2 A + \sin A - 2\sin^3 A\\ &\equiv 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A\\ &\equiv 3\sin A - 4\sin^3 A\end{aligned}$$

4. Find the general solution of the equation $\cos 2x + 3\sin x = 2$

When a trig equation involves different multiples of an angle, it is usually sensible to express the equation in a form where the trig ratios are all of the same angle and, when possible, only one trig ratio is included.

Using $\cos 2x \equiv 1 - 2\sin^2 x$ gives

$$1 - 2\sin^2 x + 3\sin x = 2$$

$$\Rightarrow \qquad 2\sin^2 x - 3\sin x + 1 = 0$$

$$\Rightarrow \qquad (2\sin x - 1)(\sin x - 1) = 0$$

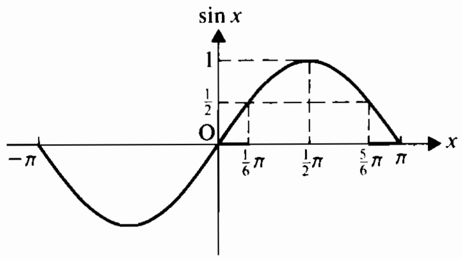
$$\therefore \qquad \sin x = \tfrac{1}{2} \quad\text{or}\quad \sin x = 1$$

When $\sin x = \frac{1}{2}$,

$$x = \begin{cases}\frac{1}{6}\pi + 2n\pi\\ \frac{5}{6}\pi + 2n\pi\end{cases}$$

When $\sin x = 1$,

$$x = \tfrac{1}{2}\pi + 2n\pi$$



The general solution is therefore $x = \frac{1}{6}\pi + 2n\pi,\ \frac{5}{6}\pi + 2n\pi,\ \frac{1}{2}\pi + 2n\pi$

EXERCISE 18c

Simplify, giving an exact value where this is possible.

1. $2\sin 15° \cos 15°$
2. $\cos^2 \frac{1}{8}\pi - \sin^2 \frac{1}{8}\pi$
3. $\sin\theta\cos\theta$
4. $1 - 2\sin^2 4\theta$
5. $\dfrac{2\tan 75°}{1-\tan^2 75°}$
6. $\dfrac{2\tan 3\theta}{1-\tan^2 3\theta}$
7. $\sqrt{(1+\cos 6\theta)}$
8. $2\cos^2 \frac{3}{8}\pi - 1$
9. $\dfrac{1+\tan x}{1-\tan x}$ (Hint. $\tan 45° = 1$)
10. $1 - 2\sin^2 \frac{1}{8}\pi$
11. Find the value of $\cos 2\theta$ and $\sin 2\theta$ when θ is acute and when
    (a) $\cos\theta = \frac{3}{5}$ (b) $\sin\theta = \frac{7}{25}$ (c) $\tan\theta = \frac{12}{5}$
12. If $\tan\theta = -\frac{7}{24}$ and θ is obtuse, find
    (a) $\tan 2\theta$ (b) $\cos 2\theta$ (c) $\sin 2\theta$ (d) $\cos 4\theta$
13. Eliminate θ from the following pairs of equations.
    (a) $x = \tan 2\theta,\ y = \tan\theta$ (b) $x = \cos 2\theta,\ y = \cos\theta$
    (c) $x = \cos 2\theta,\ y = \operatorname{cosec}\theta$ (d) $x = \sin 2\theta,\ y = \sec 4\theta$
14. Prove the following identities.
    (a) $\dfrac{1-\cos 2A}{\sin 2A} \equiv \tan A$
    (b) $\sec 2A + \tan 2A \equiv \dfrac{\cos A + \sin A}{\cos A - \sin A}$
    (c) $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$
    (d) $\cos 4A \equiv 8\cos^4 A - 8\cos^2 A + 1$
    (e) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
    (f) $\dfrac{1-\cos 2A + \sin 2A}{1+\cos 2A + \sin 2A} \equiv \tan A$
15. Find the general solutions of the following equations.
    Give answers in radians if they are exact: otherwise give them in degrees to 1 d.p.
    (a) $\cos 2x = \sin x$ (b) $\sin 2x + \cos x = 0$
    (c) $4 - 5\cos\theta = 2\sin^2\theta$ (d) $\tan 2\theta \tan\theta = 2$
    (e) $\sin 2\theta - 1 = \cos 2\theta$ (f) $5\cos x \sin 2x + 4\sin^2 x = 4$

THE HALF-ANGLE IDENTITIES

If we replace A by $\frac{1}{2}\theta$ in the double angle formula for $\tan 2A$, we get

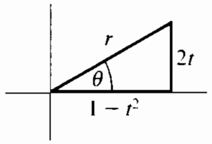
$$\tan\theta \equiv \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta} \qquad [1]$$

If we now make the substitution $t = \tan\frac{1}{2}\theta$, [1] becomes

$$\tan\theta \equiv \frac{2t}{1-t^2}$$

From the diagram, and using Pythagoras' theorem,

$$r = 1 + t^2$$



Hence we can express $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t, i.e.

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2} \qquad \text{where } t = \tan\tfrac{1}{2}\theta$$

$$\tan\theta = \frac{2t}{1-t^2}$$

These identities allow all the trig ratios of any one angle to be expressed in terms of a common variable t. This group can be helpful in problems where it is not possible to apply any of the identities used previously.

There is one other group of identities that can be useful, and these are derived from the different forms of the cosine double angle formulae.

Starting with $\cos 2A \equiv 2\cos^2 A - 1$ we have

$$\cos^2 A \equiv \tfrac{1}{2}(1+\cos 2A)$$

Similarly starting with $\cos 2A \equiv 1 - 2\sin^2 A$, we get

$$\sin^2 A \equiv \tfrac{1}{2}(1-\cos 2A)$$

Examples 18d

1. Express $\sin\theta + 2\cos\theta$ in terms of t, where $t = \tan\frac{1}{2}\theta$. Hence solve the equation $\sin\theta + 2\cos\theta = 1$ for values of θ in the range $0 \leqslant \theta \leqslant 360°$

$$\sin\theta + 2\cos\theta \equiv \frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right) \equiv \frac{2+2t-2t^2}{1+t^2}$$

Hence $\quad \sin\theta + 2\cos\theta = 1 \quad \Rightarrow \quad \frac{2+2t-2t^2}{1+t^2} = 1$

$$\Rightarrow \quad 3t^2 - 2t - 1 = 0$$

$$\Rightarrow \quad (3t+1)(t-1) = 0$$

$\therefore \quad t = -\frac{1}{3}$ or $t = 1$

i.e. $\tan\frac{1}{2}\theta = -\frac{1}{3}$ or 1

The general solution is $\frac{1}{2}\theta = -18.43° + 180n°$ or $\frac{1}{2}\theta = 45° + 180n°$

$$\theta = -36.9° + 360n° \quad \text{or} \quad \theta = 90° + 360n°$$

$\therefore$ in the specified range, $\theta = 90°, 323.1°$

The next worked example deals with a very similar equation and it illustrates that extra care is needed when using this method to solve equations.

---

2. Find the general solution of the equation $\sin\theta - \cos\theta = 1$

Using $t = \tan\frac{1}{2}\theta$, $\sin\theta - \cos\theta = 1$

$$\Rightarrow \quad \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$$

$$\Rightarrow \quad 2t - 1 + t^2 = 1 + t^2$$

It was seen on page 55 that if we can cancel the two t^2 terms, the solution $t = \infty$ is lost. In this problem we cannot ignore this solution as $t = \infty$ corresponds to a real value of θ. So we proceed as follows.

$\Rightarrow \quad t = \infty$ or $t = 1$

$\therefore \quad \tan\frac{1}{2}\theta = \infty$ or $\tan\frac{1}{2}\theta = 1$

$\Rightarrow \quad \frac{1}{2}\theta = \frac{1}{2}\pi + n\pi$ or $\frac{1}{2}\theta = \frac{1}{4}\pi + n\pi$

$$\Rightarrow \quad \theta = \begin{cases} \frac{1}{2}\pi + 2n\pi \\ (2n+1)\pi \end{cases}$$

---

3. If $\tan\theta = \frac{3}{4}$, find the possible values of
(a) $\tan\frac{1}{2}\theta$ (b) $\cos 2\theta$

(a) Using $\tan\frac{1}{2}\theta = t$ gives $\frac{3}{4} = \frac{2t}{1-t^2}$

$$\Rightarrow \quad 3t^2 + 8t - 3 = 0$$

$$\Rightarrow \quad (3t-1)(t+3) = 0$$

$$\Rightarrow \quad t = \tfrac{1}{3} \text{ or } -3$$

i.e. $\tan\frac{1}{2}\theta = \frac{1}{3}$ or $\tan\frac{1}{2}\theta = -3$

(b) The half-angle formulae are valid for any angle where t is the tan of half the given angle. In this case the given angle is 2θ so $t = \tan\theta$

Using $\quad \cos 2\theta = \frac{1-t^2}{1+t^2}$ with $t = \tan\theta$

$$\Rightarrow \quad \cos 2\theta = \frac{1-(\frac{3}{4})^2}{1+(\frac{3}{4})^2} = \frac{7}{25}$$

---

4. Express $\sqrt{\left(\frac{1-\sin 2\theta}{1+\sin 2\theta}\right)}$ in terms of $\tan\theta$

If we double the angle in the formula for $\sin\theta$ in terms of t, we get

$\sin 2\theta = \frac{2t}{1+t^2}$ where $t = \tan\theta$

$$\therefore \quad 1 - \sin 2\theta = 1 - \frac{2t}{1+t^2} = \frac{1+t^2-2t}{1+t^2} = \frac{(1-t)^2}{1+t^2}$$

and

$$1 + \sin 2\theta = 1 + \frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2} = \frac{(1+t)^2}{1+t^2}$$

Hence $$\frac{1-\sin 2\theta}{1+\sin 2\theta} = \frac{(1-t)^2}{1+t^2}\Big/\frac{(1+t)^2}{1+t^2} = \left(\frac{1-t}{1+t}\right)^2$$

$$\therefore \quad \sqrt{\left(\frac{1-\sin 2\theta}{1+\sin 2\theta}\right)} = \frac{1-t}{1+t} = \frac{1-\tan\theta}{1+\tan\theta}$$

EXERCISE 18d

1. If $\tan\theta = \frac{4}{3}$, find the possible values of
   (a) $\sin 2\theta$ (b) $\cot 2\theta$ (c) $\tan\frac{1}{2}\theta$ (d) $\cos\frac{1}{2}\theta$

Expressing the following in terms of t where $t = \tan\frac{1}{2}\theta$

2. $\dfrac{1-\cos\theta}{1+\cos\theta}$

3. $\dfrac{\sin\theta}{1-\cos\theta}$

4. $\cot\theta\cot\frac{1}{2}\theta$

5. $\dfrac{\cos^2(\frac{1}{2}\theta)}{3\sin\theta+4\cos\theta-1}$

6. Prove that $\operatorname{cosec} A + \cot A \equiv \cot\frac{1}{2}A$

7. If $\sec\theta - \tan\theta = x$ prove that $\tan\frac{1}{2}\theta = \dfrac{1-x}{1+x}$

Using $t = \tan\frac{1}{2}\theta$, solve the following equations for values of θ in the interval $[-180°, 180°]$.

8. $3\cos\theta + 2\sin\theta = 3$

9. $5\cos\theta - \sin\theta + 4 = 0$

10. $\cos\theta + 7\sin\theta = 5$

11. $2\cos\theta - \sin\theta = 1$

MIXED EXERCISE 18

1. Eliminate θ from the equations $x = \sin\theta$ and $y = \cos 2\theta$

2. Prove the identity $\dfrac{\sin 2\theta}{1+\cos 2\theta} \equiv \tan\theta$

3. Prove that $\tan(\theta+\frac{1}{4}\pi)\tan(\frac{1}{4}\pi-\theta) \equiv -1$

4. If $\cos A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$ find the possible values of $\cos(A+B)$

5. Eliminate θ from the equations $x = \cos 2\theta$ and $y = \cos^2\theta$

6. Solve the equation $8\sin\theta\cos\theta = 3$ for values of θ from $-180°$ to $180°$

7. Find the general solution of the equation $\cos^2\theta - \sin^2\theta = 1$

8. Prove the identity $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$

9. Simplify the expression $\dfrac{1+\cos 2x}{1-\cos 2x}$

10. Find the values of A between 0 and 360° for which $\sin(60° - A) + \sin(120° - A) = 0$

CHAPTER 19

DIFFERENTIATION OF COMPOUND FUNCTIONS

DIFFERENTIATING A FUNCTION OF A FUNCTION

Suppose that we want to differentiate $(2x-1)^3$. We could expand the bracket and differentiate term by term, but this is tedious and, for powers higher than three, very long and not easy. We obviously need a more direct method for differentiating an expression of this kind.

Now $(2x-1)^3$ is a cubic function of the linear function $(2x-1)$, i.e. it is a *function of a function*.

A function of this type is of the form $\text{gf}(x)$, i.e. $\text{g}\{\text{f}(x)\}$

For example

$(x^2-3)^3$ is a cubic function, g, of a quadratic function, f
$\sqrt{(1+x^4)}$ is a square root function, g, of a quartic function, f

Consider any equation of the form $y = \text{gf}(x)$
If we make the substitution $u = \text{f}(x)$ then $y = \text{gf}(x)$ can be expressed in two simple parts, i.e.

$$u = \text{f}(x) \quad \text{and} \quad y = \text{g}(u)$$

A small increase of δx in the value of x causes a corresponding small increase of δu in the value of u.
Then if $\delta x \to 0$, it follows that $\delta u \to 0$

Hence $$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\right)\left(\frac{\delta u}{\delta x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\lim_{\delta u \to 0}\frac{\delta y}{\delta u}\right) \times \left(\lim_{\delta x \to 0}\frac{\delta u}{\delta x}\right)$$

i.e. $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is known as *the chain rule*.

Examples 19a

1. Find $\dfrac{dy}{dx}$ if $y = (2x-4)^4$

If $u = 2x-4$ then $y = u^4$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ gives

$$\frac{dy}{dx} = (4u^3)(2) = 8u^3$$

But $u = 2x-4$

$$\therefore \quad \frac{dy}{dx} = 8(2x-4)^3$$

Example 1 is a particular case of the equation $y = (ax+b)^n$. Similar working shows that, in general,

$$\text{if } y = (ax+b)^n \text{ then } \frac{dy}{dx} = an(ax+b)^{n-1}$$

This fact is needed very often and is quotable.

---

2. Given $y = (x^3+1)^4$ find $\dfrac{dy}{dx}$

If $u = x^3+1$ then $y = u^4$

Using $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ gives

$$\frac{dy}{dx} = (4u^3)(3x^2) = 12x^2u^3$$

Replacing u by x^3+1 we have

$$\frac{dy}{dx} = 12x^2(x^3+1)^3$$

---

3. Differentiate w.r.t. x the function $\dfrac{1}{(1-x^2)^5}$

$$y = (1-x^2)^{-5} \quad \Rightarrow \quad y = u^{-5} \quad \text{where} \quad u = 1-x^2$$

Now
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\therefore \quad \frac{d}{dx}\left[\frac{1}{(1-x^2)^5}\right] = (-5u^{-6})(-2x) = 10x(u^{-6})$$
$$= \frac{10x}{(1-x^2)^6}$$

After some time the reader will find that in most cases the necessary substitution can be done mentally and the answer written down directly, e.g. to differentiate $(x^3-x)^{3/2}$ we mentally use the substitutions $u = x^3 - x$ and $y = u^{3/2}$ giving

$$\frac{d}{dx}(x^3-x)^{3/2} = [\tfrac{3}{2}(x^3-x)^{1/2}](3x^2-1)$$

This skill is important and well worth the practice required for its achievement.

EXERCISE 19a

Use a substitute to differentiate each function with respect to x

1. $(3x+1)^2$
2. $(3-x)^4$
3. $(4x-5)^5$
4. $(x^2+1)^3$
5. $(2+3x)^7$
6. $(2-6x)^3$
7. $(2x^4-5)^{1/2}$
8. $(x^2+3)^{-1}$
9. $\sqrt{(3x^3-4)}$
10. $\dfrac{1}{(\sqrt{x}+3x)}$
11. $\dfrac{3}{\sqrt{(4-x^2)}}$
12. $\dfrac{7}{(x^3+3x)^{1/3}}$

Differentiate each function directly.

13. $(4-2x)^5$
14. $(x^2+3)^2$
15. $(3x-4)^7$
16. $(x^2+4)^2$
17. $(1-2x^2)$
18. $(2-x^3)^4$
19. $(2+x^2)^{3/4}$
20. $\sqrt[3]{(x^2-x)}$
21. $(2-3x^2)^{-1}$
22. $(4-x^2)^{-2}$
23. $(x^5-3)^{-1/2}$
24. $\sqrt[4]{(6-\sqrt{x})}$

DIFFERENTIATING A PRODUCT

Suppose that $y = uv$ where u and v are both functions of x, e.g. $y = x^2(x^4-1)$

It is dangerously tempting to think that $\dfrac{dy}{dx}$ is given by $\left(\dfrac{du}{dx}\right)\left(\dfrac{dv}{dx}\right)$

But this is *not so* as is clearly shown by a simple example such as $y = (x^2)(x^3)$ where, because $y = x^5$, we know that $\dfrac{dy}{dx} = 5x^4$ which is *not* equal to $(2x)(3x^2)$.

i.e. **differentiation is *not* distributive across a product.**

Returning to $y = uv$ where $u = f(x)$ and $v = g(x)$, we see that if x increases by a small amount δx then there are corresponding small increases of δu, δv and δy in the values of u, v and y

$$\therefore \quad y + \delta y = (u+\delta u)(v+\delta v)$$
$$= uv + u\delta v + v\delta u + \delta u\delta v$$

But $y = uv$

$$\therefore \quad \delta y = u\delta v + v\delta u + \delta u\delta v$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$$

Now as $\delta x \to 0$, $\dfrac{\delta v}{\delta x} \to \dfrac{dv}{dx}$, $\dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}$ and $\delta u \to 0$

Therefore
$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = u\frac{dv}{dx} + v\frac{du}{dx} + 0$$

i.e.
$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

This formula is verified by the simple example we considered above, i.e. $y = (x^2)(x^3)$.

Using $u = x^2$ and $v = x^3$ gives $\dfrac{dy}{dx} = (x^3)(2x) + (x^2)(3x^2) = 5x^4$ which is correct.

Example 19b

Differentiate with respect to x

(a) $(x+1)^3(2x-5)^2$ **(b)** $\dfrac{(x-1)^2}{(x+2)}$

(a) If $u = (x+1)^3$, $\dfrac{du}{dx} = 3(x+1)^2$

and if $v = (2x-5)^2$, $\dfrac{dv}{dx} = 2(2x-5)$

$\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ gives

$$\frac{d}{dx}(x+1)^3(2x-5)^2 = \{(2x-5)^2\}\{3(x+1)^2\} + \{(x+1)^3\}\{2(2)(2x-5)\}$$

$$= (2x-5)(x+1)^2\{3(2x-5)+4(x+1)\}$$

$$= (2x-5)(x+1)^2(10x-11)$$

(b) If we write $\dfrac{(x-1)^2}{(x+2)}$ as $(x-1)^2(x+2)^{-1}$

then $u = (x-1)^2$ gives $\dfrac{du}{dx} = 2(x-1)$

and $v = (x+2)^{-1}$ gives $\dfrac{dv}{dx} = -(x+2)^{-2}$

Using $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ we have

$$\frac{d}{dx}\left[\frac{(x-1)^2}{(x+2)}\right] = (x+2)^{-1}\{2(x-1)\} + (x-1)^2\{-(x+2)^{-2}\}$$

$$= \frac{(x-1)}{(x+2)^2}\{2(x+2)-(x-1)\}$$

$$= \frac{(x-1)(x+5)}{(x+2)^2}$$

EXERCISE 19b

Differentiate each function with respect to x

1. $x^2(x-3)^2$ **2.** $x\sqrt{(x-6)}$ **3.** $(x+2)(x-2)^5$

4. $x(2x+3)^3$ **5.** $(x+1)^2(x-1)^4$ **6.** $\sqrt{x}(x-3)^3$

7. $\dfrac{(x+5)^4}{(x-3)}$ **8.** $\dfrac{x}{(3x+2)^2}$ **9.** $\dfrac{(2x-7)^2}{\sqrt{x}}$

10. $x^3\sqrt{(x-1)}$ **11.** $x(x+3)^{-1}$ **12.** $x^2(2x-3)^2$

DIFFERENTIATING A QUOTIENT

To differentiate a function of the form u/v, where u and v are both functions of x, it is sometimes convenient to rewrite the function as uv^{-1} and differentiate it as a product. This method was used in part (b) of the previous worked example but it is not always the neatest way to differentiate a quotient. The alternative is to apply the formula derived below.

When a function is of the form u/v, where u and v are both functions of x, a small increase of δx in the value of x causes corresponding small increases of δu and δv in the values of u and v. Then, as $\delta x \to 0$, δu and δv also tend to zero.

If $y = \dfrac{u}{v}$ then $y + \delta y = \dfrac{(u+\delta u)}{(v+\delta v)}$

$$\therefore \quad \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v(v+\delta v)}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \left(v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}\right)\Big/ v(v+\delta v)$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Big/ v^2$$

i.e.
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 19c

If $y = \dfrac{(4x-3)^6}{(x+2)}$ find $\dfrac{dy}{dx}$

Using $u = (4x-3)^6$ gives $\dfrac{du}{dx} = 24(4x-3)^5$

and $v = x + 2$ gives $\dfrac{dv}{dx} = 1$

Then
$$\frac{dy}{dx} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\bigg/v^2$$
$$= \frac{(x+2)\{24(4x-3)^5\} - (4x-3)^6}{(x+2)^2}$$
$$= \frac{(4x-3)^5(20x+51)}{(x+2)^2}$$

EXERCISE 19c

Use the quotient formula to differentiate each of the following functions with respect to x

1. $\dfrac{(x-3)^2}{x}$ 2. $\dfrac{x^2}{(x+3)}$ 3. $\dfrac{(4-x)}{x^2}$

4. $\dfrac{(x+1)^2}{x^3}$ 5. $\dfrac{4x}{(1-x)^3}$ 6. $\dfrac{2x^2}{(x-2)}$

7. $\dfrac{x^{5/3}}{(3x-2)}$ 8. $\dfrac{(1-2x)^3}{x^3}$ 9. $\dfrac{\sqrt{(x+1)^5}}{x}$

IDENTIFYING THE CATEGORY OF A FUNCTION

Before any of the techniques explained earlier can be used to differentiate a given function, it is important to recognise the category to which the function belongs, i.e. is it a product or a function of a function or, if it is a fraction, is it one which would better be expressed as a product.

A product comprises two parts, each of which is an *independent* function of x, whereas *if one operation is carried out on another function of* x we have a function of a function.

MIXED EXERCISE 19

This exercise contains a mixture of compound functions. In each case first identify the type of function and then use the appropriate method to find its derivative.

1. $x\sqrt{(x+1)}$ 2. $(x^2-8)^3$ 3. $x/(x^2+1)$

4. $\sqrt[3]{(2-x^4)}$ 5. $(x^2+1)/(x^2+2)$ 6. $x^2(\sqrt{x}-2)$

7. $(x^2-2)^3$ 8. $\sqrt{(x-x^2)}$ 9. $x/(\sqrt{x}+1)$

10. $x^2\sqrt{(x-2)}$ 11. $\sqrt{(x+1)}/x^2$ 12. $(x^4+x^2)^3$

13. $\sqrt{(x^2-8)}$ 14. $x^3(x^2-6)$ 15. $(x^2-6)^3$

16. $x/(x^2-6)$ 17. $(x^4+3)^{-2}$ 18. $\sqrt{x}(2-x)^3$

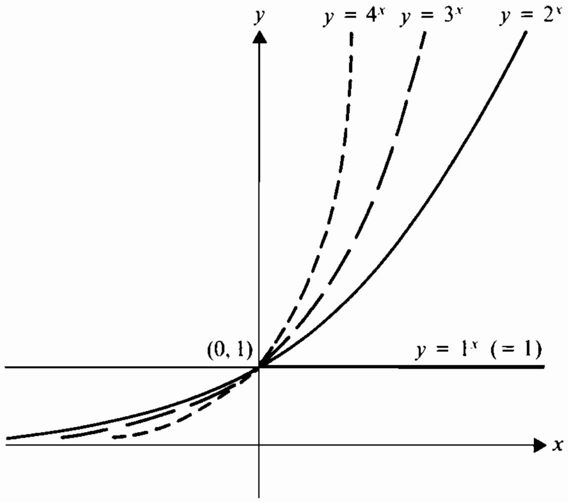
19. $\sqrt{x}/(2-x)^3$ 20. $(x-1)(x-2)^2$ 21. $(2x^3+4)^5$

CHAPTER 20

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

THE EXPONENTIAL FUNCTION

The general shape of an exponential curve was seen in Chapter 12. The next diagram shows a few more members of the exponential family.



Note that these curves and, in fact, *all* exponential curves, pass through the point $(0, 1)$.
This is because, for any positive base a,

$$\text{when } x \text{ is } 0, \quad y = a^x = a^0 = 1$$

Each exponential curve has a unique property which the reader can discover experimentally by using an accurate plot of the curve $y = 2^x$. Choose three or four points on the curve and, at each one, draw the tangent as accurately as possible and determine its gradient. Then complete the following table.

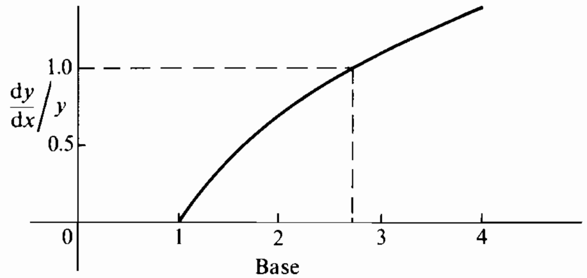
Point	Gradient of tangent, i.e. $\frac{dy}{dx}$	y-coordinate	$\frac{dy}{dx} \div y$
1			
2			
3			
4			

An accurate drawing should result in numbers in the last column that are all reasonably close to 0.7

When this experiment is carried out for 3^x and 4^x we find again that $\frac{dy}{dx} \div y$ has a constant value; for 3^x the constant is about 1.1 and for 4^x it is about 1.4 So we have

Base	2	3	4
$\frac{dy}{dx} \div y$	0.7	1.1	1.4

$\Rightarrow$



From this graph it can be seen that there is a base, somewhere between 2 and 3, for which $\frac{dy}{dx} \div y = 1$, i.e. $\frac{dy}{dx} = y$

Calling this base e we have

$$\text{if } y = e^x \text{ then } \frac{dy}{dx} = e^x$$

The function e^x is the only function which is unchanged when differentiated.

In the early eighteenth century, a number of mathematicians, working along different lines of investigation, all discovered the number e at about the same time.

The number e is irrational, i.e. like π, $\sqrt{2}$, etc., it cannot be given an exact decimal value but, to 4 significant figures, $e = 2.718$ The value of various powers of e, such as e^2, e^3, e^4, can be obtained from a calculator.

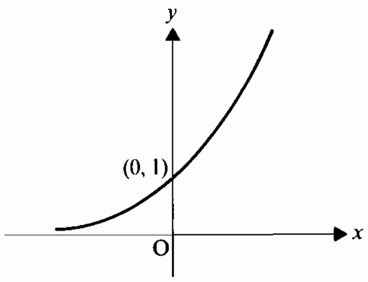
Summing up:

for any value of a $(a > 0)$, a^x is *an* exponential function

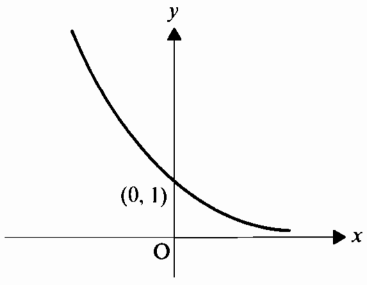
for the base e $(e \approx 2.718)$, e^x is *the* exponential function

$$\frac{d}{dx}(e^x) = e^x$$

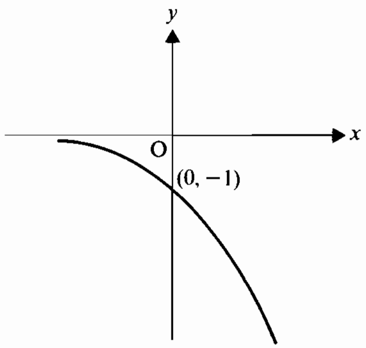
The following diagrams show sketches of $y = e^x$ and of some simple variations.



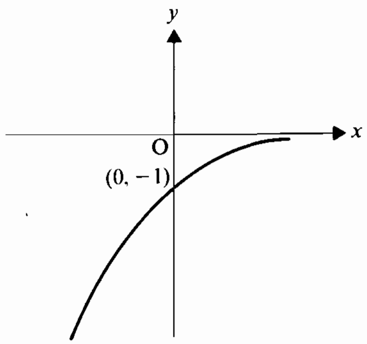
$y = e^x$



$y = e^{-x}$



$y = -e^x$



$y = -e^{-x}$

Example 20a

Find the coordinates of the stationary point on the curve $y = e^x - x$, and determine its type. Sketch the curve showing the stationary point clearly.

$$y = e^x - x \quad \Rightarrow \quad \frac{dy}{dx} = e^x - 1$$

At a stationary point, $\frac{dy}{dx} = 0$ therefore $e^x - 1 = 0$

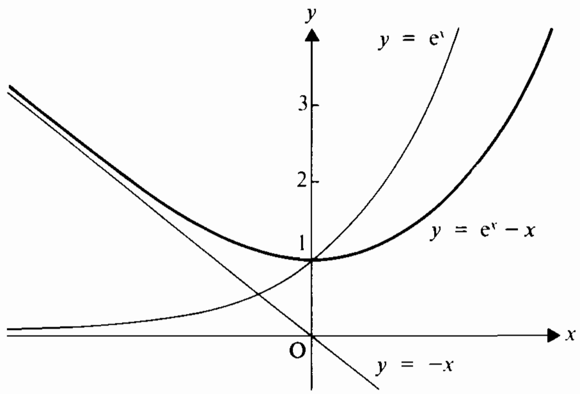
i.e. $\quad e^x = 1 \quad \Rightarrow \quad x = 0$

When $x = 0$, $y = e^0 - 0 = 1$

Therefore $(0, 1)$ is a stationary point.

$$\frac{d^2y}{dx^2} = e^x \text{ and this is positive when } x = 0$$

Therefore $(0, 1)$ is a minimum point.



This curve is made up from separate sketches of $y = e^x$ and $y = -x$ by adding their ordinates.

EXERCISE 20a

1. Evaluate, correct to 3 s.f.
   (a) e^2 (b) e^{-1} (c) $e^{1.5}$ (d) $e^{-0.3}$

2. Write down the derivative of
   (a) $2e^x$ (b) $x^2 - e^x$ (c) e^x

In questions 3 to 5 find the gradient of each curve at the specified value of x

3. $y = e^x - 2x$ where $x = 2$
4. $y = x^2 + 2e^x$ where $x = 1$
5. $y = e^x - 3x^3$ where $x = 0$
6. Find the value of x at which the function $e^x - x$ has a stationary value.
7. Sketch each given curve.
   (a) $y = 1 - e^x$ (b) $y = e^x + 1$ (c) $y = x - e^x$
   (d) $y = 1 - e^{-x}$ (e) $y = 1 + e^{-x}$ (f) $y = x^2 + e^x$

NATURAL (NAPERIAN) LOGARITHMS

Suppose that the equation $e^x = 0.59$ has to be solved.

We know that this equation can be written in logarithmic form

i.e. $$x = \log_e 0.59$$

Logarithms to the base e are called *natural* or *Naperian* logarithms. To avoid having to insert the base e in every natural logarithm, the notation ln is used, i.e.

$\log_e a$ is written $\ln a$

i.e. $$\ln a = b \iff a = e^b$$

Values of natural logs can be found by using a scientific calculator.

Returning to the equation $e^x = 0.59$ we now have

$$e^x = 0.59 \Rightarrow x = \ln 0.59 = -0.528 \text{ to 3 s.f.}$$

Logarithms to the base 10 are called *common logarithms* and are denoted by log or lg. They used to be an important tool for calculations but calculators have eliminated their usefulness in that respect.

The laws used for working with logarithms to a general base, given in Chapter 3, apply equally well to natural logarithms, i.e.

$$\ln a + \ln b = \ln ab$$
$$\ln a - \ln b = \ln a/b$$
$$\ln a^n = n \ln a$$

One further rule can be added to this list. It is needed when we want to change the base of a logarithm.

Changing the Base of a Logarithm

Suppose that $x = \log_a c$ and that we wish to express x as a logarithm to the base b

$$\log_a c = x \Rightarrow c = a^x$$

Now taking logs to the base b gives

$$\log_b c = x \log_b a \Rightarrow x = \frac{\log_b c}{\log_b a}$$

i.e. $$\log_a c = \frac{\log_b c}{\log_b a}$$

In the special case when $c = b$, i.e. when $\log_b c = 1$, this relationship becomes

$$\log_a b = \frac{1}{\log_b a}$$

When the change of base rule is used to convert a logarithm into a natural logarithm, i.e. to change the base of a log to e, we have

$$\log_a c = \frac{\ln c}{\ln a}$$

The base of an exponential function can be changed in a similar way. Suppose that we wish to express 3^x as a power of e.

Using $\quad 3^x = e^p \quad$ gives $\quad x\ln 3 = p$

$\therefore \quad 3^x = e^{x\ln 3}$

In general $\quad a^x = e^{x\ln a}$

Examples 20b

1. Separate $\ln(\tan x)$ into two terms.

$$\ln(\tan x) = \ln\left(\frac{\sin x}{\cos x}\right) = \ln\sin x - \ln\cos x$$

2. Express $4\ln(x+1) - \frac{1}{2}\ln x$ as a single logarithm.

$$4\ln(x+1) - \tfrac{1}{2}\ln x = \ln(x+1)^4 - \ln\sqrt{x} = \ln\left(\frac{(x+1)^4}{\sqrt{x}}\right)$$

EXERCISE 20b

1. Evaluate
(a) $\ln 3.451$ (b) $\ln 1.201$ (c) $\ln 17.3$

2. Express as a sum or difference of logarithms or as a product
(a) $\ln\dfrac{x}{x-1}$ (b) $\ln(5x^2)$ (c) $\ln(x^2-4)$
(d) $\ln\tan x$ (e) $\ln(\sin^2 x)$ (f) $\ln\sqrt{\left(\dfrac{x+1}{x-1}\right)}$

3. Express as a single logarithm
(a) $\ln x - 2\ln(1-x)$ (b) $1 - \ln x$
(c) $\ln\sin x + \ln\cos x$ (d) $2\ln x + \frac{1}{2}\ln(x-1)$

4. Given that $\ln a = 3$
(a) express $\log_a x^2$ as a simple natural logarithm
(b) express as a single logarithm $\ln x^3 + 6\log_a x$

5. Solve the following equations for x
(a) $e^x = 8.2$ (b) $e^{2x} + e^x - 2 = 0$ (Hint. Use $e^{2x} = (e^x)^2$)
(c) $e^{2x-1} = 3$ (d) $e^{4x} - e^x = 0$

6. Given that $\ln a = 2$ solve the following equations for x
(a) $a^x = e^2$ (b) $a^x = e^6$ (c) $a^x = 1$

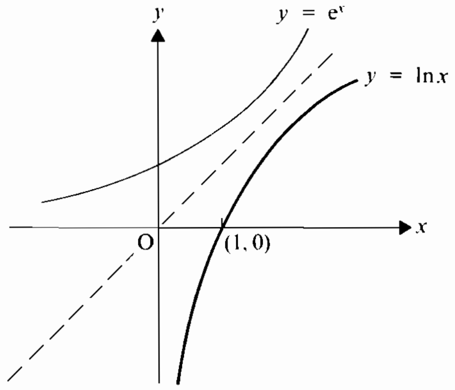
THE LOGARITHMIC FUNCTION

Consider the curve with equation $y = f(x)$ where $f(x) = \ln x$

If $y = \ln x$ then $x = e^y$,

i.e. **the logarithmic function is the inverse of the exponential function.**

It follows that the curve $y = \ln x$ is the reflection of the curve $y = e^x$ in the line $y = x$



There is no part of the curve $y = \ln x$ in the second and third quadrants. This is because, if $x = e^y$ (i.e. if $y = \ln x$), x is positive for all real values of y. Therefore

$\ln x$ does not exist for negative values of x

THE DERIVATIVE OF ln x

We know that $y = \ln x \iff x = e^y$ and we also know how to differentiate the exponential function. So a relationship between $\frac{d}{dx}(y)$ and $\frac{d}{dy}(x)$ would help in finding the derivative of $\ln x$

Consider the equation $y = f(x)$ where $f(x)$ is any function of x

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(1 \Big/ \frac{\delta x}{\delta y}\right)$$

Now $\delta y \to 0$ as $\delta x \to 0$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\delta y \to 0} \left(1 \Big/ \frac{\delta x}{\delta y}\right)$$

i.e.

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$$

This relationship can be used to find the derivative of *any* function if the derivative of its inverse is known. We will now apply it to differentiate $\ln x$.

$$y = \ln x \iff x = e^y$$

Differentiating e^y w.r.t. y gives

$$\frac{dx}{dy} = e^y = x$$

Therefore $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{x}$

i.e.

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

This result can be used to differentiate many log functions if they are first simplified by applying the laws given on page 337.

Examples 20c

1. Find the derivative of (a) $\ln(1/x^3)$ (b) $\ln(4\sqrt{x})$

(a) $f(x) = \ln(1/x^3) = \ln(x^{-3}) = -3\ln x$

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}\{-3\ln x\} = \frac{-3}{x}$$

(b) $f(x) = \ln(4\sqrt{x}) = \ln 4 + \ln(\sqrt{x}) = \ln 4 + \frac{1}{2}\ln x$

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(\ln 4) + \frac{d}{dx}(\tfrac{1}{2}\ln x)$$

$$= 0 + \frac{\frac{1}{2}}{x} = \frac{1}{2x}$$

2. Find $\frac{dy}{dx}$ if $y = \log_a x^2$

We only know how to differentiate natural logs, so first we must change the base from a to e.

$$y = \log_a x^2 = \frac{\ln x^2}{\ln a} = \frac{2\ln x}{\ln a}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{2}{x \ln a}$$

EXERCISE 20c

1. Write down the derivative of each of the following functions.
(a) $\ln x^3$ (b) $\ln(3x)$ (c) $\ln(x^{-2})$ (d) $\ln(3/\sqrt{x})$
(e) $\ln(1/x^5)$ (f) $\ln(2x^{1/2})$ (g) $\ln(x^{-3/2})$ (h) $\ln(x^3/\sqrt{x})$

2. Locate the stationary points on each curve.
(a) $y = \ln x - x$ (b) $y = x^3 - 2\ln x^3$ (c) $y = \ln x - \sqrt{x}$

3. Sketch each of the following curves.
(a) $y = -\ln x$ (b) $y = \ln(-x)$ (c) $y = 2 + \ln x$
(d) $y = \ln x^2$

FURTHER DIFFERENTIATION OF EXPONENTIAL AND LOG FUNCTIONS

Methods were introduced in Chapter 19 for differentiating a product, a quotient and a function of a function, i.e.

if $y = uv$ then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)\Big/v^2$

if $y = g(u)$ and $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

(Remember that the substitution can often be done mentally.)

These techniques can now be applied when exponential and log functions are involved.

The function of a function result is particularly useful as many derivatives of this type can be written down directly and easily.

To Differentiate e^u where $u = f(x)$

If $y = e^u$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ gives

$$\frac{dy}{dx} = e^u \times \frac{du}{dx}$$

This can also be expressed in the form

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

i.e. $$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

e.g. if $y = e^{(x^2+1)}$ then $\dfrac{dy}{dx} = 2xe^{(x^2+1)}$

The case when u is a linear function of x is particularly useful,

i.e. $y = e^{(ax+b)} \Rightarrow \dfrac{dy}{dx} = ae^{(ax+b)}$

To Differentiate $\ln u$ where $u = f(x)$

If $y = \ln u$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ gives

$$\frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx}$$

This can also be expressed in the form

$$\frac{d}{dx}\{\ln f(x)\} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

e.g. if $y = \ln(2+x^3)$ then $\dfrac{dy}{dx} = \dfrac{3x^2}{2+x^3}$

Again the case when u is $ax+b$ occurs frequently and is worth noting,

i.e. $y = \ln(ax+b) \Rightarrow \dfrac{dy}{dx} = \dfrac{a}{ax+b}$

Compound Exponential and Logarithmic Functions

For any given function the first step is to identify its category, e.g. x^2e^x and $(1+x)\ln x$ are both products.
Whereas e^{x^2} and $\ln(1-x^2)$ are both functions of a function.

EXERCISE 20d

In Questions 1 to 9

(a) identify the type of function

(b) express the function in terms of u and/or v, stating clearly the substitutions that have been made.

1. $e^x(x^2+1)$	**2.** $e^{(x^2+1)}$	**3.** $x\ln x$
4. $\sqrt{\{e^{(x+1)}\}}$	**5.** $e^x\ln x$	**6.** $\ln(3-x^2)$
7. $(\ln x)^2$	**8.** e^{-2x}	**9.** $1/\ln x$

10. If f and g are the functions defined by $f: x \to x^2$ and $g: x \to e^x$ write down the functions $fg(x)$ and $gf(x)$

11. The functions f, g and h are defined as follows

$$f: x \to x^2 \qquad g: x \to 1/x \qquad h: x \to \ln x$$

Write down the functions

(a) $fg(x)$ (b) $hf(x)$ (c) $hg(x)$

(d) $fh(x)$ (e) $hfg(x)$ (f) $fg^{-1}(x)$

Examples 20e

1. Find the derivative of x^3e^x

$y = x^3e^x$ becomes $y = uv$ if $u = x^3$ and $v = e^x$

$$\Rightarrow \qquad \frac{du}{dx} = 3x^2 \text{ and } \frac{dv}{dx} = e^x$$

$$\therefore \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = e^x(3x^2) + x^3(e^x)$$

$$\text{i.e.} \qquad \frac{dy}{dx} = x^2(3+x)e^x$$

2. Differentiate $\ln(x\sqrt{\{x^2-4\}})$, w.r.t. x

First we simplify the log expression by changing it into a sum.

$$\ln(x\sqrt{\{x^2-4\}}) = \ln x + \ln\sqrt{(x^2-4)} = \ln x + \tfrac{1}{2}\ln(x^2-4)$$

$$\therefore \qquad \frac{d}{dx}\ln\{x\sqrt{(x^2-4)}\} = \frac{d}{dx}\ln x + \frac{d}{dx}\{\tfrac{1}{2}\ln(x^2-4)\}$$

$$= \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2-4}\right)$$

$$= \frac{1}{x} + \frac{x}{x^2-4}$$

Simplifying the given function at the start, made the differentiation in this problem much easier. *Before differentiating any function, all possible simplification should be done*, particularly when complicated log expressions are involved.

EXERCISE 20e

Differentiate the following functions with respect to x

1. xe^x 2. $x^2\ln x$ 3. $e^x(x^3-2)$

4. $x^2\ln(x-2)^6$ 5. $(x-1)e^x$ 6. $(x^2+4)\ln\sqrt{x}$

7. $x\sqrt{(2+x)}$ 8. $x\ln\sqrt{(x-5)}$ 9. $(x^2-2)e^x$

10. $\dfrac{x}{e^x}$ 11. $\dfrac{e^x}{x^2}$ 12. $\dfrac{(\ln x)}{x^3}$

13. $\dfrac{\sqrt{(x+1)}}{\ln x}$ 14. $\dfrac{e^x}{x^2-1}$ 15. $\dfrac{e^x}{e^x-e^{-x}}$

16. e^{4x} 17. $\ln(x^2-1)$ 18. e^{x^2}

19. $6e^{(1-x)}$ 20. $e^{(x^2+1)}$ 21. $\ln\sqrt{(x+2)}$

22. $(\ln x)^2$ 23. $1/(\ln x)$ 24. $\sqrt{(e^x)}$

MIXED EXERCISE 20

In this exercise a variety of functions are to be differentiated. In each case identify the type of function and then use the appropriate method to find the derivative. Some of the given functions can be differentiated by using one of the basic rules so do not assume that special techniques are always needed.

In Questions 1 to 18 differentiate the given function with respect to x

1. $x\ln x$ 2. $(4x-1)^{2/3}$ 3. $\dfrac{e^x}{x-1}$

4. $\dfrac{\sqrt{(1+x^3)}}{x^2}$ 5. $\dfrac{\ln x}{\ln(x-1)}$ 6. 10^{3x}

7. $\dfrac{(1+2x^2)}{1+x^2}$ 8. $e^{-2/x}$ 9. $\ln(1-e^x)$

10. $e^{3x}x^3$ 11. $\dfrac{2x}{(2x-1)(x-3)}$ 12. $\dfrac{e^{x/2}}{x^5}$

13. $\ln\left[\dfrac{x^2}{(x+3)(x^2-1)}\right]$ **14.** $\ln 4x^3(x+3)^2$ **15.** $(\ln x)^4$

16. $\dfrac{(x+3)^3}{x^2+2}$ **17.** $\sqrt{(e^x - x)}$ **18.** $4\ln(x^2+1)$

Find and simplify $\dfrac{dy}{dx}$ and hence find $\dfrac{d^2y}{dx^2}$ if

19. $y = \dfrac{(1+2x)}{(1-2x)}$ **20.** $y = \ln\dfrac{x}{x+1}$ **21.** $y = \dfrac{e^x}{e^x - 4}$

CHAPTER 21

FURTHER TRIGONOMETRIC IDENTITIES

THE FACTOR FORMULAE

The last set of identities considered in this book are called the factor formulae. They convert expressions such as $\sin A + \sin B$ into a product, so *factorising* the expression.

Consider the compound angle identities

$$\sin A\cos B + \cos A\sin B \equiv \sin(A+B)$$
$$\sin A\cos B - \cos A\sin B \equiv \sin(A-B)$$

Adding these gives $\quad 2\sin A\cos B \equiv \sin(A+B) + \sin(A-B)\quad$ [1]

Subtracting gives $\quad 2\cos A\sin B \equiv \sin(A+B) - \sin(A-B)\quad$ [2]

Working similarly with the compound angle identities for $\cos(A+B)$ and $\cos(A-B)$ gives

$$2\cos A\cos B \equiv \cos(A+B) + \cos(A-B) \quad [3]$$

and $\quad -2\sin A\sin B \equiv \cos(A+B) - \cos(A-B)\quad$ [4]

Identities [1] to [4] can be used when a product has to be expressed as a sum or difference. For example, to express $2\cos 7\theta \cos 2\theta$ as a sum we would use [3] to give

$$2\cos 7\theta \cos 2\theta \equiv \cos(7\theta + 2\theta) + \cos(7\theta - 2\theta) \equiv \cos 9\theta + \cos 5\theta$$

However, when a sum or difference has to be expressed as a product, these identities are more easily remembered when they are expressed in an alternative form as follows.

Using $\left.\begin{array}{l} A+B = P \\ A-B = Q \end{array}\right\}$ gives $A = \frac{1}{2}(P+Q)$ and $B = \frac{1}{2}(P-Q)$

Then equations [1] to [4] become

$$\sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q) \qquad [5]$$

$$\sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q) \qquad [6]$$

$$\cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q) \qquad [7]$$

$$\cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q) \qquad [8]$$

For example, to express $\sin 6\theta - \sin 4\theta$ as a product, we would use [6], to give

$$\sin 6\theta - \sin 4\theta \equiv 2\cos\tfrac{1}{2}(6\theta + 4\theta)\sin\tfrac{1}{2}(6\theta - 4\theta) \equiv 2\cos 5\theta \sin\theta$$

Many people find that these identities are more easily remembered in words than in symbols, for example [5] can be remembered as

sum of two sines ≡ twice sin (half sum) cos (half difference)

However, remembering every one of these identities in detail is not necessary but it is *important* to know that they exist and that they provide a powerful tool for dealing with trig functions. Formulae books provide details when it is known what is being looked for.

This group of identities can now be used to solve equations, simplify expressions, prove further identities and so on.

Examples 21a

1. Find the general solution of the equation $\sin 5x - \sin 3x = 0$

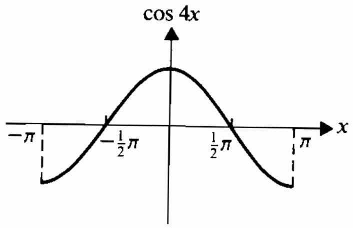
Using identity [6], the LHS of the equation can be expressed as a product.

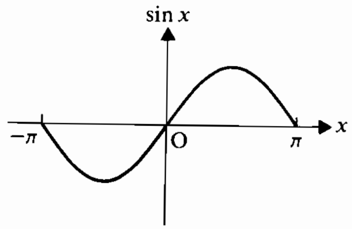
$$\sin 5x - \sin 3x = 0$$

$$\Rightarrow \quad 2\cos\tfrac{1}{2}(5x+3x)\sin\tfrac{1}{2}(5x-3x) = 0$$

$$\Rightarrow \quad \cos 4x \sin x = 0$$

$$\Rightarrow \quad \cos 4x = 0 \quad \text{or} \quad \sin x = 0$$





$$\therefore \quad 4x = \pm\tfrac{1}{2}\pi + 2n\pi$$

$$\Rightarrow \quad x = \pm\tfrac{1}{8}\pi + \tfrac{1}{2}n\pi \quad \text{or} \quad x = n\pi$$

The general solution is therefore $x = \frac{1}{8}\pi(4n \pm 1),\ n\pi$

2. Factorise $\cos\theta - \cos 3\theta - \cos 5\theta + \cos 7\theta$

As a first step, we take two terms and factorise them, and then take the remaining two terms and factorise them. With a bit of forethought, we can arrange the pairs of terms so that both pairs are in the form (cos + cos). In any case, the terms should be rearranged so that this first factorisation results in a common factor.

$$(\cos 7\theta + \cos\theta) - (\cos 5\theta + \cos 3\theta)$$

$$\equiv 2\cos 4\theta\cos 3\theta - 2\cos 4\theta\cos\theta$$

$$\equiv 2\cos 4\theta(\cos 3\theta - \cos\theta)$$

$$\equiv 2\cos 4\theta(-2\sin 2\theta\sin\theta) \equiv -4\cos 4\theta\sin 2\theta\sin\theta$$

3. Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} \equiv \tan\dfrac{A+B}{2}$

$$\text{LHS} \equiv \frac{\sin A + \sin B}{\cos A + \cos B} \equiv \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$
$$\equiv \tan\tfrac{1}{2}(A+B)$$

4. If A, B and C are the angles of a triangle, show that

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

$$\text{LHS} = (\sin A + \sin B) + \sin C$$
$$= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$

Now $A + B + C = 180° \Rightarrow \frac{1}{2}(A+B) + \frac{1}{2}C = 90°$

i.e. $\frac{1}{2}(A+B)$ and $\frac{1}{2}C$ are complementary angles

$\Rightarrow$ $\sin\frac{1}{2}(A+B) = \cos\frac{1}{2}C$ and $\sin\frac{1}{2}C = \cos\frac{1}{2}(A+B)$

$\therefore$ $\sin A + \sin B + \sin C$

$$= 2\cos\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B) + 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C$$
$$= 2\cos\tfrac{1}{2}C\,\{\cos\tfrac{1}{2}(A-B) + \cos\tfrac{1}{2}(A+B)\}$$
$$= 2\cos\tfrac{1}{2}C\,\{2\cos\tfrac{1}{2}A\cos(-\tfrac{1}{2}B)\}$$
$$= 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C \qquad (\cos -\theta = \cos\theta)$$

EXERCISE 21a

1. Express as a product of two trig functions
   (a) $\sin 3\theta + \sin\theta$ (b) $\cos 5\theta + \cos 3\theta$ (c) $\sin 4\theta - \sin 2\theta$
   (d) $\cos 7\theta - \cos\theta$ (e) $\sin 3A - \sin 5A$ (f) $\cos A - \cos 5A$
   (g) $\sin 2A + \sin 20°$ (h) $\sin 2\theta + 1$ (*Hint.* $\sin 90° = 1$)

2. Express as a sum or difference of two trig functions
   (a) $2\sin 2\theta\cos\theta$ (b) $2\cos 3\theta\cos 2\theta$ (c) $2\cos\theta\sin 4\theta$
   (d) $-2\sin 3\theta\sin\theta$ (e) $2\sin 4\theta\sin 2\theta$ (f) $\cos\theta\cos 4\theta$

3. Simplify
   (a) $\cos(\theta - 60°) + \cos(\theta + 60°)$ (b) $\sin(x - 45°) - \sin(x + 45°)$
   (c) $\sqrt{3}\cos\theta - \sin(\theta + 60°) - \sin(\theta + 120°)$

4. Evaluate, leaving your answers in surd form,
   (a) $\sin 105° + \sin 15°$ (b) $\cos 15° + \cos 75°$

5. Find the general solutions of the following equations.
   (a) $\cos 2x + \cos 4x = 0$ (b) $\sin 3x - \sin x = 0$
   (c) $\cos x - \cos 3x = 0$ (d) $\sin 4\theta + \sin 2\theta = 0$

6. Factorise the expression $\cos\theta + \cos 3\theta + \cos 2\theta$. Hence find the general solution of the equation $\cos\theta + \cos 3\theta + \cos 2\theta = 0$

7. Factorise the expression $\sin 3\theta - \sin\theta + \sin 7\theta - \sin 5\theta$. Hence find the general solution of the equation $\sin 3\theta - \sin\theta = \sin 5\theta - \sin 7\theta$.

8. Solve the following equations for angles from 0 to 180°
   (a) $\cos x = \cos 2x + \cos 4x$
   (b) $\cos x + \cos 3x = \sin x + \sin 3x$
   (c) $\sin 3\theta + \sin 6\theta + \sin 9\theta = 0$
   (d) $\sin 3\theta - \sin\theta = \cos 2\theta$
   (e) $\cos 5\theta - \cos\theta = \sin 3\theta$
   (f) $\cos 2x = \cos(30° - x)$

9. Prove the following identities.
   (a) $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} \equiv \dfrac{\tan(A+B)}{\tan(A-B)}$
   (b) $\dfrac{\cos 2A + \cos 2B}{\cos 2B - \cos 2A} \equiv \cot(A+B)\cot(A-B)$
   (c) $\dfrac{\sin A\sin 2A + \sin 3A\sin 6A}{\sin A\cos 2A + \sin 3A\cos 6A} \equiv \tan 5A$
   (d) $\dfrac{\sin 3x + \sin 5x}{\sin 4x + \sin 6x} \equiv \dfrac{\sin 4x}{\sin 5x}$
   (e) $\sin\theta + \sin 2\theta + \sin 3\theta \equiv (1 + \cos 2\theta)\sin 2\theta$
   (f) $1 + 2\cos 2\alpha + \cos 4\alpha \equiv 4\cos^2\alpha\cos 2\alpha$

10. If A, B and C are the angles of a triangle, prove that
(a) $\cos(B+C) = -\cos A$ (b) $\sin C = \sin(A+B)$
(c) $\sin\frac{1}{2}(A+B) = \cos\frac{1}{2}C$ (d) $\sin\frac{1}{2}B = \cos\frac{1}{2}(A+C)$
(e) $\sin B + \sin(A-C) = 2\sin A\cos C$
(f) $\cos(A-B) - \cos C = 2\cos A\cos B$
(g) $\sin(A+B) + \sin(B+C) = 2\cos\frac{1}{2}B\cos\frac{1}{2}(A-C)$
(h) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$

The next exercise contains questions that may require any of the identities from Chapters 17 and 18 as well as those from this one.

MIXED EXERCISE 21

1. If A is acute and $\sin A = \frac{1}{2}$ and if B is obtuse and $\sin B = \frac{1}{3}$, find in surd form the value of
(a) $\tan(A+B)$ (b) $\cos(A-B)$

2. Find the values of θ in the range $-90° \leqslant \theta \leqslant 90°$ for which $\tan^2\theta + \sec\theta = 11$

3. Find the general solution of the equation $\sin 3x - \sin x = \cos 2x$

4. By writing $\frac{3}{10}\pi$ as $\frac{1}{2}\pi - \frac{1}{5}\pi$, show that $\cos\frac{3}{10}\pi = \sin\frac{1}{5}\pi$

5. Find the general solution of the equation $\tan 2\theta + 2\sin\theta = 0$

6. Prove that if $\sec\alpha = \cos\beta + \sin\beta$ then $\tan^2\alpha = \sin 2\beta$

7. Find the values of θ from 0 to 180° for which $\sin 5\theta - \sin\theta = \cos 3\theta$

8. Prove the identity $\dfrac{1}{\sin x} + \dfrac{1}{\tan x} \equiv \cot\frac{1}{2}x$ and hence find in surd form the value of $\tan\frac{1}{8}\pi$

9. Find the Cartesian equation of the curve whose parametric equations are $x = \cos 2\theta$ and $y = \sin\theta$

10. Solve the simultaneous equations $\cos x - \cos y = 1$ and $\sin^2 x - \sin^2 y = 0$ for values of x and of y from 0 to 360°

11. Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$

12. Find the general solution of the equation $\tan x\sec x = \sqrt{2}$

13. Use the substitution $t \equiv \tan\frac{1}{2}x$ to find the values of x from 0 to 360° for which $4\cos x - 6\sin x = 5$

14. Find the Cartesian equation of the curve whose parametric equations are $x = 2\sin(\theta + \frac{1}{3}\pi)$ and $y = \cos(\theta - \frac{1}{6}\pi)$

15. Find the values of θ from 0 to 360° for which $\sin 2\theta = \sin 3\theta$

16. Find the Cartesian equation of the curve given parametrically by $x = a\cos 2t$ and $y = a\sin t$

17. Find the values of θ from 0 to 360° for which $\cos\theta = 2\sin(\theta - 30°)$

18. Use the substitution $t = \tan\theta$ to show that

$$\frac{1}{3+5\sin 2\theta} = \frac{1+t^2}{(3t+1)(t+3)}$$

19. Find the general solution of the equation $\sin 2x - \sin 4x = 0$

20. Express $\sin\theta$ in the form $\cos\alpha$ giving α in terms of θ. Hence solve the equation $\cos 3\theta = \sin\theta$ for values of θ between 0 and 360°

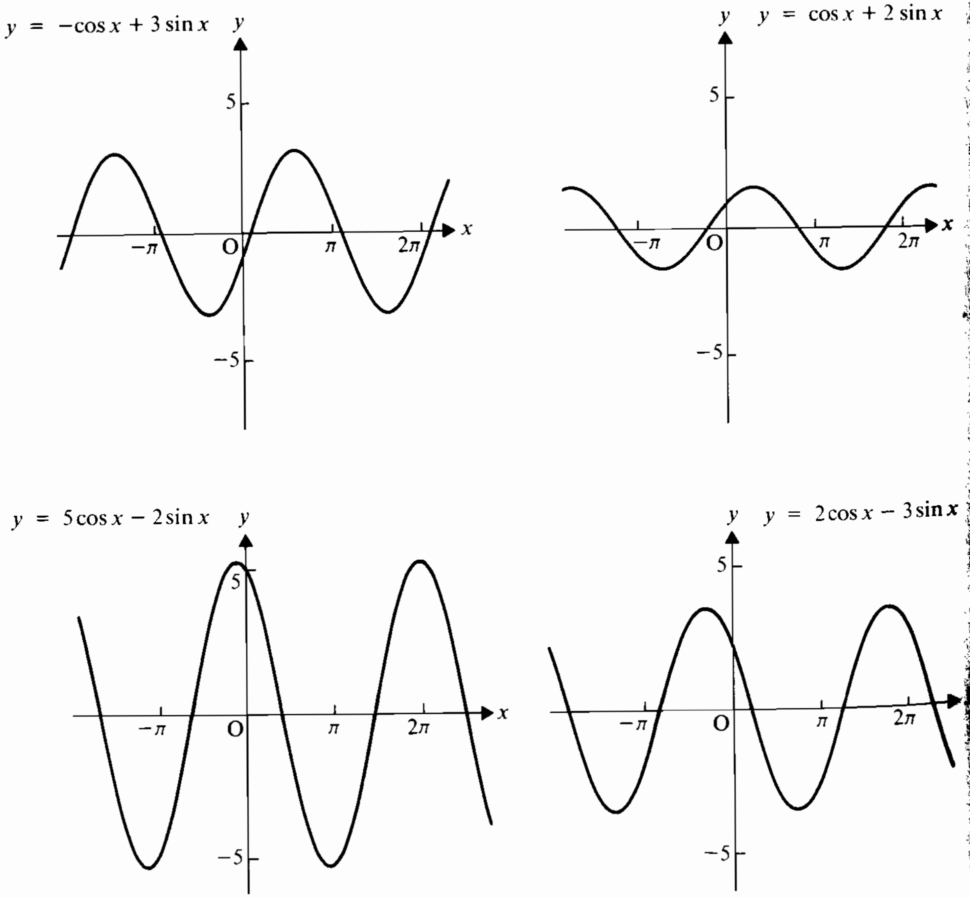
CHAPTER 22

FURTHER TRIGONOMETRIC FUNCTIONS

$f(\theta) = a\cos\theta + b\sin\theta$

SG

The diagrams below show the graphs of $f(\theta) = a\cos\theta + b\sin\theta$ for a variety of values of a and b.



If the reader has access to the appropriate technology, we suggest that the graph of $f(\theta)$ is drawn for some other values of a and b. Each of these graphs is a sine wave, although with differing amplitude and phase shift.

These diagrams suggest that it is possible to express $a\cos\theta + b\sin\theta$ as $r\sin(\theta + \alpha)$ where the values of r and α depend on the values of a and b. This is possible provided that we can find values of r and α such that

$$r\sin(\theta + \alpha) \equiv a\cos\theta + b\sin\theta$$

i.e. $$r\,\underline{\sin\theta}\cos\alpha + r\,\underline{\underline{\cos\theta}}\sin\alpha \equiv a\,\underline{\underline{\cos\theta}} + b\,\underline{\sin\theta}$$

Since this is an identity we can compare coefficients of $\sin\theta$ and of $\cos\theta$

$\Rightarrow$ $$r\sin\alpha = a \qquad [1]$$

and $$r\cos\alpha = b \qquad [2]$$

Equations [1] and [2] can now be solved to give r and α in terms of a and b.

Squaring and adding equations [1] and [2] gives

$$r^2(\sin^2\alpha + \cos^2\alpha) = a^2 + b^2 \quad\Rightarrow\quad r = \sqrt{(a^2 + b^2)}$$

Dividing equation [1] by equation [2] gives

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{a}{b} \quad\Rightarrow\quad \tan\alpha = \frac{a}{b}$$

Therefore $$r\sin(\theta + \alpha) \equiv a\cos\theta + b\sin\theta$$

$$\text{where}\quad r = \sqrt{(a^2 + b^2)} \quad\text{and}\quad \tan\alpha = \frac{a}{b}$$

It is also possible to express $a\cos\theta + b\sin\theta$ as $r\sin(\theta - \alpha)$ or as $r\cos(\theta \pm \alpha)$, using a similar method.

Examples 22a

1. Express $3\sin\theta - 2\cos\theta$ as $r\sin(\theta - \alpha)$

$$3\sin\theta - 2\cos\theta \equiv r\sin(\theta-\alpha)$$

$$\Rightarrow \quad 3\,\underline{\sin}\,\theta - 2\,\underline{\underline{\cos}}\,\theta \equiv r\,\underline{\sin}\,\theta\cos\alpha - r\,\underline{\underline{\cos}}\,\theta\sin\alpha$$

Comparing coefficients of $\sin\theta$ and of $\cos\theta$ gives

$$\left.\begin{aligned} 3 &= r\cos\alpha \\ 2 &= r\sin\alpha \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} 13 &= r^2 &\Rightarrow\quad r &= \sqrt{13} \\ \tan\alpha &= \tfrac{2}{3} &\Rightarrow\quad \alpha &= 33.7^\circ \end{aligned}\right.$$

$$\therefore \quad 3\sin\theta - 2\cos\theta = \sqrt{13}\sin(\theta - 33.7^\circ)$$

---

2. Find the maximum value of $f(x) = 3\cos x + 4\sin x$ and the smallest positive value of x at which it occurs.

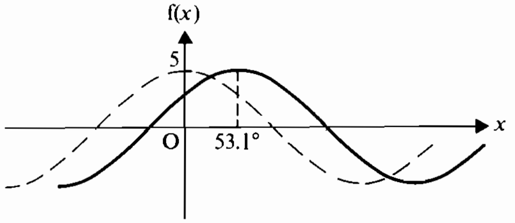
Expressing $f(x)$ in the form $r\sin(x+\alpha)$ enables us to 'read' its maximum value, and the values of x at which they occur, from the resulting sine wave. Note also that in this question there is a choice of form in which to express $f(x)$: in this case it is sensible to choose $r\cos(x-\alpha)$ as this fits $f(x)$ better than $r\sin(x+\alpha)$

$$3\,\underline{\underline{\cos}}\,x + 4\,\underline{\sin}\,x \equiv r\cos(x-\alpha) \equiv r\,\underline{\underline{\cos}}\,x\cos\alpha + r\,\underline{\sin}\,x\sin\alpha$$

$$\text{Hence}\quad \left.\begin{aligned} r\cos\alpha &= 3 \\ r\sin\alpha &= 4 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} r^2 &= 25 &\Rightarrow\quad r &= 5 \\ \tan\alpha &= \tfrac{4}{3} &\Rightarrow\quad \alpha &= 53.1^\circ \end{aligned}\right.$$

$$\therefore \quad f(x) \equiv 5\cos(x - 53.1^\circ)$$

The graph of $f(x)$ is a cosine wave with amplitude 5 and phase shift 53.1°



$\therefore$ $f(x)$ has a maximum value of 5 and, from the sketch, the smallest positive value of x at which it occurs is 53.1°

---

3. Find the greatest and least values of $\dfrac{2}{\sin x - \cos x}$

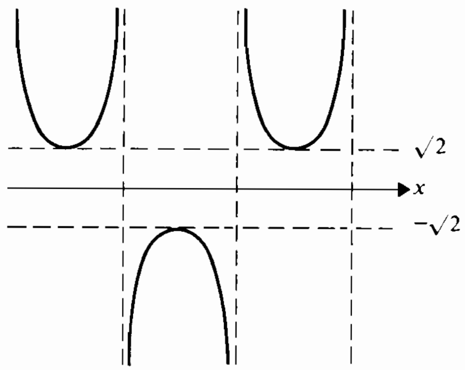
We first express $\sin x - \cos x$ in the form $r\sin(x-\alpha)$ then the given function can be expressed as a cosec function and we can sketch its graph. Note that values of x are not required so we do not need the value of α. Note also that the greatest and least values of a function are not necessarily the same as the maximum and minimum value of that function.

If $f(x) \equiv \underline{\sin}\,x - \underline{\underline{\cos}}\,x \equiv r\sin(x-\alpha) \equiv r\,\underline{\sin}\,x\cos\alpha - r\,\underline{\underline{\cos}}\,x\sin\alpha$

$$\text{then}\quad \left.\begin{aligned} r\cos\alpha &= 1 \\ r\sin\alpha &= 1 \end{aligned}\right\} \Rightarrow r^2 = 2, \text{ i.e. } r = \sqrt{2}$$

$$\therefore \quad \sin x - \cos x \equiv \sqrt{2}\sin(x-\alpha)$$

$$\text{Hence}\quad \frac{2}{f(x)} \equiv \frac{2}{\sqrt{2}\sin(x-\alpha)} \equiv \sqrt{2}\operatorname{cosec}(x-\alpha)$$



From the sketch, the greatest value of $\dfrac{2}{\sin x - \cos x}$ is ∞ and the least value is $-\infty$

Note that $\dfrac{2}{f(x)}$ has a maximum value of $-\sqrt{2}$ and a minimum value of $\sqrt{2}$

THE EQUATION $a\cos x + b\sin x = c$

One way of solving an equation of this type was covered in Chapter 18; it uses the half angle formulae involving t where $t = \tan\frac{1}{2}x$. An alternative method is to express the LHS of the equation in the form $r\cos(x+\alpha)$. This method, which has the advantage that solutions are not easily lost, is illustrated in the next example.

Examples 22a (continued)

4. Find, in radians, the general solution of the equation

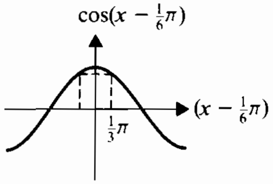
$$\sqrt{3}\cos x + \sin x = 1$$

If $\sqrt{3}\cos x + \sin x \equiv r\cos(x-\alpha) \equiv r\cos x\cos\alpha + r\sin x\sin\alpha$

then $\begin{cases} r\cos\alpha = \sqrt{3} \\ r\sin\alpha = 1 \end{cases} \Rightarrow \begin{cases} r^2 = 4 \\ \tan\alpha = \frac{1}{\sqrt{3}} \end{cases} \Rightarrow \begin{matrix} r = 2 \\ \alpha = \frac{1}{6}\pi \end{matrix}$

i.e. $\sqrt{3}\cos x + \sin x \equiv 2\cos(x - \frac{1}{6}\pi)$

$\therefore$ the equation becomes



$$2\cos(x - \tfrac{1}{6}\pi) = 1$$

$$\Rightarrow \quad \cos(x - \tfrac{1}{6}\pi) = \tfrac{1}{2}$$

$$\Rightarrow \quad x - \tfrac{1}{6}\pi = \pm\tfrac{1}{3}\pi + 2n\pi$$

$$\therefore \quad x = \tfrac{1}{2}\pi + 2n\pi, \ -\tfrac{1}{6}\pi + 2n\pi$$

EXERCISE 22a

1. Find the values of r and α for which
   (a) $\sqrt{3}\cos\theta - \sin\theta \equiv r\cos(\theta+\alpha)$
   (b) $\cos\theta + 3\sin\theta \equiv r\cos(\theta-\alpha)$
   (c) $4\sin\theta - 3\cos\theta \equiv r\sin(\theta-\alpha)$

2. Express $\cos 2\theta - \sin 2\theta$ in the form $r\cos(2\theta+\alpha)$

3. Express $2\cos 3\theta + 5\sin 3\theta$ in the form $r\sin(3\theta+\alpha)$

4. Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $r\sin(\theta-\alpha)$. Hence sketch the graph of $f(\theta) = \cos\theta - \sqrt{3}\sin\theta$. Give the maximum and minimum values of $f(\theta)$ and the values of θ between 0 and 360° at which they occur.

5. Express $7\cos\theta - 24\sin\theta$ in the form $r\cos(\theta+\alpha)$. Hence sketch the graph of $f(\theta) = 7\cos\theta - 24\sin\theta + 3$ and give the maximum and minimum values of $f(\theta)$ and the values of θ between 0 and 360° at which they occur.

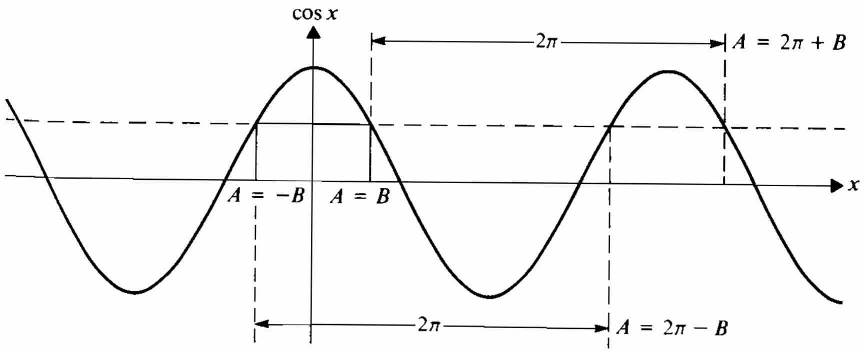
6. Find the greatest and least values of $\cos x + \sin x$. Hence find the maximum and minimum values of $\dfrac{1}{\cos x + \sin x}$

7. Find the maximum and minimum values of $\dfrac{\sqrt{2}}{\cos\theta - \sqrt{2}\sin\theta}$

8. Find the general solution of the following equations.
   (a) $\cos x + \sin x = \sqrt{2}$ (b) $7\cos x + 6\sin x = 2$
   (c) $\cos x - 3\sin x = 1$ (d) $2\cos x - \sin x = 2$

THE EQUATION $\cos A = \cos B$

An equation such as $\cos 2x = \cos x$ can be solved by using a factor formula, but there is another very simple method of solution, based on consideration of the graph of $\cos x$



If $\cos A = \cos B$

then, from the graph, we see that $A = 2n\pi \pm B$

i.e. $\cos A = \cos B \quad \Rightarrow \quad A = 2n\pi \pm B$

A similar conclusion is reached from the graphs of $\tan x$ and of $\sin x$

$$\tan A = \tan B \quad \Rightarrow \quad A = B + n\pi$$

$$\sin A = \sin B \quad \Rightarrow \quad A = \begin{cases} 2n\pi + B \\ (2n+1)\pi - B \end{cases}$$

This method can also be used for equations of the form $\cos A = \sin B$, and this is illustrated in Examples 22b, number 2.

Examples 22b

1. Find the general solution of the equation $\cos 4\theta = \cos\theta$

Using only the conclusion to the argument above, we can write

$$4\theta = 2n\pi \pm \theta$$

hence $5\theta = 2n\pi$
or $3\theta = 2n\pi$ $\Rightarrow \quad \theta = \frac{2}{5}n\pi, \frac{2}{3}n\pi$

---

2. Find the general solution of the equation $\cos 3x = \sin x$

We know that $\sin x = \cos(\frac{1}{2}\pi - x)$, so the equation can be written

$$\cos 3x = \cos(\tfrac{1}{2}\pi - x)$$

$\therefore \quad 3x = 2n\pi \pm (\frac{1}{2}\pi - x)$

Hence $4x = 2n\pi + \frac{1}{2}\pi$
or $2x = 2n\pi - \frac{1}{2}\pi$ $\Rightarrow \quad x = \frac{1}{2}n\pi + \frac{1}{8}\pi,\ n\pi - \frac{1}{4}\pi$

---

3. Find the values of θ between 0 and 360° for which $\tan(3\theta - 40°) = \tan\theta$

The general solution is $3\theta - 40° = 180n° + \theta$

$\Rightarrow \quad \theta = 90n° + 20°$

For $0 \leqslant \theta \leqslant 360°$, $n = 0, 1, 2$ and 3

$\Rightarrow \quad \theta = 20°, 110°, 200°, 290°$

EXERCISE 22b

Find the general solution of the following equations.

1. $\cos 4\theta = \cos 3\theta$
2. $\tan 7\theta = \tan 2\theta$
3. $\cos 3\theta = \sin 2\theta$
4. $\cot 4\theta = \tan 5\theta$
5. $\sin 4\theta = \sin 3\theta$
6. $\cos 5\theta \sec\theta = 1$
7. $\cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(4\theta + \frac{\pi}{4}\right)$
8. $\tan 2\theta = \cot\theta$

Find the solutions, from 0 to π inclusive, of the following equations.

9. $\cos 3\theta = \cos 7\theta$
10. $\tan 3\theta = \cot 2\theta$
11. $\sin 7\theta = \sin 2\theta$
12. $\sec 6\theta = \sec 5\theta$

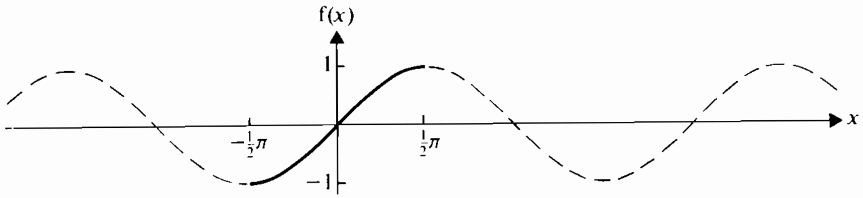
Solve the following equations giving values from $-180°$ to $+180°$.

13. $\cos(2\theta + 60°) = \cos\theta$
14. $\tan 3\theta = \tan(\theta - 50°)$
15. $\sin\theta = \cos(2\theta + 60°)$
16. $\cos(30° - \theta) = \cos 2\theta$
17. $\tan(2\theta + 60°) = \tan 4\theta$
18. $\sin 3\theta = \sin(\theta + 80°)$

THE INVERSE TRIGONOMETRIC FUNCTIONS

Consider the function given by $f: x \to \sin x$ for $x \in \mathbb{R}$

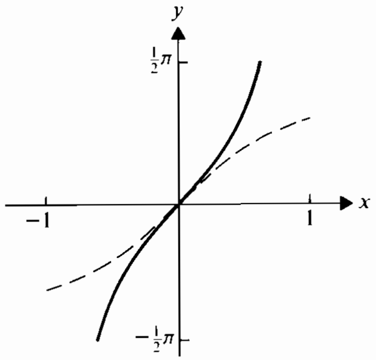
The inverse mapping is given by $\sin x \to x$ but this is not a function because one value of $\sin x$ maps to many values of x, i.e. $f(x) = \sin x$ does not have an inverse function for the domain $x \in \mathbb{R}$



However, if we now consider the function $f: x \to \sin x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ then the reverse mapping, $\sin x \to x$, is such that one value of $\sin x$ maps to only one value of x

Therefore $f: x \to \sin x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ does have an inverse, i.e. f^{-1} exists.

Now the equation of the graph of f is $y = \sin x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and the curve $y = f^{-1}(x)$ is obtained by reflecting $y = \sin x$ in the line $y = x$



Therefore interchanging x and y gives the equation of this curve, i.e. $\sin y = x$, so $y =$ the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is x

Using *arcsin* to mean 'the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is', we have $y = \arcsin x$

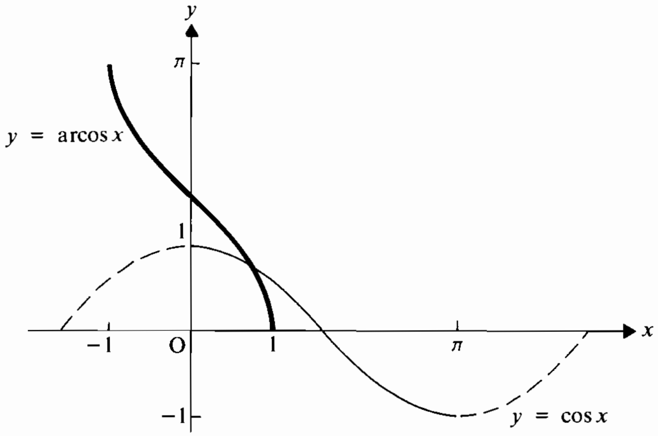
Thus	if	$f: x \to \sin x$	$-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$
	then	$f^{-1}: x \to \arcsin x$	$-1 \leqslant x \leqslant 1$

It is important to realise that $\arcsin x$ is an angle, and further that this angle is in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$

Thus, for example, $\arcsin 0.5$ is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is 0.5, i.e. $\arcsin 0.5 = \frac{1}{6}\pi$

Note that the alternative notation $\sin^{-1}$ is sometimes used to mean 'the angle whose sine is', and if this notation is adopted, care is needed not to confuse it with $1/\sin x$

Now consider the function given by $f: x \to \cos x$, $0 \leqslant x \leqslant \pi$



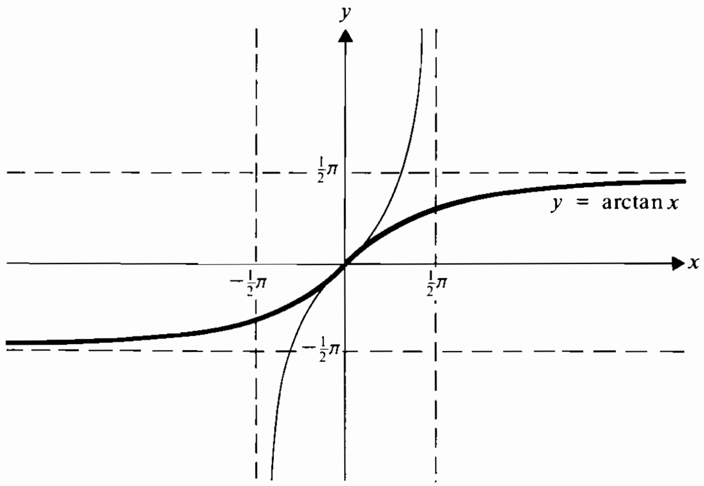
From the diagram, we see that f^{-1} exists and it is denoted by arccos where $\arccos x$ means 'the angle between 0 and π whose cosine is x'

Thus	if	$f: x \to \cos x$	$0 \leqslant x \leqslant \pi$
	then	$f^{-1}: x \to \arccos x$	$-1 \leqslant x \leqslant 1$

Note that $\arccos x$ is an angle in the range $0 \leqslant x \leqslant \pi$

Thus, for example, $\arccos -0.5 = x \quad \Rightarrow \quad x = \frac{2}{3}\pi$

Similarly, if $f: x \to \tan x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$, then f^{-1} exists and is written *arctan* where $\arctan x$ means 'the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is x'



Examples 22c

1. Find the exact value of $\arctan\frac{3}{4} + \arctan\frac{5}{12}$

If $\alpha = \arctan\frac{3}{4}$ and $\beta = \arctan\frac{5}{12}$

then $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{5}{12}$

Now $\arctan\frac{3}{4} + \arctan\frac{5}{12} = \alpha + \beta$

and
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(\frac{3}{4}) + (\frac{5}{12})}{1 - (\frac{3}{4})(\frac{5}{12})}$$
$$= \frac{56}{33}$$

$\therefore$
$$(\alpha+\beta) = \arctan\frac{56}{33}$$

i.e. $\arctan\frac{3}{4} + \arctan\frac{5}{12} = \arctan\frac{56}{33}$

---

2. Solve the equation $\arcsin x + \arccos\frac{1}{2}x = \frac{5}{6}\pi$

If $\arcsin x = \theta$ and $\arccos\frac{1}{2}x = \phi$

then $x = \sin\theta$ and $\frac{1}{2}x = \cos\phi$

Therefore the given equation becomes $\theta + \phi = \frac{5}{6}\pi$

Hence
$$\sin(\theta+\phi) = \frac{1}{2}$$

$\Rightarrow$
$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{2} \qquad [1]$$

Now $\sin\theta = x$ so $\cos\theta = \sqrt{(1-x^2)}$

and $\cos\phi = \frac{1}{2}x$ so $\sin\phi = \dfrac{\sqrt{(4-x^2)}}{2}$

$\therefore$ [1] becomes
$$x\left(\frac{1}{2}x\right) + \sqrt{(1-x^2)}\left(\frac{\sqrt{(4-x^2)}}{2}\right) = \frac{1}{2}$$

$\Rightarrow$
$$\sqrt{(1-x^2)}\sqrt{(4-x^2)} = 1 - x^2$$

Squaring both sides of the equation gets rid of the square roots, but it can introduce extra solutions which do not satisfy the original equation. It is therefore *essential* that all solutions are tested to see if they satisfy the original equation.

$$\therefore \quad (1-x^2)(4-x^2) = (1-x^2)^2$$
$$\Rightarrow \quad (1-x^2)(4-x^2-1+x^2) = 0$$
$$\Rightarrow \quad (1-x^2) = 0 \quad \text{or} \quad (4-x^2-1+x^2) = 0$$
$$\therefore \quad x = \pm 1 \quad \text{or} \quad x = \infty$$

Now $x = \sin\theta$, $\therefore$ $x = \infty$ is not a possible solution.

Substituting $x = 1$ into the LHS of the original equation gives

$$\arcsin 1 + \arccos\tfrac{1}{2} = \tfrac{1}{2}\pi + \tfrac{1}{3}\pi = \tfrac{5}{6}\pi = \text{RHS}$$

$\therefore$ $x = 1$ is a solution.

Substituting $x = -1$ into the LHS of the original equation gives

$$\arcsin -1 + \arccos -\tfrac{1}{2} = -\tfrac{1}{2}\pi + \tfrac{2}{3}\pi \neq \text{RHS}$$

$\therefore$ $x = -1$ is not a solution.

So the only solution is $x = 1$

---

3. Prove that $\arctan x + \arctan\left(\dfrac{1-x}{1+x}\right) \equiv \dfrac{1}{4}\pi$

First we will simplify the LHS of the identity.

If $\alpha = \arctan x$ then $x = \tan\alpha$

and if $\beta = \arctan\left(\dfrac{1-x}{1+x}\right)$ then $\dfrac{1-x}{1+x} = \tan\beta$

Now $\arctan x + \arctan\left(\dfrac{1-x}{1+x}\right) = \alpha + \beta$

and
$$\tan(\alpha+\beta) \equiv \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x + \left(\frac{1-x}{1+x}\right)}{1 - x\left(\frac{1-x}{1+x}\right)}$$
$$= \frac{x^2+1}{1+x^2} = 1$$

Hence $\alpha + \beta = \arctan 1 = \frac{1}{4}\pi$, i.e. $\arctan x + \arctan\left(\dfrac{1-x}{1+x}\right) \equiv \dfrac{1}{4}\pi$

EXERCISE 22c

1. Find the value of the following in terms of π
   (a) $\arctan\sqrt{3}$ (b) $\arcsin -1$ (c) $\arccos 0$
   (d) $\arcsin -\frac{\sqrt{3}}{2}$ (e) $\arccos -\frac{1}{2}$ (f) $\arctan -1$

2. Find the value of the following in terms of π
   (a) $\arctan\frac{1}{3} + \arctan\frac{1}{2}$ (b) $\arcsin\frac{1}{2} + \arccos\frac{1}{2}$

3. Simplify
   (a) $\arcsin x + \arccos x$ (b) $\sin(2\arctan x)$
   (c) $\arctan x + \arctan\frac{1}{x}$ (d) $\tan(\arctan\frac{1}{3} + \arctan\frac{1}{4})$

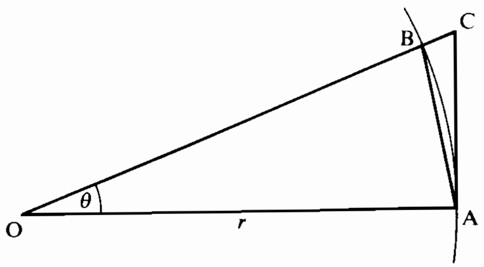
4. Prove that
   (a) $\cos(2\arcsin x) \equiv 1 - 2x^2$ (b) $\sin(\arccos x) \equiv \sqrt{(1-x^2)}$

5. Solve the following equations.
   (a) $\tan^{-1}2 = \tan^{-1}4 - \tan^{-1}x$
   (b) $\sin^{-1}\left(\frac{x}{x-1}\right) + 2\tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$
   (c) $\arctan(1+x) + \arctan(1-x) = \arctan 2$

SMALL ANGLES

Using a calculator to find the sine and tangent of small angles measured in radians, we find that $\sin\theta$ and $\tan\theta$ are approximately equal to θ. For example, correct to three significant figures $\sin 0.1 = 0.100$ and $\tan 0.1 = 0.100$

This can be proved as follows.
In the diagram a small angle, θ radians, is subtended by the arc AB at the centre O of a circle of radius r. AB is a chord of the circle and AC is the tangent to the circle at A, cutting OB produced at C.



Now area $\triangle OAB <$ area sector OAB $<$ area $\triangle OAC$

i.e. $\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$

Dividing by $\frac{1}{2}r^2$, which is positive, gives

$$\sin\theta < \theta < \tan\theta \qquad [1]$$

Dividing by $\sin\theta$, which is positive as θ is a small positive angle, gives

$$1 < \frac{\theta}{\sin\theta} < \sec\theta$$

Now as $\theta \to 0$, $\sec\theta \to 1$ so $\frac{\theta}{\sin\theta}$ lies between 1 and a number that approaches 1 as θ gets smaller,

i.e. $$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1$$

Similarly, dividing [1] by $\tan\theta$, we can show that

$$\lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1$$

These limiting values verify that, for small positive values of θ,

$$\sin\theta \approx \theta \quad \text{and} \quad \tan\theta \approx \theta$$

To find an approximate value for $\cos\theta$ when θ is small, we use the identity

$$\cos\theta \equiv 1 - 2\sin^2\tfrac{1}{2}\theta$$

But if $\frac{1}{2}\theta$ is small, $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, so $\cos\theta \approx 1 - 2(\frac{1}{2}\theta)^2$

i.e. $$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

Collecting these results, we have

> **when θ is small and measured in radians,**
> **$\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$**

These approximations are correct to 3 s.f. for angles in the range -0.105 rad $< \theta < 0.105$ rad, i.e. $-6° < \theta < 6°$

Example 22d

Find, as a rational function of θ, an approximation for the expression $\dfrac{\sin 3\theta}{1+\cos 2\theta}$ when 3θ is small.

When 3θ is small, $\sin 3\theta \approx 3\theta$

and 2θ is also small, so $\cos 2\theta \approx 1 - \dfrac{(2\theta)^2}{2}$

$$\therefore \qquad \frac{\sin 3\theta}{1+\cos 2\theta} \approx \frac{3\theta}{2(1-\theta^2)}$$

EXERCISE 22d

1. If θ is small enough to regard 4θ as small, find approximations for the following expressions.

   (a) $\dfrac{2\theta}{\sin 4\theta}$ (b) $\dfrac{\theta \sin\theta}{\cos 2\theta}$ (c) $\sin \frac{1}{2}\theta \sec\theta$

   (d) $\dfrac{\theta\tan\theta}{1-\cos\theta}$ (e) $\dfrac{2\sin\frac{1}{2}\theta}{\theta}$ (f) $\dfrac{\sin\theta\tan\theta}{\theta^2}$

2. If θ is small enough for θ^2 to be neglected (i.e. $\theta^2 \approx 0$), show that

   (a) $2\cos(\frac{1}{3}\pi + \theta) \approx 1 - \theta\sqrt{3}$ (b) $4\sin(\frac{1}{4}\pi - \theta) \approx 2\sqrt{2}(1-\theta)$

3. Find the limiting value as $\theta \to 0$ of the following expressions.

   (a) $\dfrac{\sin\theta}{2\theta}$ (b) $\dfrac{\tan 2\theta}{\sin 3\theta}$

   (c) $\sin(\frac{1}{3}\pi + \theta)$ (d) $\tan(\frac{1}{4}\pi + \theta)$

MIXED EXERCISE 22

1. Express $4\sin\theta - 3\cos\theta$ in the form $r\sin(\theta - \alpha)$. Hence find the maximum and minimum values of $\dfrac{7}{4\sin\theta - 3\cos\theta + 2}$
   State the greatest and least values.

2. If $\alpha = \arctan\frac{1}{2} - \arctan\frac{1}{4}$, find the value, in surd form, of

   (a) $\tan\alpha$ (b) $\cos\alpha$ (c) $\sin\alpha$

3. Express $\sin 2\theta - \cos 2\theta$ in the form $r\sin(2\theta - \alpha)$. Hence find the smallest positive value of θ for which $\sin 2\theta - \cos 2\theta$ has a maximum value.

4. If $\arctan x + \arctan y = \frac{1}{4}\pi$, show that $x + y = 1 - xy$

5. Find the limiting value as $\theta \to 0$ of $\dfrac{\frac{1}{2}(1-\sin\theta-\cos\theta)}{2\theta}$

6. Express $\cos x + \sin x$ in the form $r\cos(x-\alpha)$. Hence find the smallest positive value of x for which $\dfrac{1}{(\cos x + \sin x)}$ has a minimum value.

7. Find all the values of x in the range $0 \leqslant x \leqslant 2\pi$ for which $\cos 3x = \cos 2x$

8. Find all the values of x between 0 and 180° for which $\sin(x - 30°) = \cos 4x$

9. Show that $x = \frac{1}{14}\pi$ is a solution of the equation $\sin 3x = \cos 4x$

10. Find the general solution of the equation $\tan 5\theta = \cot 2\theta$

11. Express $3\cos x - 4\sin x$ in the form $r\cos(x+\alpha)$. Hence express $4 + \dfrac{10}{3\cos x - 4\sin x}$ in the form $4 + k\sec(x+\alpha)$ and sketch the graph of $4 + \dfrac{10}{3\cos x - 4\sin x}$

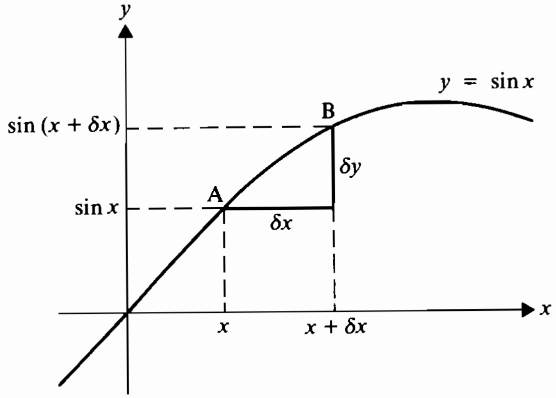
CHAPTER 23

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

THE DERIVATIVE OF sin *x*

110 The derivative of any new function, $f(x)$, can be found by differentiating from first principles. This involves finding the limit of the gradient of the chord joining any two neighbouring points on the curve $y = f(x)$

Consider the curve $y = \sin x$



A is any point (x, y) on the curve $y = \sin x$ and at the nearby point B the x-coordinate is $x + \delta x$

At A, $y = \sin x$ and at B, $y = \sin(x + \delta x)$

Now $$\frac{\delta y}{\delta x} = \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \frac{2\cos(x+\frac{1}{2}\delta x)\sin(\frac{1}{2}\delta x)}{\delta x} \quad \text{(using a factor formula)}$$

$$= \frac{\cos(x+\frac{1}{2}\delta x)\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x}$$

Then, *provided that x is measured in radians*,

$$\text{as} \quad \delta x \to 0, \quad \cos(x+\tfrac{1}{2}\delta x) \to \cos x \quad \text{and} \quad \frac{\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x} \to 1$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\cos(x+\frac{1}{2}\delta x)\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x} = \cos x$$

i.e. $$\text{if} \quad y = \sin x \quad \text{then} \quad \frac{dy}{dx} = \cos x$$

In a similar way it can be shown that

$$\text{if} \quad y = \cos x \quad \text{then} \quad \frac{dy}{dx} = -\sin x$$

These two results can be quoted whenever they are needed.

It is important to realise that they are valid only when x is measured in radians.

Throughout all subsequent work on the calculus of trig functions in this book, the angle is measured in radians unless it is stated otherwise.

Examples 23a

1. Find the smallest positive value of x for which there is a stationary value of the function $x + 2\cos x$

$f(x) = x + 2\cos x \quad \Rightarrow \quad f'(x) = 1 - 2\sin x$

where $f'(x) = \dfrac{d}{dx}f(x)$

For stationary values $f'(x) = 0$

i.e. $1 - 2\sin x = 0 \quad \Rightarrow \quad \sin x = \frac{1}{2}$

The smallest positive angle with a sine of $\frac{1}{2}$ is $\frac{1}{6}\pi$

NOTE that the answer *must* be given in radians because the rule used to differentiate $\cos x$ is valid only for an angle in radians.

2. Find the smallest positive value of θ for which the curve $y = 2\theta - 3\sin\theta$ has a gradient of $\frac{1}{2}$

$$y = 2\theta - 3\sin\theta \quad \text{gives} \quad \frac{dy}{d\theta} = 2 - 3\cos\theta$$

when $\frac{dy}{d\theta} = \frac{1}{2}$

$$2 - 3\cos\theta = \tfrac{1}{2}$$
$$3\cos\theta = \tfrac{3}{2}$$
$$\cos\theta = \tfrac{1}{2}$$

The smallest positive value of θ for which $\cos\theta = \frac{1}{2}$, is $\frac{1}{3}\pi$

EXERCISE 23a

1. By differentiating from first principles, show that $\frac{d}{dx}(\cos x) = -\sin x$

2. Write down the derivative of each of the following expressions.
   (a) $\sin x - \cos x$ (b) $\sin\theta + 4$ (c) $3\cos\theta$
   (d) $5\sin\theta - 6$ (e) $2\cos\theta + 3\sin\theta$ (f) $4\sin x - 5 - 6\cos x$

3. Find the gradient of each curve at the point whose x-coordinate is given.
   (a) $y = \cos x$; $\frac{1}{2}\pi$ (b) $y = \sin x$; 0
   (c) $y = \cos x + \sin x$; π (d) $y = x - \sin x$; $\frac{1}{2}\pi$
   (e) $y = 2\sin x - x^2$; $-\pi$ (f) $y = -4\cos x$; $\frac{1}{2}\pi$

4. For each of the following curves find the smallest positive value of θ at which the gradient of the curve has the given value.
   (a) $y = 2\cos\theta$; -1 (b) $y = \theta + \cos\theta$; $\frac{1}{2}$
   (c) $y = \sin\theta + \cos\theta$; 0 (d) $y = \sin\theta + 2\theta$; 1

5. Considering only positive values of x, locate the first two turning points on each of the following curves and determine whether they are maximum or minimum points.
   (a) $2\sin x - x$ (b) $x + 2\cos x$
   In each case illustrate your solution by a sketch.

6. Find the equation of the tangent to the curve $y = \cos\theta + 3\sin\theta$ at the point where $\theta = \frac{1}{2}\pi$

7. Find the equation of the normal to the curve $y = x^2 + \cos x$ at the point where $x = \pi$

8. Find the coordinates of a point on the curve $y = \sin x + \cos x$ at which the tangent is parallel to the line $y = x$

COMPOUND FUNCTIONS

The variety of functions which can be handled when they occur in products, quotients and functions of a function, now includes the sine and cosine ratios.

Differentiation of sin f(x)

If $y = \sin f(x)$ then using $u = f(x)$ gives $y = \sin u$

Then $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \cos u \frac{du}{dx}$$

i.e. $$\frac{d}{dx}\{\sin f(x)\} = f'(x)\cos f(x)$$

Similarly $$\frac{d}{dx}\{\cos f(x)\} = -f'(x)\sin f(x)$$

e.g. $$\frac{d}{dx}\sin e^x = e^x\cos e^x \quad \text{and} \quad \frac{d}{dx}\cos\ln x = -\frac{1}{x}\sin\ln x$$

In particular $$\frac{d}{dx}(\sin ax) = a\cos ax$$

and $$\frac{d}{dx}(\cos ax) = -a\sin ax$$

These results are quotable.

Examples 23b

1. Differentiate $\cos\left(\frac{1}{6}\pi - 3x\right)$ with respect to x

$$\frac{d}{dx}\{\cos\left(\tfrac{1}{6}\pi - 3x\right)\} = -(-3)\sin\left(\tfrac{1}{6}\pi - 3x\right)$$

$$= 3\sin\left(\tfrac{1}{6}\pi - 3x\right)$$

---

2. Find the derivative of $\dfrac{e^x}{\sin x}$

$$y = \frac{e^x}{\sin x} = \frac{u}{v}$$

where $u = e^x$ and $v = \sin x$

$$\Rightarrow \quad \frac{du}{dx} = e^x \quad \text{and} \quad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Big/ v^2 = \frac{e^x \sin x - e^x \cos x}{\sin^2 x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{e^x}{\sin x}\right) = \frac{e^x}{\sin^2 x}(\sin x - \cos x)$$

---

3. Find $\dfrac{dy}{d\theta}$ if $y = \cos^3\theta$

$$y = \cos^3\theta = [\cos\theta]^3$$

$$\therefore \quad y = u^3 \quad \text{where} \quad u = \cos\theta$$

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = (3u^2)(-\sin\theta) = 3(\cos\theta)^2(-\sin\theta)$$

$$\therefore \quad y = \cos^3\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = -3\cos^2\theta\sin\theta$$

This is one example of a general rule, i.e.

$$\text{if } y = \cos^n x \text{ then } \frac{dy}{dx} = -n\cos^{n-1}x\sin x$$

and

$$\text{if } y = \sin^n x \text{ then } \frac{dy}{dx} = n\sin^{n-1}x\cos x$$

EXERCISE 23b

Differentiate each of the following functions with respect to x

1. $\sin 4x$
2. $\cos(\pi - 2x)$
3. $\sin\left(\frac{1}{2}x + \pi\right)$
4. $\dfrac{\sin x}{x}$
5. $\dfrac{\cos x}{e^x}$
6. $\sqrt{\sin x}$
7. $\sin^2 x$
8. $\sin x \cos x$
9. $e^{\sin x}$
10. $\ln(\cos x)$
11. $e^x \cos x$
12. $x^2 \sin x$
13. $\sin x^2$
14. $e^{\cos x}$
15. $\ln \sin^3 x$
16. $\sec x$, i.e. $\dfrac{1}{\cos x}$
17. $\tan x$, i.e. $\dfrac{\sin x}{\cos x}$
18. $\operatorname{cosec} x$
19. $\cot x$

Using the answers to Questions 9 to 12, we can now make a complete list of the derivatives of the basic trig functions:

function	derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

DIFFERENTIATING INVERSE TRIG FUNCTIONS

In the previous section the derivative of a log function was found, without differentiating from first principles, by treating it as the inverse of an exponential function. A similar approach is used to find the derivative of each of the inverse trig functions.

Differentiation of arcsin x

If $y = \arcsin x$, the inverse relationship is $x = \sin y$ and this can be differentiated.

We can also use the property proved on page 340

that $$\frac{dy}{dx} = 1\bigg/\frac{dx}{dy}$$

Hence if $x = \sin y$ then $\frac{dx}{dy} = \cos y \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$

Now $\cos^2 y = 1 - \sin^2 y = 1 - x^2$

Therefore $$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

i.e. $$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{(1-x^2)}}$$

Remember that $y = \arcsin x$ exists only within the range $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$. For these values of y, $\cos y \geqslant 0$, therefore $\cos y = \sqrt{(1-x^2)}$ and not $-\sqrt{(1-x^2)}$

Similarly $$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{(1-x^2)}}$$

and $$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

It is left to the reader to prove these results in the next exercise.

Example 23c

Differentiate with respect to x

(a) $\arcsin e^x$ (b) $x \arctan x$

(a) $\arcsin e^x$ is an inverse trig function of an exponential function so the chain rule can be used.

$$\frac{d}{dx}(\arcsin e^x) = \frac{1}{\sqrt{\{1-(e^x)^2\}}}e^x = \frac{e^x}{\sqrt{(1-e^{2x})}}$$

(b) $$\frac{d}{dx}(x \arctan x) = \arctan x + x\frac{d}{dx}\arctan x = \arctan x + \frac{x}{1+x^2}$$

EXERCISE 23c

1. Show that $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{(1-x^2)}}$

2. Show that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Differentiate each function with respect to x

3. $x^2 \arcsin x$ 4. $(\arctan x)^2$ 5. $\arcsin 3x$

6. $e^{\arctan x}$ 7. $\ln(\arcsin x)$ 8. $\arccos x^3$

EXTENDING THE CHAIN RULE

We have already seen that the chain rule can be used to write down directly the derivative of $y = fg(x)$ where $u = g(x)$

i.e. $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x)$$

e.g. $$\frac{d}{dx}\ln(\tan x) = \frac{1}{\tan x}(\sec^2 x)$$

This direct differentiation of a succession of functions of x can be extended to deal quickly and easily with $y = \text{fgh}(x)$ where $y = \text{f}(u)$, $u = \text{g}(v)$ and $v = \text{h}(x)$,

i.e. $$\frac{d}{dx}\{\text{fgh}(x)\} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = \text{f}'(u)\text{g}'(v)\text{h}'(x)$$

e.g. $$\frac{d}{dx}\left(e^{\sin x^2}\right) = \left(e^{\sin x^2}\right)(\cos x^2)(2x)$$

and $$\frac{d}{dx}\{\ln(\cos x)\}^3 = 3\{\ln(\cos x)\}^2\left(\frac{1}{\cos x}\right)(-\sin x)$$

This extended use of the chain rule allows quite complex functions to be differentiated without *writing down* detailed substitutions.
The reader will find it helpful to express the given function in words before applying the chain rule as this automatically places the operators in the correct order for differentiation.

Consider, for example, the function $\ln(\sin e^x)$, which is

a log function of a sine function of an exponential function.

Using the chain rule to write down the derivative, we apply in succession,

the rule for differentiating a log function
the rule for differentiating a sine function
the rule for differentiating an exponential function

i.e. $$\frac{d}{dx}\{\ln(\sin e^x)\} = \left(\frac{1}{\sin e^x}\right)(\cos e^x)(e^x)$$

The reader has already had practice in using the chain rule to differentiate simple expressions of the type $\text{gf}(x)$. The real benefit of the chain rule is that the derivatives of apparently complicated functions, $\text{fgh}(x)$ or even $\text{fghj}(x)$, can be written down just as easily as for simpler ones.

EXERCISE 23d

Questions 1 to 4 are simple expressions. The reader is advised to revise the use of the chain rule on these easy examples before going on to the rest of the exercise. Some of the remaining questions look more complicated than would be found in most A-level examination papers but they are fun to try and surprisingly easy.

Using the chain rule to differentiate each function

1. $\sqrt{(1 + \ln x)}$ **2.** $\cos(x^2 + 3)$

3. $e^{(x^3 - x)}$ **4.** $\sin(\ln x)$

5. $\ln(\cos x^2)$ **6.** $\left(1 + e^{x^2}\right)^2$

7. $\sqrt{(3 - \sin^2 x)}$ **8.** $\{\ln(\tan x)\}^2$

9. $e^{\sqrt{(2 - x^2)}}$ **10.** $\cos^2(x^2 + 1)$

Compound Functions of More Than One Type

Example 23e

Differentiate $e^{x^2}\sin x$ with respect to x

If $y = e^{x^2}\sin x$ then, using $u = e^{x^2}$ and $v = \sin x$ gives $y = uv$

Now u is a function of a function which can be differentiated mentally giving $\frac{du}{dx} = 2xe^{x^2}$. Then the product formula can be used.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = \{\sin x\}\left\{2xe^{x^2}\right\} + e^{x^2}\cos x$$

EXERCISE 23e

Differentiate each function with respect to x

1. $e^{x^2}\sin x$ **2.** $\dfrac{\cos^2 x}{x}$ **3.** $x(\ln\sin x)$

4. $\dfrac{x^2 - 1}{\ln x}$ **5.** $e^x\sqrt{(x^2 + 2)}$ **6.** $\dfrac{\ln x}{\sqrt{\cos x}}$

7. $(x + 1)e^{\sin x}$ **8.** $\dfrac{x^2}{\sin^2 x}$ **9.** $e^{x^2}\ln x$

MIXED EXERCISE 23

This exercise contains a variety of functions of all the types covered in Chapters 19, 20 and 23. Consider carefully what method to use in each case and do not forget to check first whether a given function has a standard derivative.

Find the derivative of each function in Questions 1 to 26.

1. (a) $-\sin 4\theta$ (b) $\theta - \cos\theta$ (c) $\sin^3\theta + \sin 3\theta$

2. (a) $x^3 + e^x$ (b) $e^{(2x+3)}$ (c) $e^x \sin x$

3. (a) $\ln \frac{1}{3}x^{-3}$ (b) $\ln 2/x^2$ (c) $\ln \sqrt{x}/4$

4. (a) $3\sin x - e^{-x}$ (b) $\ln x^{1/2} - \frac{1}{2}\cos x$
(c) $x^4 + 4e^x - \ln 4x$ (d) $\frac{1}{2}e^{-x} + x^{-1/2} - \ln \frac{1}{2}x$

5. $(x+1)\ln x$ **6.** $\sin^2 3x$ **7.** $(4x-1)^{2/3}$

8. $(3\sqrt{x} - 2x)^2$ **9.** $\dfrac{(x^4-1)}{(x+1)^3}$ **10.** $\dfrac{\ln x}{\ln(x-1)}$

11. $\ln \cot x$ **12.** $x^2 \sin x$ **13.** $\dfrac{e^x}{x-1}$

14. $\dfrac{1+\sin x}{1-\sin x}$ **15.** $x^2\sqrt{(x-1)}$ **16.** $(1-x^2)(1-x)^2$

17. $\ln\sqrt{\dfrac{(x+3)^3}{(x^2+2)}}$ **18.** $\dfrac{e^x \cos x}{x^5}$ **19.** $\sin x \cos^3 x$

20. $e^{\cos^2 x}$ **21.** $e^{\arcsin x}$ **22.** $(\arccos x)^3$

23. $\arctan e^x$ **24.** xa^x **25.** $\log_a(x^2+1)$

26. Find the value(s) of x for which the following functions have stationary values.
(a) $3x - e^x$ (b) $x^2 - 2\ln x$ (c) $\ln 1/x + 4x$

In each Question from 27 to 30, find
(a) the gradient of the curve at the given point,
(b) the equation of the tangent to the curve at that point,
(c) the equation of the normal to the curve at that point.

27. $y = \sin x - \cos x$; $x = \frac{1}{2}\pi$ **28.** $y = x + e^x$; $x = 1$

29. $y = 1 + x + \sin x$; $x = 0$ **30.** $y = 3 - x^2 + \ln x$; $x = 1$

31. Considering only positive values of θ, locate the first two turning points, if there are two, on each of the following curves and determine whether they are maximum or minimum points.
(a) $y = 1 - \sin x$ (b) $y = \frac{1}{2}x + \cos x$ (c) $y = e^x - 3x$

32. Find the coordinates of a point on the curve where the tangent is parallel to the given line.
(a) $y = 3x - 2\cos x$; $y = 4x$ (b) $y = 2\ln x - x$; $y = x$

CHAPTER 24

DIFFERENTIATING IMPLICIT AND PARAMETRIC FUNCTIONS

IMPLICIT FUNCTIONS

All the differentiation carried out so far has involved equations that could be expressed in the form $y = f(x)$
However the equations of some curves, for example $x^2 - y^2 + y = 1$, cannot easily be written in this way, as it is too difficult to isolate y
A relationship of this type, where y is not given explicitly as a function of x, is called an *implicit function*, i.e. it is *implied* in the equation that $y = f(x)$

TO DIFFERENTIATE AN IMPLICIT FUNCTION

The method we use is to differentiate, term by term, with respect to x, but first we need to know how to differentiate terms like y^2 with respect to x

If $\quad g(y) = y^2$ and $y = f(x)$

then $\quad g(y) = \{f(x)\}^2$ which is a function of a function

Using the mental substitution $u = f(x)$ we have

$$\frac{d}{dx}\{f(x)\}^2 = 2\{f(x)\}\left(\frac{d}{dx}f(x)\right) = 2y\left(\frac{dy}{dx}\right) = \left(\frac{d}{dy}g(y)\right)\left(\frac{dy}{dx}\right)$$

In general, $$\frac{d}{dx}g(y) = \left(\frac{d}{dy}g(y)\right)\left(\frac{dy}{dx}\right)$$

e.g. $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ and $\frac{d}{dx}e^y = e^y\frac{dy}{dx}$

We can now differentiate, term by term with respect to x, the example considered above, i.e.

if $$x^2 - y^2 + y = 1$$

then $$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + \frac{dy}{dx} = \frac{d}{dx}(1)$$

$$\Rightarrow \quad 2x - 2y\frac{dy}{dx} + \frac{dy}{dx} = 0$$

Hence $$2x = \frac{dy}{dx}(2y - 1) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{(2y-1)}$$

Examples 24a

1. Differentiate each equation with respect to x and hence find $\frac{dy}{dx}$ in terms of x and y.
   (a) $x^3 + xy^2 - y^3 = 5$ (b) $y = xe^y$

(a) If $x^3 + xy^2 - y^3 = 5$ then, differentiating term by term,

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(xy^2) - \frac{d}{dx}(y^3) = \frac{d}{dx}(5)$$

The term xy^2 is a product so we differentiate it using the product rule, i.e.

$$\frac{d}{dx}(xy^2) = y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2) = y^2 + (x)(2y)\frac{dy}{dx}$$

$$\therefore \quad 3x^2 + y^2 + 2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

Hence $$\frac{dy}{dx} = \frac{(3x^2 + y^2)}{y(3y - 2x)}$$

(b) If $y = xe^y$ then $$\frac{dy}{dx} = \frac{d}{dx}(xe^y)$$

$$= e^y\frac{d}{dx}(x) + x\frac{d}{dx}(e^y)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^y + xe^y\frac{dy}{dx}$$

Hence $$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$$

2. If $e^x y = \sin x$ show that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

In a problem of this type it is tempting to express $e^x y = \sin x$ in the form $y = e^{-x}\sin x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and show that they satisfy the given equation, which is called a *differential equation*. However it is much more direct to differentiate the implicit equation as given.

Differentiating $e^x y = \sin x$ w.r.t. x gives

$$e^x y + e^x\frac{dy}{dx} = \cos x$$

Differentiating again w.r.t. x gives

$$\left(e^x y + e^x\frac{dy}{dx}\right) + \left(e^x\frac{dy}{dx} + e^x\frac{d^2y}{dx^2}\right) = -\sin x = -e^x y$$

Hence $e^x\dfrac{d^2y}{dx^2} + 2e^x\dfrac{dy}{dx} + 2e^x y = 0$

There is no finite value of x for which $e^x = 0$ so we can divide the equation by e^x

i.e. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

---

3. Find the equation of the tangent at the point (x_1, y_1) to the curve with equation $x^2 - 2y^2 - 6y = 0$

To find the equation of the tangent we need the gradient of the curve and in this case it must be found by implicit differentiation.

$$x^2 - 2y^2 - 6y = 0 \quad\Rightarrow\quad 2x - 2\left(2y\frac{dy}{dx}\right) - 6\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{(3+2y)}$$

$\therefore$ the gradient of the tangent at the point (x_1, y_1) is $\dfrac{x_1}{(3+2y_1)}$ and the equation of the tangent is $y - y_1 = \dfrac{x_1}{(3+2y_1)}(x - x_1)$ which simplifies to $xx_1 - 2yy_1 - 3(y + y_1) = x_1^2 - 2y_1^2 - 6y_1$

Because (x_1, y_1) is on the given curve, $x_1^2 - 2y_1^2 - 6y_1 = 0$, so the equation of the tangent becomes

$$xx_1 - 2yy_1 - 3(y + y_1) = 0$$

Note that in the last example the equation of the curve can be converted into the equation of the tangent by changing x^2 into xx_1, y^2 into yy_1 and y into $\frac{1}{2}(y + y_1)$.

In fact, for any curve whose equation is of degree two, the equation of the tangent at (x_1, y_1) can be written down directly by making the replacements listed above, together with two more, i.e.

$$x \rightarrow \tfrac{1}{2}(x + x_1) \quad\text{and}\quad xy \rightarrow \tfrac{1}{2}(xy_1 + x_1y)$$

This property can be applied to advantage when the numerical values of the coordinates of the point of contact are known,

e.g. the equation of the tangent at the point $(1, -1)$ to the curve

$$3x^2 - 7y^2 + 4xy - 8x = 0$$

can be written down as

$$3x(1) - 7y(-1) + 2\{x(-1) + (1)y\} - 4(x + 1) = 0$$

i.e. $\qquad 9y - 3x = 4$

Question 18 in the following exercise gives the reader the opportunity to justify using these mechanical replacements in the equation of a curve, to give the equation of a tangent.

Note that, although this method allows the equation of a tangent to be *written down*, its use is not suitable when the *derivation* of the equation is required.

EXERCISE 24a

Differentiate the following equations with respect to x

1. $x^2 + y^2 = 4$
2. $x^2 + xy + y^2 = 0$
3. $x(x+y) = y^2$
4. $\frac{1}{x} + \frac{1}{y} = e^y$
5. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$
6. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
7. $\sin x + \sin y = 1$
8. $\sin x \cos y = 2$
9. $xe^y = x + 1$
10. $\sqrt{\{(1+y)(1+x)\}} = x$
11. Find $\frac{dy}{dx}$ as a function of x if $y^2 = 2x + 1$
12. Find $\frac{d^2y}{dx^2}$ as a function of x if $\sin y + \cos y = x$
13. Find the gradient of $x^2 + y^2 = 9$ at the points where $x = 1$
14. If $y \cos x = e^x$ show that $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} - 2y = 0$
15. Write down the equation of the tangent to
    (a) $x^2 - 3y^2 = 4y$ (b) $x^2 + xy + y^2 = 3$
    at the point (x_1, y_1).
16. Show that the equation of the tangent to $x^2 + xy + y = 0$ at the point (x_1, y_1) is
$$x(2x_1 + y_1) + y(x_1 + 1) + y_1 = 0$$
17. Write down the equation of the tangent at $(1, \frac{1}{3})$ to the curve whose equation is
$$2x^2 + 3y^2 - 3x + 2y = 0$$
18. Show that the equation of the tangent at (x_1, y_1) to the curve $ax^2 + by^2 + cxy + dx = 0$ is
$$axx_1 + byy_1 + \tfrac{1}{2}c(xy_1 + yx_1) + \tfrac{1}{2}d(x + x_1) = 0$$

19. Given that $\sin y = 2\sin x$ show that $\left(\frac{dy}{dx}\right)^2 = 1 + 3\sec^2 y$. By differentiating this equation with respect to x show that
$$\frac{d^2y}{dx^2} = 3\sec^2 y \tan y$$
and hence that $\cot y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 1 = 0$

LOGARITHMIC DIFFERENTIATION

The advantage of simplifying a logarithmic expression before attempting to differentiate it has already been noted.

We are now going to examine some equations which are awkward to differentiate as they stand but which are much easier to deal with if we first take logs of both sides of the equation. The types of equation where this method is particularly important are

1) those in which the variable is an index

2) complicated functions involving fractions, roots, products, etc.

The process used is called *logarithmic differentiation*.

Examples 24b

1. Differentiate x^x with respect to x

$$y = x^x$$
$$\ln y = x \ln x$$

thus $\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$

$\Rightarrow$ $\frac{dy}{dx} = y(1 + \ln x)$

Therefore $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$

2. If $y = \dfrac{x\sqrt{(x^2-1)}}{x+2}$ find $\dfrac{dy}{dx}$

$$y = \frac{x\sqrt{(x^2-1)}}{x+2} \quad \Rightarrow \quad \ln y = \ln x + \tfrac{1}{2}\ln(x^2-1) - \ln(x+2)$$

$$\therefore \qquad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2-1}\right) - \frac{1}{x+2}$$

$$\Rightarrow \qquad \frac{dy}{dx} = (y)\left(\frac{x^3+4x^2-2}{x(x^2-1)(x+2)}\right)$$

$$= \left(\frac{x\sqrt{(x^2-1)}}{x+2}\right)\left(\frac{x^3+4x^2-2}{x(x^2-1)(x+2)}\right)$$

$$\text{i.e.} \qquad \frac{dy}{dx} = \frac{x^3+4x^2-2}{(x+2)^2\sqrt{(x^2-1)}}$$

---

3. Differentiate the equation $x = y^x$ with respect to x

$$x = y^x \quad \Rightarrow \quad \ln x = x\ln y$$

$$\therefore \qquad \frac{1}{x} = \ln y + (x)\left(\frac{1}{y}\frac{dy}{dx}\right)$$

$$\text{i.e.} \qquad x^2\frac{dy}{dx} + xy\ln y = y$$

Note that in the third example it is not easy to express $\dfrac{dy}{dx}$ as a function of x, because it is difficult in the first place to find y in terms of x. So although the usual practice is to give a derived function in terms of x it is not always possible, or sensible, to do so.

Differentiation of a^x where a is a Constant

The basic rule for differentiating an exponential function applies when the base is e but not for any other base. So for a^x, where we need another approach, we use logarithmic differentiation.

Using $\qquad y = a^x$ gives $\ln y = x\ln a$

Differentiating w.r.t. x gives $\qquad \dfrac{1}{y}\dfrac{dy}{dx} = \ln a$

Hence $\qquad \dfrac{dy}{dx} = y\ln a = a^x\ln a$

i.e. $\qquad \dfrac{d}{dx}a^x = a^x\ln a$

This result is quotable.

EXERCISE 24b

Differentiate each equation with respect to x

1. $x^y = e^x$ 2. $x^y = (y+1)$ 3. $y = (x+x^2)^x$

Find $\dfrac{dy}{dx}$ if

4. $y = \dfrac{x}{(x+2)(x-4)}$
5. $y = \dfrac{x^2}{(x-1)(x-3)}$
6. $y = (1-x)^5(x^2+2)$
7. $y = \sqrt{\{(x+1)(x-3)^3\}}$
8. $y = \dfrac{x}{(x+2)^2(x^2-1)}$
9. $y = \dfrac{1}{\sqrt{\{(x^2+4)(3x-2)\}}}$

PARAMETRIC EQUATIONS

SG

Sometimes a direct relationship between x and y is awkward to analyse; in such cases it is often easier to express x and y each in terms of a third variable, called a *parameter*.

Consider, for example, the equations

$$x = t^3$$
$$y = t^2 - t$$

The direct relationship between x and y can be found by eliminating t from these two *parametric equations*. It is $y = x^{2/3} - x^{1/3}$

While the gradient and general shape of the curve, as well as the equation of a tangent or a normal, can be obtained from the cartesian equation, they are often more simply derived from the parametric equations.

Sketching a Curve Given in Parametric Form

To get an idea of the shape of the curve whose parametric equations are

$$x = t^3 \quad \text{and} \quad y = t^2 - t$$

we can

1) *plot* a small number of points by calculating the values of x and y that correspond to certain chosen values of t,

t	−2	−1	0	1	2
x	−8	−1	0	1	8
y	6	2	0	0	2

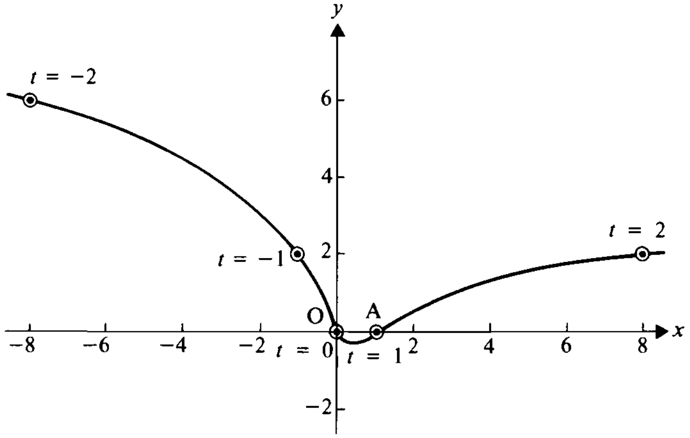
2) examine the behaviour of the curve as $t \to \pm\infty$,

as $t \to \infty$, $x \to \infty$ and $y \to \infty$

as $t \to -\infty$, $x \to -\infty$ and $y \to \infty$

There is no finite value of t for which either x or y is undefined so it is reasonable to assume that the curve is continuous.

Based on all this information, a sketch of the curve can now be made.



The location of turning points is a further aid to curve sketching. To use this we need to be able to find the gradient of a curve given parametrically.

FINDING THE GRADIENT FUNCTION USING PARAMETRIC EQUATIONS

If both x and y are given as functions of t then a small increase of δt in the value of t results in corresponding small increases of δx and δy in the values of x and y

As $\delta t \to 0$, δx and δy also approach zero, therefore

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

But $\dfrac{dt}{dx} = 1 \Big/ \dfrac{dx}{dt}$

Therefore $$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Hence, for the parametric equations considered above, i.e. $x = t^3$ and $y = t^2 - t$, we have

$$\frac{dy}{dx} = \frac{(2t-1)}{3t^2}$$

Each point on the curve is defined by a value of t which also gives the value of $\dfrac{dy}{dx}$ at that point. Similarly, the value(s) of t where $\dfrac{dy}{dx}$ has a special value lead to the coordinates of the relevant point(s) on the curve.

At turning point(s) $\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad 2t - 1 = 0 \quad \Rightarrow \quad t = \frac{1}{2}$

$t = \frac{1}{2} \Rightarrow x = \frac{1}{8}$ and $y = -\frac{1}{4}$ therefore $(\frac{1}{8}, -\frac{1}{4})$ is a turning point.

In this case the curve sketched before the turning point was known is now only marginally improved, showing that it is not always necessary to carry out this investigation. Some curves, on the other hand, would be difficult to draw without finding their turning points.

Examples 24c

1. Find the cartesian equation of the curve whose parametric equations are

(a) $x = t^2$, $y = 2t$ (b) $x = \cos\theta$, $y = \sin\theta$ (c) $x = 2t$, $y = 2/t$

(a) $y = 2t \quad \Rightarrow \quad t = \frac{1}{2}y$

$\therefore \quad x = t^2 \quad \Rightarrow \quad x = (\frac{1}{2}y)^2 = \frac{1}{4}y^2 \quad \Rightarrow \quad y^2 = 4x$

(b) Using $\cos^2\theta + \sin^2\theta = 1$ where $\cos\theta = x$ and $\sin\theta = y$ gives

$$x^2 + y^2 = 1$$

(c) $y = 2/t \quad \Rightarrow \quad t = 2/y$

$\therefore \quad x = 2t \quad \Rightarrow \quad x = 4/y \quad \Rightarrow \quad xy = 4$

2. Find the stationary point on the curve whose parametric equations are $x = t^3$, $y = (t+1)^2$ and determine its nature. Sketch the curve, showing the stationary point and the behaviour of the curve as $x \to \pm\infty$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2(t+1)}{3t^2}$$

At stationary points $\frac{dy}{dx} = 0$ i.e. $t = -1$

When $t = -1$, $x = -1$ and $y = 0$

Therefore the stationary point is $(-1, 0)$

To determine the nature of the stationary point we examine the sign of $\frac{dy}{dx}$ in the neighbourhood of the point by first choosing appropriate values for x and then finding the corresponding values of t

The equations $x = t^3$ and $y = (t+1)^2$ show that there is no finite value of t for which either x or y is not defined, so the curve is continuous. Also there is no other stationary point.

Value of t	$-\sqrt[3]{2}$	-1	0
Sign of $\frac{dy}{dx}$	$-$ \	0 —	$+$ /

Hence $(-1, 0)$ is a minimum point.

Also

t	-3	-2	1	2	3
x	-27	-8	1	8	27
y	4	1	4	9	16

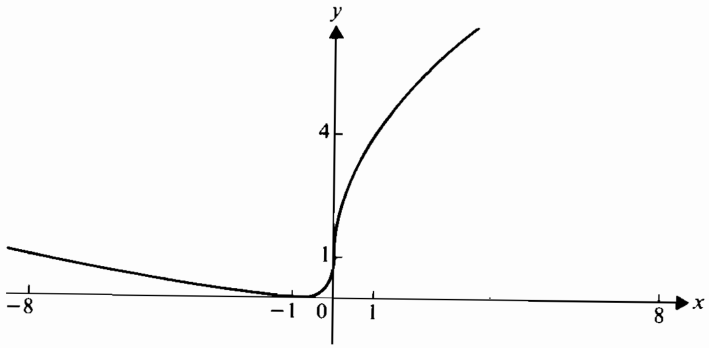
The equations $x = t^3$ and $y = (t+1)^2$ show that there is no finite value of t for which either x or y is undefined, so the curve is continuous.

Also we see that,

as $t \to \infty$, $x \to \infty$ and $y \to \infty$

and as $t \to -\infty$, $x \to -\infty$ and $y \to \infty$

The curve can now be sketched.



3. The parametric equations of a curve are

$$x = \sin^2\theta, \quad y = 1 + 2\sin\theta$$

Show that the equation of the tangent to the curve at the point $P(\sin^2\theta, 1 + 2\sin\theta)$ is $y = x + \sin\theta + \sin^2\theta$ and find the point(s) where the tangent is parallel to the y-axis.

$$y = 1 + 2\sin\theta \quad \text{and} \quad x = \sin^2\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2\cos\theta \quad \text{and} \quad \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2\cos\theta}{2\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta} \qquad \text{(provided that } \cos\theta \neq 0)$$

The equation of the tangent is given by $y - y_1 = m(x - x_1)$, i.e.

$$y - (1 + 2\sin\theta) = \frac{1}{\sin\theta}(x - \sin^2\theta)$$

$$\Rightarrow \quad y\sin\theta = x - \sin^2\theta + \sin\theta + 2\sin^2\theta$$

i.e. $$y\sin\theta = x + \sin\theta + \sin^2\theta$$

*This is the **general** equation of the tangent because it is the equation of the tangent to the given curve at **any** point on the curve.*

If the tangent is parallel to the y-axis, the value of $\frac{dy}{dx}$ is infinitely large,

i.e. $$\frac{1}{\sin\theta} \to \infty \quad \Rightarrow \quad \sin\theta = 0$$

$$\Rightarrow \quad x = 0 \quad \text{and} \quad y = 1$$

Therefore the tangent is parallel to the y-axis at the point $(0, 1)$

---

4. Find the equation of the normal to the curve $x = t^2$, $y = t + 2/t$, at the point where $t = 1$. Show, without sketching the curve, that this normal does not cross the curve again.

$$x = t^2 \quad \text{and} \quad y = t + \frac{2}{t} \quad \text{give} \quad \frac{dy}{dt} = 1 - \frac{2}{t^2} \quad \text{and} \quad \frac{dx}{dt} = 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1 - 2/t^2}{2t} = \frac{t^2 - 2}{2t^3}$$

When $t = 1$; $x = 1$, $y = 3$ and $\frac{dy}{dx} = -\frac{1}{2}$

Therefore the gradient of the normal at $P(1, 3)$ is $\frac{-1}{-\frac{1}{2}} = 2$

The equation of this normal is $\quad y - 3 = 2(x - 1)$

i.e. $\quad y = 2x + 1$

All points for which $x = t^2$ and $y = t + 2/t$ are on the given curve, therefore, for any point that is on both the curve and the normal, these coordinates also satisfy the equation of the normal,
i.e., at points common to the curve and the normal,

$$t + \frac{2}{t} = 2t^2 + 1 \quad \Rightarrow \quad 2t^3 - t^2 + t - 2 = 0 \qquad [1]$$

If a cubic equation can be factorised, each factor equated to zero gives a root of the equation, just as in the case of a quadratic equation.

Now we know that one point where the curve and normal meet is the point where $t = 1$, so $t = 1$ is a root of [1] and $(t - 1)$ is a factor of the LHS,

i.e. $$(t - 1)(2t^2 + t + 2) = 0$$

Therefore, at any other point where the normal meets the curve, the value of t is a root of the equation $2t^2 + t + 2 = 0$

Checking the value of $b^2 - 4ac$ shows that this equation has no real roots so there are no more points where the normal meets the curve.

EXERCISE 24c

1. Find the gradient function of each of the following curves in terms of the parameter.
(a) $x = 2t^2$, $y = t$ (b) $x = \sin\theta$, $y = \cos\theta$
(c) $x = t$, $y = 4/t$

2. If $x = \dfrac{t}{1-t}$ and $y = \dfrac{t^2}{1-t}$, find $\dfrac{dy}{dx}$ in terms of t. What is the value of $\dfrac{dy}{dx}$ at the point where $x = 1$?

3. (a) If $x = t^2$ and $y = t^3$, find $\dfrac{dy}{dx}$ in terms of t
(b) If $y = x^{3/2}$, find $\dfrac{dy}{dx}$
(c) Explain the connection between these two results.

4. Find the cartesian equation of each of the curves given in Question 1 and hence find $\dfrac{dy}{dx}$. Show in each case that $\dfrac{dy}{dx}$ agrees with the gradient function found in Question 1.

5. Find the turning points of the curve whose parametric equations are $x = t$, $y = t^3 - t$, and distinguish between them.

6. A curve has parametric equations $x = \theta - \cos\theta$, $y = \sin\theta$
Find the coordinates of the points at which the gradient of this curve is zero.

7. Find the equation of the tangent to the curve $x = t^2$, $y = 4t$ at the point where $t = -1$

8. Find the equation of the general normal to the curve $x = t$, $y = 1/t$

9. Find the equation of the general tangent to the curve $x = t^2$, $y = 4t$

10. Find the equation of the normal to the curve $x = \cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{1}{4}\pi$. Find the coordinates of the point where this normal cuts the curve again.

11. The parametric equations of a curve are $x = e^t$ and $y = \sin t$.
(a) Find the gradient function in terms of t
(b) Find the equation of the curve in the form $y = f(x)$
(c) Find $\dfrac{dy}{dx}$
(d) Eliminate x from the result obtained in part (c) and compare with the answer to part (a).

12. The parametric equations of a curve are $x = t$ and $y = 1/t$
Find the general equation of the tangent to this curve (i.e. the equation at the point $(t, 1/t)$).
Find, in terms of t, the coordinates of the points at which the tangent cuts the x and y axes. Hence show that the area enclosed by this tangent and the coordinate axes is constant.

13. A curve has parametric equations $x = t^2$, $y = 4t$. Find the equation of the normal to this curve at the point $(t^2, 4t)$.
Find the coordinates of the point where this normal cuts the coordinate axes. Hence find, in terms of t, the area of the triangle enclosed by the normal and the axes.

MIXED EXERCISE 24

1. Differentiate with respect to x
(a) y^4 (b) xy^2 (c) $1/y$ (d) $x\ln y$
(e) $\sin y$ (f) e^y (g) $y\cos x$ (h) $y\cos y$

In each Question from 2 to 13, find $\dfrac{dy}{dx}$

2. $x^2 - 2y^2 = 4$

3. $1/x + 1/y = 2$

4. $x^2y^3 = 9$

5. $y = \dfrac{x^4(2-x^2)}{(1+x)^3}$

6. $yx^5 = \dfrac{(x-1)^4}{(x+3)}$

7. $x^2y^2 = \dfrac{(y+1)}{(x+1)}$

8. $x = t^2$, $y = t^3$

9. $x = (t+1)^2$, $y = t^2 - 1$

10. $x = \sin^2\theta$, $y = \cos^3\theta$

11. $x = 4t$, $y = 4/t$

12. $y^2 - 2xy + 3y = 7x$ **13.** $x = \dfrac{t}{1-t}, \quad y = \dfrac{t^2}{1-t}$

14. If $x = \sin t$ and $y = \cos 2t$, find $\dfrac{dy}{dx}$ in terms of x and prove that $\dfrac{d^2y}{dx^2} + 4 = 0$

15. If $x = e^t - t$ and $y = e^{2t} - 2t$, show that $\dfrac{dy}{dx} = 2(e^t + 1)$

CHAPTER 25

APPLICATIONS OF DIFFERENTIATION

RATE OF INCREASE

When the variation of y depends upon another variable x, then

$\dfrac{dy}{dx}$ gives the rate at which y increases compared with x

This fact forms the basis of methods which can be used to analyse practical situations in which two variables are related.

SMALL INCREMENTS

Consider two variables, x and y, related by the equation $y = f(x)$

If x increases by a small increment δx

then y increases by a corresponding small amount δy

Now $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

so,

provided that δx is small, $\dfrac{\delta y}{\delta x} \approx \dfrac{dy}{dx} \quad \Rightarrow \quad \delta y \approx \dfrac{dy}{dx}(\delta x)$

or, alternatively, $\delta\{f(x)\} \approx f'(x)\,\delta x$

This approximation can be used to estimate the value of a function close to a known value, i.e. $y + \delta y$ can be estimated if y is known at a particular value of x

For example, knowing that $\ln 1 = 0$, an approximate value for $\ln 1.1$ can be found from $y = \ln x$ as follows

$$y = \ln x \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

so
$$\delta y \approx \frac{dy}{dx}(\delta x) = \frac{1}{x}(\delta x)$$

Now x increases from 1 to 1.1, i.e. $\delta x = 0.1$

$\therefore$
$$\delta y \approx \tfrac{1}{1}(0.1)$$

Hence
$$\ln 1.1 = y + \delta y \approx (\ln 1) + 0.1$$

but $\ln 1 = 0$

$\therefore$
$$\ln 1.1 \approx 0.1$$

Examples 25a

1. Using $y = \sqrt{x}$, estimate the value of $\sqrt{101}$

$$y = \sqrt{x} \quad \Rightarrow \quad \frac{dy}{dx} = \tfrac{1}{2}x^{-1/2}$$

$$\delta y \approx \frac{dy}{dx}\delta x \quad \text{gives} \quad \delta y \approx \frac{1}{2\sqrt{x}}\delta x$$

So that the value of y can be written down, the value we take for x must be a number with a known square root.

Taking $x = 100$, $y = \sqrt{100}$ and $\delta x = 1$ gives

$$\delta y \approx \frac{1}{2\sqrt{100}}(1) = \frac{1}{20}$$

Then
$$\sqrt{101} = y + \delta y \approx \sqrt{100} + \tfrac{1}{20}$$

i.e. $\sqrt{101} \approx 10.05$

2. Given that $1° = 0.0175$ rad and that $\cos 30° = 0.8660$, use $f(\theta) = \sin\theta$ to find an approximate value for

   (a) $\sin 31°$ (b) $\sin 29°$

Using $\delta\{f(\theta)\} \approx f'(\theta)\,\delta\theta$ gives $\delta(\sin\theta) \approx (\cos\theta)\,\delta\theta$

(a) Taking $\theta = \frac{1}{6}\pi$, $\sin\theta = \frac{1}{2}$ and $\delta\theta = 0.0175$ gives

$$\delta(\sin\theta) \approx (\cos\tfrac{1}{6}\pi)(0.0175) \quad \text{and} \quad \sin 31° \approx \sin 30° + \delta(\sin\theta)$$

Hence
$$\sin 31° \approx 0.5 + (0.8660)(0.0175)$$

i.e.
$$\sin 31° \approx 0.515$$

(b) Again $\theta = \frac{1}{6}\pi$ and $\sin\theta = \frac{1}{2}$, but this time, because the angle *decreases* from 30° to 29°, $\delta\theta = -0.0175$

$$\delta(\sin\theta) \approx (\cos\tfrac{1}{6}\pi)(-0.0175)$$

Hence $\sin 29° \approx \sin 30° + \delta(\sin\theta) = 0.5 + (0.8660)(-0.0175)$

i.e. $\sin 29° \approx 0.485$

Small Percentage Increases

In order to adapt the method used above to estimate the percentage change in a dependent variable caused by a small change in the independent variable we use the additional fact that

$$\text{if } x \text{ increases by } r\% \text{ then } \delta x = \frac{r}{100}(x)$$

and the corresponding percentage increase in y is $\dfrac{\delta y}{y} \times 100$

Examples 25a (continued)

3. The period, T, of a simple pendulum is calculated from the formula $T = 2\pi\sqrt{(l/g)}$, where l is the length of the pendulum and g is the constant gravitational acceleration. Find the percentage change in the period caused by lengthening the pendulum by 2 per cent.

$$T = 2\pi\sqrt{\left(\frac{l}{g}\right)} \quad \Rightarrow \quad \frac{dT}{dl} = \left(\frac{2\pi}{\sqrt{g}}\right)\left(\tfrac{1}{2}l^{-1/2}\right) = \frac{\pi}{\sqrt{(lg)}}$$

The length *increases* so the small increment in the length is positive and is given by

$$\delta l = \left(\tfrac{2}{100}\right)(l) = \tfrac{1}{50}l$$

Using $\delta T \approx \dfrac{dT}{dl}\delta l$ gives

$$\delta T \approx \left(\tfrac{1}{50}l\right)\{\pi/\sqrt{(lg)}\} = \left(\tfrac{1}{50}\right)\pi\sqrt{(l/g)}$$

The percentage change in the period is given by $\dfrac{\delta T}{T} \times 100$

i.e. $\tfrac{1}{50}\{\pi\sqrt{(l/g)}\} \div \{2\pi\sqrt{(l/g)}\} \times 100\% = 1\%$

This is a positive change, so we see that the period *increases* by 1%

EXERCISE 25a

1. Using $y = \sqrt[3]{x}$, find, *without using a calculator*, an approximate value for
   (a) $\sqrt[3]{1001}$ (b) $\sqrt[3]{9}$ (c) $\sqrt[3]{63}$
   Work to 6 d.p.
   Now use a calculator to find the accuracy of each approximation.

2. Given that $1° = 0.0175$ rad, $\sin 60° = 0.8660$ and $\sin 45° = 0.7071$, use $f(\theta) = \cos\theta$ to find an approximate value for
   (a) $\cos 31°$ (b) $\cos 59°$ (c) $\cos 44°$

3. If $f(x) = x\ln(1+x)$ find an approximation for the increase in $f(x)$ when x increases by δx
   Hence estimate the value of $\ln(2.1)$ given that $\ln 2 = 0.6931$

4. If $y = \tan x$ find an approximation for δy when x is increased by δx and use it to estimate, in terms of π, the value of $\tan \frac{9}{32}\pi$

5. Use $f(x) = \sqrt[5]{x}$ to find the approximate value of $\sqrt[5]{33}$

6. Given that $y = \sqrt{\left(\dfrac{x-2}{x-1}\right)}$ determine the value of $\dfrac{dy}{dx}$ when $x = 3$ Deduce the approximate increase in the value of y when x increases from 3 to $3 + \alpha$ where α is small.

COMPARATIVE RATES OF CHANGE

Some problems involving the rate of change of one variable compared with another do not provide a direct relationship between these two variables. Instead, each of them is related to a third variable.

The identity $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$ is useful in solving problems of this type.

Suppose, for instance, that the radius, r, of a circle is increasing at a rate of 1 mm per second. This means that $\dfrac{dr}{dt} = 1$. The rate at which the area, A, of the circle is increasing is $\dfrac{dA}{dt}$

We do not know A as a function of t but we do know that $A = \pi r^2$ and that

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Then $\dfrac{dA}{dt}$ can be calculated, as $\dfrac{dr}{dt}$ is given and $\dfrac{dA}{dr}$ can be found from $A = \pi r^2$

In some cases, more than three variables may be involved but the same approach is used with a relationship of the form

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dq} \times \frac{dq}{dx}$$

Examples 25b

1. A spherical balloon is being blown up so that its volume increases at a constant rate of $1.5\ \text{cm}^3/\text{s}$. Find the rate of increase of the radius when the volume of the balloon is $56\ \text{cm}^3$.

If, at time t, the radius of the balloon is r and the volume is V then

$$V = \tfrac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

We are looking for $\dfrac{dr}{dt}$ and we are given $\dfrac{dV}{dt} = 1.5$ so we use

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{dV}{dt} \div \frac{dV}{dr} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{1.5}{4\pi r^2} = \frac{3}{8\pi r^2}$$

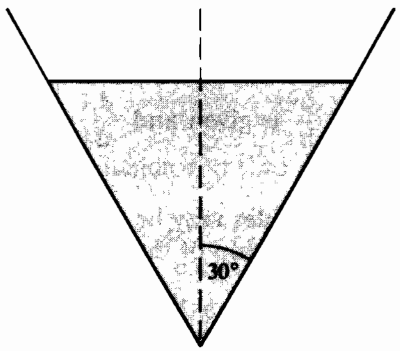
Now substituting $V = 56$ in $V = \frac{4}{3}\pi r^3$ gives $r = 2.373$ to 4 s.f.

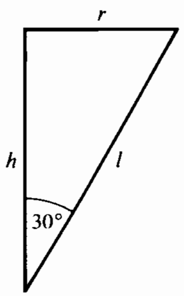
Therefore, when $V = 56$, $\quad \dfrac{dr}{dt} = \dfrac{3}{8\pi(2.373)^2} = 0.021\,20$

i.e. the radius is increasing at a rate of 0.0212 cm/s (correct to 3 s.f.)

---

2. A vessel containing liquid is in the form of an inverted hollow cone with a semi-vertical angle of 30°. The liquid is running out of a small hole at the vertex of the cone, at a constant rate of $3\ \text{cm}^3/\text{s}$. Find the rate at which the surface area which is in contact with the liquid is changing, at the instant when the volume of liquid left in the vessel is $81\pi\ \text{cm}^3$.





At any time, t, the volume, V, of liquid in the vessel is given by

$$V = \tfrac{1}{3}\pi r^2 h$$

In this equation, V, r and h are all variables so it cannot yet be differentiated.

Using $r = l \sin 30° = \frac{1}{2}l$ and $h = l\cos 30° = \frac{1}{2}\sqrt{3}l$ gives

$$V = \tfrac{1}{24}\pi l^3\sqrt{3} \quad \Rightarrow \quad \frac{dV}{dl} = \tfrac{1}{8}\pi l^2\sqrt{3}$$

The surface area, S, in contact with the liquid at any time is given by

$$S = \pi r l = \tfrac{1}{2}\pi l^2 \quad \Rightarrow \quad \frac{dS}{dl} = \pi l$$

Now $\dfrac{dS}{dt}$ is required and $\dfrac{dV}{dt}$ is given as -3 (negative because the volume is decreasing). There is no equation from which $\dfrac{dS}{dV}$ can be obtained so this time we use a three-step link,

i.e. $$\frac{dS}{dt} = \frac{dS}{dl} \times \frac{dl}{dV} \times \frac{dV}{dt} = \frac{dS}{dl} \div \frac{dV}{dl} \times \frac{dV}{dt}$$

$$= (\pi l)\left(\frac{8}{\pi l^2\sqrt{3}}\right)(-3) = \frac{-8\sqrt{3}}{l}$$

At the instant that the value of $\dfrac{dS}{dt}$ is required, $V = 81\pi$

i.e. $\quad \frac{1}{24}\pi l^3\sqrt{3} = 81\pi \quad \Rightarrow \quad l = 6\sqrt{3}$

At this instant, $\quad \dfrac{dS}{dt} = \dfrac{-8\sqrt{3}}{6\sqrt{3}} = -\dfrac{4}{3}$

i.e. the wet surface area is decreasing at $1\frac{1}{3}\ \text{cm}^2/\text{s}$.

EXERCISE 25b

1. Ink is dropped on to blotting paper forming a circular stain which increases in area at a rate of $2.5\ \text{cm}^2/\text{s}$. Find the rate at which the radius is changing when the area of the stain is $16\pi\ \text{cm}^2$.

2. The surface area of a cube is increasing at a rate of $10\ \text{cm}^2/\text{s}$. Find the rate of increase of the volume of the cube when the edge is of length 12 cm.

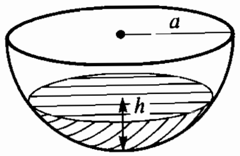
3. The circumference of a circular patch of oil on the surface of a pond is increasing at 2 m/s. When the radius is 4 m, at what rate is the area of the oil changing?

4. A container in the form of a right circular cone of height 16 cm and base radius 4 cm is held vertex downward and filled with liquid. If the liquid leaks out from the vertex at a rate of $4\text{ cm}^3/\text{s}$, find the rate of change of the depth of the liquid in the cone when half of the liquid has leaked out.

5. A right circular cone has a constant volume. The height h and the base radius r can both vary. Find the rate at which h is changing with respect to r at the instant when r and h are equal.

6. The radius of a hemispherical bowl is a cm. The bowl is being filled with water at a steady rate of $3\pi a^3\text{ cm}^3$ per minute. Find, in terms of a, the rate at which the water is rising when the depth of water in the bowl is $\frac{1}{2}a$ cm.

(The volume of the shaded part of this hemisphere is $\frac{1}{3}\pi h^2(3a - h)$)



CONSOLIDATION D

SUMMARY

Throughout this summary, a and b represent constant quantities.

TRIGONOMETRY

General Solutions

If $\cos\theta = \cos\alpha$ then $\theta = 2n\pi \pm \alpha$

If $\tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$

If $\sin\theta = \sin\alpha$ then $\theta = 2n\pi + \alpha$ or $(2n-1)\pi - \alpha$

Factor Formulae

Before using these formulae it is wise to ensure that $A > B$, rearranging the given expression if necessary.

In the form for converting a sum or difference into a product,

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

In the form for converting a product into a sum or difference,

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

Expressing $a\cos\theta \pm b\sin\theta$ as a Single Term

For various values of a and b, $a\cos\theta \pm b\sin\theta$ can be expressed as

$$r\cos(\theta \pm \alpha) \quad \text{or} \quad r\sin(\theta \pm \alpha)$$

where $r = \sqrt{(a^2 + b^2)}$ and $\tan\alpha$ is either a/b or b/a

Small Angles

When θ is a small angle measured in radians, then

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

and $$\lim_{\theta \to 0}\left[\frac{\sin\theta}{\theta}\right] = 1$$

LOGARITHMS

For any positive numbers a and b the formula for changing the base of a logarithm is $$\log_a x = \frac{\log_b x}{\log_b a}$$

Natural logarithms obey the standard laws of logarithms.

$$\ln a + \ln b = \ln(ab)$$
$$\ln a - \ln b = \ln(a/b)$$
$$\ln a^p = p\ln a$$
$$\log_p a = \frac{\ln a}{\ln p}$$

FUNCTIONS

Exponential functions:

$f:x \to a^x$ is an exponential function

$f:x \to e^x$ is *the* exponential function: $e = 2.71828\ldots$

Logarithmic functions:

$f:x \to \log_n x$, is a logarithmic function

$f:x \to \log_e x = \ln x$ is the natural logarithmic function

Inverse trigonometric functions:

The inverse trig functions are $\arcsin x$, $\arccos x$, $\arctan x$

$\arcsin x$ means 'the angle in the range $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$ whose sine is x'

$\arccos x$ means 'the angle in the range $0 \leqslant \theta \leqslant \pi$ whose cosine is x'

$\arctan x$ means 'the angle in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ whose tangent is x'

DIFFERENTIATION

Standard Results

$f(x)$	$\frac{d}{dx}f(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

$f(x)$	$\frac{d}{dx}f(x)$
e^x	e^x
$\ln x$	$1/x$
a^x	$a^x \ln a$
$\arcsin x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\arccos x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\arctan x$	$\frac{1}{(1+x^2)}$

Further Quotable Results

$f(x)$	$\frac{d}{dx}f(x)$
$\sin ax$	$a\cos ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$1/x$ (*not* a/x)
$\arcsin \frac{x}{a}$	$\frac{1}{\sqrt{(a^2-x^2)}}$
$\arctan \frac{x}{a}$	$\frac{a}{(a^2+x^2)}$

COMPOUND FUNCTIONS

If u and v are both functions of x then

$$y = uv \quad \Rightarrow \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \Big/ v^2$$

If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

This process, known as the Chain Rule, can be extended, e.g. if $y = f(u)$, $u = g(v)$, $v = h(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dv} \times \dfrac{dv}{dx}$

IMPLICIT DIFFERENTIATION

When y cannot be isolated, each term can be differentiated with respect to x,

e.g. $\dfrac{d}{dx}(y^2) = (2y)\left(\dfrac{dy}{dx}\right)$ and $\dfrac{d}{dx}(xy) = y + (x)\left(\dfrac{dy}{dx}\right)$ (by product rule)

LOGARITHMIC DIFFERENTIATION

Sometimes it is easier to differentiate $y = f(x)$ with respect to x if we first take logs of both sides.

When doing this remember that $\dfrac{d}{dx}(\ln y) = \dfrac{1}{y}\dfrac{dy}{dx}$

This process is called logarithmic differentiation. It is *essential* when differentiating functions such as x^x

Parametric Differentiation

If $y = f(t)$ and $x = g(t)$ then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

Small Increments

If $y = f(x)$ and x increases by a small amount, δx, then

$$\delta y \approx \left(\frac{dy}{dx}\right)(\delta x)$$

Comparative Rates of Change

If a quantity p depends on a quantity q and the rate at which q increases with time t is known, then

$$\frac{dp}{dt} = \frac{dp}{dq} \times \frac{dq}{dt}$$

MULTIPLE CHOICE EXERCISE D

TYPE I

1. $\cos(A + B) + \cos(A - B) \equiv$

   A $2\cos A \sin B$ B $-2\sin A \cos B$ C $2\cos A \cos B$ D $-2\sin A \sin B$

2. If $x^2 + y^2 = 4$ then $\dfrac{dy}{dx}$ is

   A $2x + 2y$ B $4 - x^2$ C $-\dfrac{x}{y}$ D $\dfrac{y}{x}$

3. The approximate value, when θ is small, of the expression $\dfrac{2\theta - \sin\theta}{\sin 2\theta - \theta}$ is

   A 1 B 2 C -1 D -2

4. $\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)$ is

   A $\dfrac{-1}{(1+x)^2}$ B $\dfrac{1}{1-x}$ C $\ln(1+x)$ D $\dfrac{-1}{1+x^2}$

5. $\dfrac{d}{dx}\ln\left(\dfrac{x+1}{2x}\right)$ is

   A $\dfrac{1}{2}$ B $\dfrac{1}{x+1} - \dfrac{1}{2x}$ C $\dfrac{2x}{x+1}$ D $\dfrac{1}{x+1} - \dfrac{1}{x}$

6. $\dfrac{d}{dx}a^x$ is

   A xa^{x-1} B a^x C $x\ln a$ D $a^x \ln a$

7. If $x = \cos\theta$ and $y = \cos\theta + \sin\theta$, $\dfrac{dy}{dx}$ is

   A $1 - \cot\theta$ B $1 - \tan\theta$ C $\cot\theta - 1$ D $\cot\theta + 1$

8. The greatest value of $5\cos\theta - 4\sin\theta$ is

A 3 B 1 C $\sqrt{41}$ D ± 5

TYPE II

9. $3\cos\theta - 4\sin\theta \equiv$

A $5\cos(\theta+\alpha)$ where $\tan\alpha = \frac{3}{4}$
B $5\sin(\alpha-\theta)$ where $\tan\alpha = \frac{3}{4}$
C $5\cos(\theta+\alpha)$ where $\tan\alpha = \frac{4}{3}$
D $-5\cos(\theta-\alpha)$ where $\tan\alpha = \frac{4}{3}$

10. Given that α is a very small angle measured in radians,

A $\sin(2\pi+\alpha) = \sin\alpha$
B $\sin\alpha \approx \alpha$
C $\sin(2\pi+\alpha) \approx 2\pi+\alpha$
D $\cos\alpha \approx \alpha$

11. Given that $y = \arccos x$,

A $\cos y = x$
B $0 \leqslant x \leqslant \pi$
C $y = \sec x$
D $0 \leqslant y \leqslant \pi$

12. If $y = \ln(\ln x)$ and $x > 1$ then

A $\frac{dy}{dx} = \frac{1}{\ln x}$
B $e^y = \ln x$
C $\frac{dy}{dx} = \frac{1}{x\ln x}$
D $y = \ln x^2$

13. Given that $x = \cos^2\theta$ and $y = \sin^2\theta$,

A $x^2 + y^2 = 1$
B $\frac{dy}{dx} = \tan\theta$
C $0 \leqslant y \leqslant 1$
D $y = x - \frac{1}{2}\pi$

TYPE III

14. $\sin 3\theta = \cos 4\theta \quad \Rightarrow \quad \frac{1}{2}\pi - 3\theta = 2n\pi \pm 4\theta$

15. $\frac{d}{dx}(uv) = \frac{du}{dx} \times \frac{dv}{dx}$

16. $\frac{d}{dx}(x^2y^2) = 2xy^2 + 2x^2y$

17. Given that $y = \ln x^2$ and x increases by δx then $\delta y \approx \left(\frac{1}{x^2}\right)(\delta x)$

18. When $y = \cos 2\theta$ and $x = \sin\theta$, $\frac{dy}{dx} = -4x\sqrt{(1-x^2)}$

19. If $y = f(t)$ and $x = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

20. If $y = \ln\left(\frac{x}{1+x}\right)$ then $\frac{dy}{dx} = \frac{d}{dx}\{\ln x\} - \frac{d}{dx}\{\ln(1+x)\}$

MISCELLANEOUS EXERCISE D

1. Differentiate with respect to x

(a) $\frac{x}{2x+1}$ (b) $\arcsin(x^2)$ (C 86)

2. Find all the values of θ, such that $0 \leqslant \theta \leqslant 180°$, satisfying the equation $\cos\theta + \cos 3\theta = 0$ (C 88)

3. Differentiate with respect to x

(a) $\sin(2x^2)$ (b) 2^x (U of L 88)

4. Find the values of x, in the interval $0 \leqslant x \leqslant 2\pi$, which satisfy the equation

$$\sin 3x + \sin x = \sin 2x$$

giving your answers in terms of π (AEB 85)$_p$

5. Find $\frac{dy}{dx}$ when

(a) $y = \frac{1+\sin x}{1+\cos x}$ (b) $y = \ln\sqrt{\left(\frac{1+x}{1-x}\right)}$, $|x| < 1$

and simplify your answers as far as possible. (U of L 85)

6. A curve has parametric equations

$$x = 2t + \sin 2t \quad y = \cos 2t, \quad 0 < t < \tfrac{1}{2}\pi$$

Show that, at the point with parameter t, the gradient of the curve is $-\tan t$ (C 87)

7. Show that, when x is small,

$$(1 - \sin^2 3x)\cos 2x \approx 1 - 11x^2$$

8. Evaluate $\frac{dy}{dx}$ when $y = 1$, given that

(a) $y(x + y) = 3$ (b) $x = \frac{1}{(4-t)^2}$, $y = \frac{t}{4-t}$, $0 < t < 4$

(U of L 85)

9. Prove the identity

$$\sqrt{3}\cos\theta + \sin\theta \equiv 2\cos(\theta - \tfrac{1}{6}\pi)$$

Find, in terms of π, the general solution of the equation

$$\sqrt{3}\cos\theta + \sin\theta = 1$$ (AEB 86)

10. For a given mass of gas, the volume, V cm^3, and pressure, p cm of mercury, are related by $p = kV^n$ where k and n are constants.

(a) Prove that $\frac{dp}{dV} = \frac{np}{V}$

(b) For a particular mass of gas $n = -1.4$. At the instant when the volume is 20 cm^3, the pressure is 150 cm of mercury and the volume is decreasing at a rate of 0.5 cm^{3}s^{-1}. Calculate the rate of change of pressure at this instant, in cm of mercury per second.

(AEB 87)

11. Differentiate with respect to x

(a) $(4x - 1)^{20}$ (b) $\arctan(\sqrt{x})$

12. The parametric equations of a curve are $x = 5a\sec\theta$ and $y = 3a\tan\theta$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ and a is a positive constant. Find the coordinates of the point on the curve where the gradient is -1

13. Given that $y = e^x\ln(1 + \sin 2x)$, $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$, find $\frac{dy}{dx}$ in terms of x.

(U of L 88)

14. A curve is given by the equation

$$y = \sin x + \tfrac{1}{2}\sin 2x \quad 0 \leqslant x \leqslant 2\pi$$

Find the values of x for which y is zero.

Find the exact coordinates of the stationary points on the curve and sketch the curve. (JMB 84)

15. Find all the values of θ for which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and $\sin 8\theta = \sin 2\theta$

(U of L 88)

16. The parametric equations of a curve are

$$x = \cos 2\theta + 2\cos\theta \quad y = \sin 2\theta - 2\sin\theta$$

Show that $\frac{dy}{dx} = \tan\frac{1}{2}\theta$

Find the equation of the normal to the curve at the point where $\theta = \frac{1}{3}\pi$ (AEB 85)

17. Given that $3\cos\theta + 4\sin\theta \equiv R\cos(\theta - \alpha)$, where $R > 0$ and $0 \leqslant \alpha \leqslant \pi/2$, state the value of R and the value of $\tan\alpha$.

For each of the following equations, solve for θ in the interval $0 \leqslant \theta \leqslant 2\pi$ and give your answers in radians correct to one decimal place.

(a) $3\cos\theta + 4\sin\theta = 2$

(b) $3\cos 2\theta + 4\sin 2\theta = 5\cos\theta$

The curve with equation $y = \frac{10}{3\cos x + 4\sin x + 7}$, between $x = -\pi$ and $x = \pi$, cuts the y-axis at A, has a maximum point at B and a minimum point at C. Find the coordinates of A, B and C. (AEB 88)

18. A curve has parametric equations

$$x = 2t - \ln(2t) \quad y = t^2 - \ln(t^2)$$

where $t > 0$

Find the value of t at the point on the curve at which the gradient is 2 (C 86)

19. Write down $\sin x + \sin 3x$ as a product of factors.

Find, in terms of π, the solutions of the equation

$$\sin x + \sin 3x = \sin 2x$$

which lie in the interval $-\pi < x \leqslant \pi$.

Find also the general solution of the equation. (JMB 87)

20. A curve has parametric equations

$$x = 1 + \sqrt{(32)}\cos\theta \quad y = 5 + \sqrt{(32)}\sin\theta \quad 0 \leqslant \theta \leqslant 2\pi$$

Show that the tangent to the curve at the point with parameter θ is given by

$$(y - 5)\sin\theta + (x - 1)\cos\theta = \sqrt{(32)}$$

Find the two values of θ such that this tangent passes through the point $A(1, -3)$. Hence, or otherwise, find the equations of the two tangents to the curve from the point A. (U of L 88)

21. The equation of a curve is

$$3x^2 + y^2 = 2xy + 8x - 2$$

Find an equation connecting x, y and $\frac{dy}{dx}$ at all points on the curve.
Hence show that the coordinates of all points on the curve at which $\frac{dy}{dx} = 2$ satisfy the equation

$$x + y = 4$$

Deduce the coordinates of these points. (JMB 85)

22. A curve is defined by the parametric equations $x = \cos^3 t$, $y = \sin^3 t$, $0 \leqslant t \leqslant \frac{1}{4}\pi$.
Show that the equation of the normal to the curve at the point $P(\cos^3 t, \sin^3 t)$ is

$$x \cos t - y \sin t = \cos^4 t - \sin^4 t$$ (JMB 87)

23. Given that $y = \sqrt{\left(\frac{2+x}{3+x}\right)}$, find the value of $\frac{dy}{dx}$ when $x = 1$
Hence find the approximate increase in the value of y when x increases from 1 to $1 + \alpha$, where α is small.

24. Find approximations, when θ is very small, for the following expressions

(a) $\cos\theta + \sin\theta$ (b) $\dfrac{2\tan\theta - \theta}{\sin 2\theta}$

(c) $\cot\theta(1 - \cos\theta)$ (d) $\dfrac{\sqrt{2} - \sin\theta}{\cos\theta}$

25. The radius of a sphere is increasing at a rate of 4 cm/s. Obtain, as a multiple of π, the rate of increase of the volume of the sphere when the radius is 10 cm. (C Spec Paper)

26. A straight metal bar, of square cross-section, is expanding due to heating. After t seconds the bar has dimensions x cm by x cm by $10x$ cm. Given that the area of the cross-section is increasing at 0.024 $cm^2\,s^{-1}$ when $x = 6$, find the rate of increase of the side of the cross-section at this instant. Find also the rate of increase of the volume when $x = 6$ (JMB 86)

27. Given that x is so small that terms in x^3 and higher powers of x may be neglected, show that

$$11 \sin x - 6 \cos x + 5 = A + Bx + Cx^2$$

and state the values of the constants A, B, C. (U of L 89)

CHAPTER 26

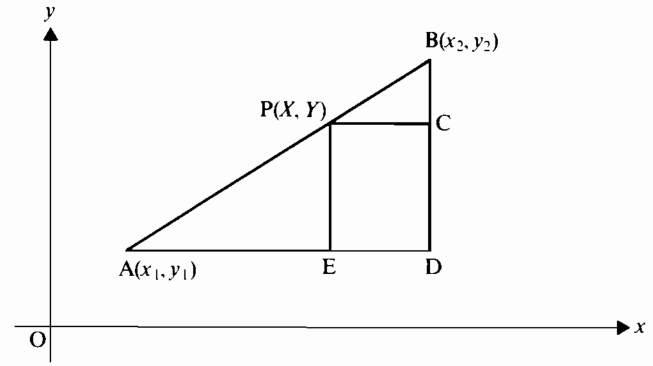
STRAIGHT LINES 2

COORDINATE GEOMETRY AND STRAIGHT LINES

In any graphical work, particularly that of a geometric nature, frequent use is made of straight lines. In Chapter 6 and in Chapter 9 a variety of methods and results involving points and straight lines are obtained using Cartesian coordinates. This chapter extends that work and before proceeding the reader should revise those earlier results.

THE COORDINATES OF A POINT DIVIDING A LINE IN A GIVEN RATIO

$A(x_1, y_1)$ and $B(x_2, y_2)$ are two fixed points and the point $P(X, Y)$ divides the line joining AB in the ratio $\lambda : \mu$



From the diagram, $\triangle APE$ and $\triangle PBC$ are similar,

$$\therefore \quad \frac{AE}{PC} = \frac{AP}{PB} = \frac{\lambda}{\mu}$$

But $AE = X - x_1$, $PC = x_2 - X$ $\quad \therefore \quad X = \dfrac{\lambda x_2 + \mu x_1}{\lambda + \mu}$

A similar result can be found for Y, so

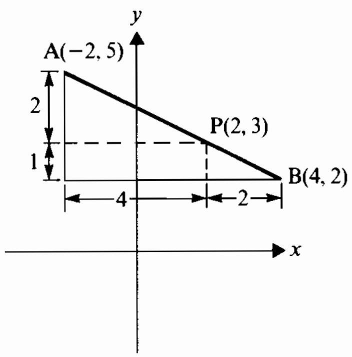
if the point P divides the line joining $A(x_1, y_1)$ to $B(x_2, y_2)$ in the ratio $\lambda : \mu$, then the coordinates of P are

$$\left(\frac{\lambda x_2 + \mu x_1}{\lambda + \mu}, \frac{\lambda y_2 + \mu y_1}{\lambda + \mu}\right)$$

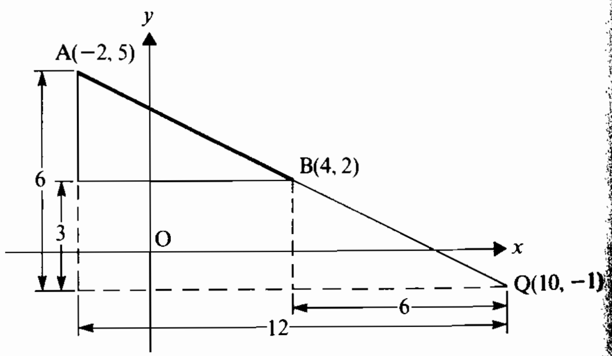
These formulae, which are quotable, apply to both internal and external division. Their use is not always necessary however. When the coordinates of A and B are known numbers, a diagram, together with simple mental arithmetic, is often adequate.

Examples 26a

1. Find the coordinates of the points P and Q which divide the line joining A(−2, 5) and B(4, 2) in the ratio 2 : 1
(a) internally (b) externally.



(a) Internal division



(b) External division

(a) From the diagram, P is the point (2, 3)

Alternatively, the formula can be used as follows.

P divides AB internally in the ratio 2 : 1 so $\lambda = 2$ and $\mu = 1$

$$\Rightarrow \quad x = \frac{2(4) + 1(-2)}{2 + 1} = 2 \quad \text{and} \quad y = \frac{2(2) + 1(5)}{2 + 1} = 3$$

(b) The diagram shows that Q is the point (10, −1)

Or, using the formula,

Q divides AB externally in the ratio 2 : 1, so $\lambda = 2$ and $\mu = -1$

$$\Rightarrow \quad x = \frac{2(4) - 1(-2)}{2 - 1} = 10 \quad \text{and} \quad y = \frac{2(2) - 1(5)}{2 - 1} = -1$$

Note that in the external division, the sign of μ is opposite to the sign of λ because the direction of the line segment QB is opposite that of the line segment AQ. For this reason, external division is sometimes denoted by a negative ratio, e.g. 2 : −1.

THE EQUATION OF A STRAIGHT LINE

The equation of any straight line can be written in the form $ax + by + c = 0$ where a, b and c are constants.

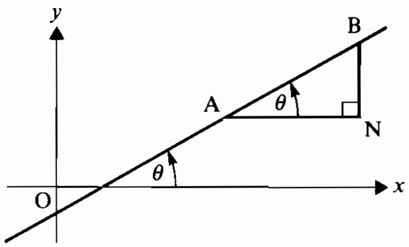
When the equation is written in standard form, i.e. $y = mx + c$, then m is the gradient of the line and c is its intercept on the y-axis.

(Note that the c in the general form of the equation and the c in the standard form are not usually the same number.)

When m is positive, the line makes an acute angle with the positive direction of the x-axis, and when m is negative, the line makes an obtuse angle with the positive direction of the x-axis.

When θ is acute

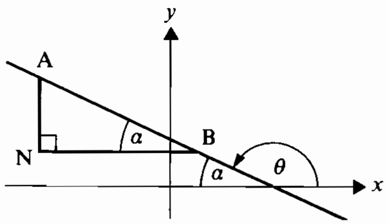
gradient of AB $= \dfrac{BN}{AN} = \tan\theta$



When θ is obtuse

gradient of AB $= \dfrac{-AN}{BN}$

$= -\tan\alpha = \tan\theta$



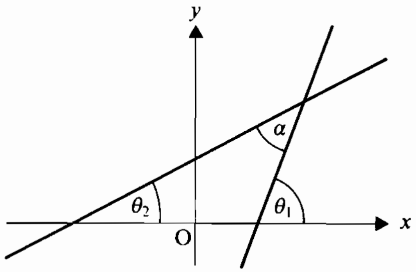
In both cases the gradient of AB $= \tan\theta$

Therefore the gradient of a line is equal to the tangent of the angle between the line and the positive direction of the x-axis.

i.e. $$m = \tan\theta$$

THE ANGLE BETWEEN TWO LINES

Consider two lines with gradients m_1 and m_2, where $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$



The angle, α, between the lines is given by $\alpha = \theta_1 - \theta_2$

Therefore $$\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}$$

i.e. $$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Examples 26a (continued)

2. Find the tangent of the angle between the lines $3x - 2y = 5$ and $4x + 5y = 1$

If m_1 is the gradient of $3x - 2y = 5$, then $m_1 = \frac{3}{2}$

If m_2 is the gradient of $4x + 5y = 1$, then $m_2 = -\frac{4}{5}$

Then the angle, α, between these lines is given by

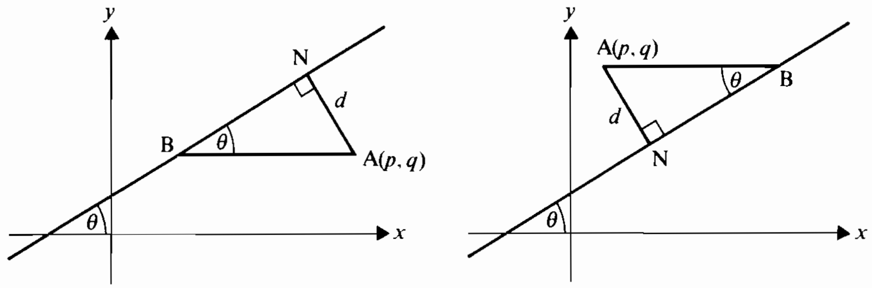
$$\tan\alpha = \frac{\frac{3}{2} - (-\frac{4}{5})}{1 + (\frac{3}{2})(-\frac{4}{5})} = \left(\frac{23}{10}\right)\Big/\left(\frac{-2}{10}\right) = -\frac{23}{2}$$

As $\tan\alpha$ is negative, α is the obtuse angle between the lines. The acute angle between them is arctan 23/2

THE DISTANCE OF A POINT FROM A STRAIGHT LINE

The 'distance of a point from a line' is understood to mean the perpendicular distance.

Consider the line with equation $ax + by + c = 0$ and any point $A(p, q)$ distant d from the line.



(i) (ii)

In both diagrams, $d = AN = AB\sin\theta$

and AB is horizontal so, at B, $y = q$ and $x = -(bq + c)/a$

Hence in diagram (i) $$AB = p - \left(\frac{-(bq + c)}{a}\right) = \frac{ap + bq + c}{a}$$

and in diagram (ii) $$AB = \left(\frac{-(bq + c)}{a}\right) - p = -\frac{ap + bq + c}{a}$$

i.e. for any point A, $$AB = \pm\frac{ap + bq + c}{a}$$

The gradient of the line, $\tan\theta$, is $-\frac{a}{b} \Rightarrow \sin\theta = \frac{a}{\pm\sqrt{(a^2 + b^2)}}$

Now $AN = AB\sin\theta$

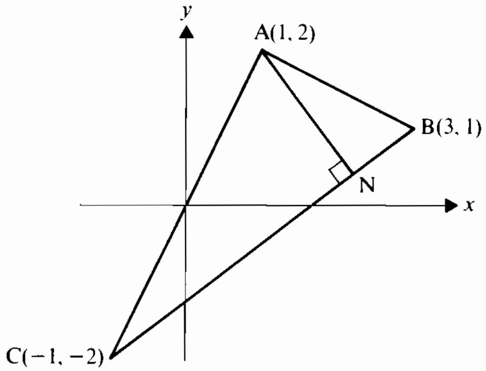
hence $$AN = \pm\left(\frac{ap + bq + c}{a}\right)\left(\frac{a}{\sqrt{(a^2 + b^2)}}\right) = \pm\left(\frac{ap + bq + c}{\sqrt{(a^2 + b^2)}}\right)$$

The two signs ($\pm$) in this formula arise from points that are on opposite sides of the line. If only the *length*, d, of AN is required, we take the positive value of this formula, i.e.

$$d = \left|\frac{ap + bq + c}{\sqrt{(a^2 + b^2)}}\right|$$

Examples 26a (continued)

3. The vertices of a triangle are the points $A(1, 2)$, $B(3, 1)$, $C(-1, -2)$. Find the length of the altitude through A and hence find the area of the triangle.



The equation of the line BC is $y = \frac{3}{4}x - \frac{5}{4}$

Writing this equation in the form $3x - 4y - 5 = 0$, then the length of the altitude AN through $A(1, 2)$ is given by

$$d = \left|\frac{3(1) - 4(2) - 5}{\sqrt{(3^2 + \{-4\}^2)}}\right| = \left|-\frac{10}{5}\right| = 2$$

The length of BC is $\sqrt{\{(3)^2 + (4)^2\}} = 5$

Therefore the area of $\triangle ABC$ is $\frac{1}{2}(CB)(AN) = 5$ sq. units.

---

4. Determine, without the aid of a diagram, whether the points $A(-3, 4)$ and $B(-2, 3)$ are on the same side of the line $y + 3x + 4 = 0$. Find the acute angle between this line and AB, giving your answer in degrees correct to three significant figures.

We need to find the *sign* of the expression used to give the distance of each point from the line.

Writing the equation of the given line as $3x + y + 4 = 0$, and using the formula $\dfrac{ap + bq + c}{\sqrt{(a^2 + b^2)}}$ gives

for A, $\dfrac{3(-3) + 4 + 1}{\sqrt{(9 + 1)}} < 0$ and for B, $\dfrac{3(-2) + 3 + 4}{\sqrt{(9 + 1)}} > 0$

The signs are opposite so A and B are on opposite sides of the line.

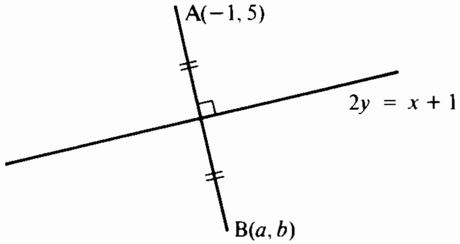
The gradient of the given line is -3, and the gradient of AB is -1. If α is an angle between the given line and AB then

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-3 + 1}{1 + 3} = -\frac{1}{2}$$

Therefore the acute angle between the lines is $\arctan\frac{1}{2}$ which is 26.6° to 3 s.f.

---

5. Find the reflection of the point $A(-1, 5)$ in the line $2y = x + 1$



If $B(a, b)$ is the reflection of A in the given line, l, then

AB is bisected by l and AB is at right angles to l

Let $C(x, y)$ be the point where AB and l cut.

As we have two facts about AB, we can use these to give two relationships between a and b. These can then be solved simultaneously to find a and b. First we can use the fact that l bisects AB.

$C(x, y)$ is the midpoint of AB, so $x = \frac{1}{2}(a - 1)$ and $y = \frac{1}{2}(b + 5)$ but $C(x, y)$ is on the line $2y = x + 1$, i.e.

$$b + 5 = \tfrac{1}{2}(a - 1) + 1 \quad \Rightarrow \quad a - 2b = 9 \qquad [1]$$

Then we can use the fact that AB and l are perpendicular, so the product of their gradients is -1

AB and l are perpendicular, so

$$\left(\frac{b - 5}{a + 1}\right)\left(\frac{1}{2}\right) = -1 \quad \Rightarrow \quad 2a + b = 3 \qquad [2]$$

Solving equations [1] and [2] simultaneously gives

$$a = 3 \quad \text{and} \quad b = -3$$

EXERCISE 26a

1. Find the coordinates of the point that divides AB in the given ratio in each of the following cases.
   (a) A(2, 4), B(−3, 9) 1:4 internally
   (b) A(−3, −4), B(3, 5) 3:1 externally
   (c) A(1, 5), B(8, −2) 4:3
   (d) A(−1, 6), B(3, −2) 3:−2

2. A is the midpoint of BC. If A is (X, Y) and B is (x_1, y_1) show that C has coordinates $(2X - x_1, 2Y - y_1)$.

3. Find the distance from A to the given line in each of the following cases.
   (a) A(3, 4); $2x - y = 3$
   (b) A(−1, −2); $3y = 4x - 1$
   (c) A(a, b); $y = mx + c$
   (d) A(4, −1); $x + y = 6$
   (e) A(x, y); $ax + by + c = 0$
   (f) A(0, 0); $ax + by + c = 0$

4. Determine whether A and B are on the same or opposite sides of the given line in each of the following cases.
   (a) A(1, 2), B(4, −3); $3x + y = 7$
   (b) A(0, 3), B(7, 6); $x - 4y + 1 = 0$
   (c) A(−5, 1), B(−2, 3); $7x + y - 6 = 0$

5. Find the tangents of the acute angles between the following pairs of lines.
   (a) $2x + 3y = 7$, $x - 6y = 5$
   (b) $x + 4y - 1 = 0$, $3x + 7y = 2$
   (c) $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$

6. A(0, 1), B(3, 7) and C(−4, −4) are the vertices of a triangle. Find the tangent of each of the three angles in the triangle and the length of each altitude.

7. Find the equations of the two lines through the origin which are inclined at 45° to the line $2x + 3y - 4 = 0$

8. Find the image of the point (5, 6) in the line
   (a) $3x - y + 1 = 0$
   (b) $y = 4x + 20$
   (c) $2x + 5y + 18 = 0$

9. A point P(X, Y) is equidistant from the line $x + 2y = 3$ and from the point (2, 0). Find an equation relating X and Y.

10. Show that A(4, 1) and B(2, −3) are equidistant from the line $2x + 5y = 1$. Is A the reflection of B in this line?

11. Write down the distance of the point P(X, Y) from each of the lines $5x - 12y + 3 = 0$ and $3x + 4y - 6 = 0$. By equating these distances find the equations of two lines that bisect the angles between the two given lines [i.e. the equations of the set of points P(X, Y)].

12. A(4, 4) and B(7, 0) are two vertices of a triangle OAB. Find the equation of the line that bisects the angle OBA. If this line meets OA at C show that C divides OA in the ratio OB:BA.

REDUCTION OF A RELATIONSHIP TO A LINEAR LAW

In this part of the chapter we look at a practical application of the equation $y = mx + c$

Linear Relationships

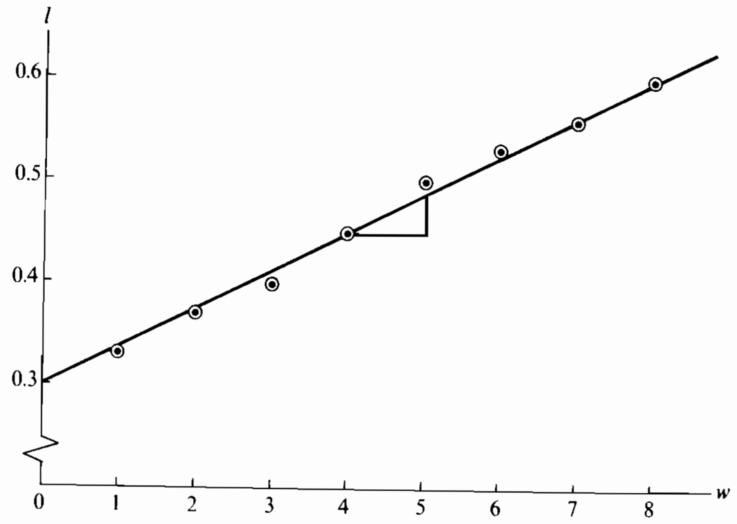
If it is thought that a certain relationship exists between two variable quantities, this hypothesis can be tested by experiment, i.e. by giving one variable certain values and measuring the corresponding values of the other variable. The experimental data collected can then be displayed graphically. If the graph shows points that lie approximately on a straight line (allowing for experimental error) then a linear relationship between the variables (i.e. a relationship of the form $Y = mX + c$) is indicated. Further, the gradient of the line (m) and the vertical axis intercept (c) provide the values of the constants.

Examples 26b

1. An elastric string is fixed at one end and a variable weight is hung on the other end. It is believed that the length of the string is related to the weight by a linear law. Use the following experimental data to confirm this belief and find the particular relationship between the length of the string and the weight.

Weight (W) in newtons	1	2	3	4	5	6	7	8
Length (l) in metres	0.33	0.37	0.4	0.45	0.5	0.53	0.56	0.6

If l and W are related by a linear law then, allowing for experimental error, we expect that the points will lie on a straight line. Plotting l against W gives the following graph.



These points do lie fairly close to a straight line.

From the graph, l and W are connected by a linear relationship, i.e. a relationship of the form $l = aW + b$

Now we draw the line of 'closest fit'. This is the line that has the points distributed above and below it as evenly as possible; it is not necessarily the line which goes through the most points.

By measurement from the graph

the gradient $= 0.04$

the intercept on the vertical axis $= 0.3$

So comparing $\left.\begin{array}{l} l = aW + b \\ Y = mX + c \end{array}\right\}$ we have $a = 0.04,\ b = 0.3$
with

i.e. within the limits of experimental accuracy

$$l = 0.04W + 0.3$$

When the gradient of a line is found from a graph, the increase in a quantity is measured from the *scale used for that quantity* and it is worth noting that the scales used for the two quantities are *not* usually the same.

The values of the constants found from calculating the gradient and intercept from a drawn graph are approximate. Apart from experimental error in the data, selecting the line of best fit is a personal judgement and so is subject to slight variations which affect the values obtained.

92 There are methods for calculating the equation of the line of best fit; these are called regression lines and computer programmes exist which will give these equations from the data. Using such a programme, the values of a and b in the last example are given as $a = 0.039$ and $b = 0.291$

Now if the relationship is not of a linear form, the points on the graph will lie on a section of a curve. It is very difficult to identify the equation of a curve from a section of it, so the form of a non-linear relationship can rarely be verified in this way.

Non-linear relationships, however, can often be reduced to a linear form. The following examples illustrate some of the relationships which can be verified by plotting experimental data in a form which gives a straight line.

Relationships of the Form $y = ax^n$

A relationship of the form $y = ax^n$ where a is a constant can be reduced to a linear relationship by taking logarithms, since

$$y = ax^n \quad \Longleftrightarrow \quad \ln y = n \ln x + \ln a$$

(Although any base can be used, it is sensible to use either e or 10 as these are built into most calculators.)

Comparing $\qquad \ln y = n \ln x + \ln a$

with $\qquad Y = mX + c$

we see that plotting values of $\ln y$ against values of $\ln x$ gives a straight line whose gradient is m and whose intercept on the vertical axis is $\ln a$

Examples 26b (continued)

2. The following data, collected from an experiment is believed to obey a law of the form $p = aq^n$. Verify this graphically and find the values of a and n.

q	1	2	3	4	5	6
p	0.5	0.63	0.72	0.8	0.85	0.9

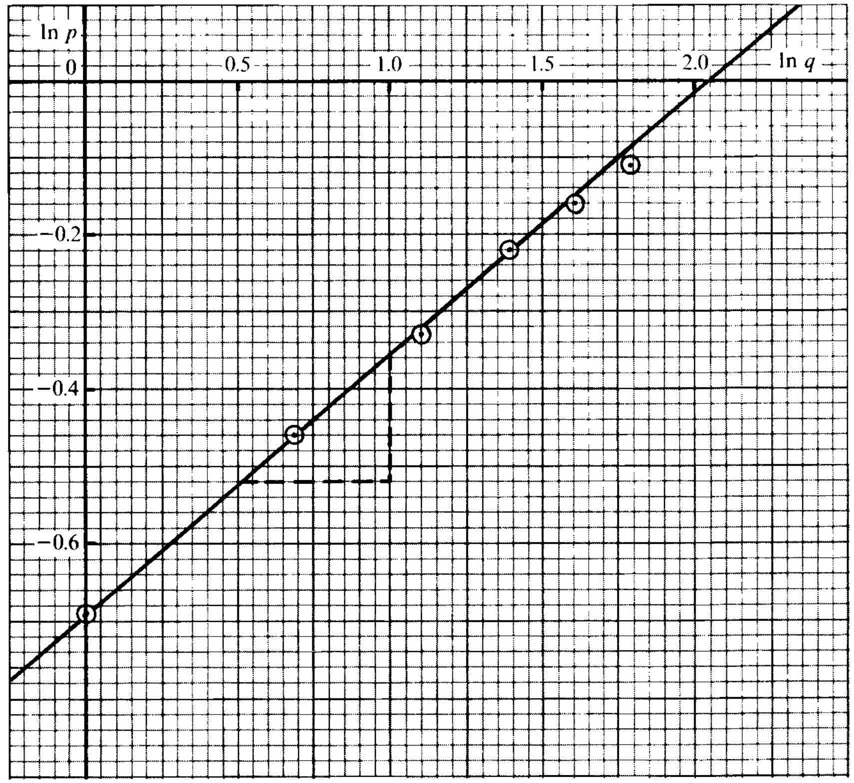
If the relationship $p = aq^n$ is correct, then $\qquad \ln p = n \ln q + \ln a$

comparing with $\qquad y = mx + c$

we see that $\ln p$ and $\ln q$ are related by a linear law.

First a table of values of $\ln p$ and $\ln q$ is needed.

$\ln q$	0	0.69	1.10	1.39	1.61	1.79
$\ln p$	−0.69	−0.46	−0.33	−0.22	−1.16	−0.11



The points lie on a straight line confirming that there is a linear relationship between $\ln q$ and $\ln p$.

From the graph, the gradient of the line is 0.33, $\Rightarrow \; n = 0.33$ and the intercept on the vertical axis is −0.69, so

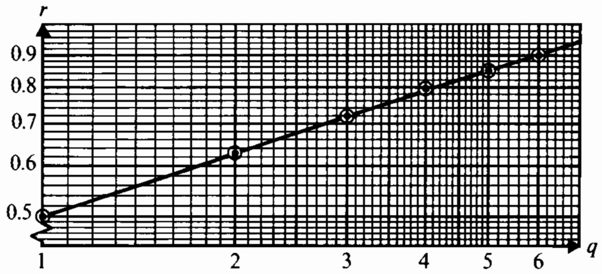
$$\ln a = -0.69 \quad \Rightarrow \quad a = 0.5$$

Therefore the data does obey a law of the form $p = aq^n$, where $a \approx 0.5$ and $n \approx 0.33$

(Using the tabulated values of $\ln q$ and $\ln p$ and a computer programme, gives $n = 0.327$ and $\ln a = -0.687$)

An alternative method for investigating relationships of this form uses log–log graph paper, i.e. graph paper where the grids and the scales marked on them are adjusted to represent the logarithms of the numbers being plotted. So to plot $\log S$ against $\log T$ say, values of S and T can be plotted directly.

The diagram illustrates the use of log–log graph paper using the data given in the previous example.



This straight line verifies that the relationship is of the form $p = aq^n$ When $q = 1$, $p = a$. So *from the graph* we see that $a = 0.5$

Reading another pair of values from the graph, (*not* from the table) and substituting these into the relationships gives $q = 2.5$ and $p = 0.68$,

then $$0.68 = (0.5)(2.5)^n \quad \Rightarrow \quad n = 0.335$$

Relationships of the Form $y = ab^x$

A relationship of the form $y = ab^x$ where a and b are constant can be reduced to a linear relationship by taking logs, since

$$y = ab^x \quad \Longleftrightarrow \quad \log y = x \log b + \log a$$

Comparing $\log y = x \log b + \log a$

with $Y = mX + c$

we see that plotting values of $\log y$ against corresponding values of x gives a straight line whose gradient is $\log b$ and whose intercept on the vertical axis is $\log a$.

Relationships of the Form $\frac{1}{y} + \frac{1}{x} = \frac{1}{a}$

If a is a constant, $\frac{1}{y} + \frac{1}{x} = \frac{1}{a}$ is a linear relationship between $(1/y)$ and $(1/x)$

i.e. if values of $(1/y)$ are plotted against corresponding values of $(1/x)$, a straight line will result.

By comparing $(1/y) = -(1/x) + (1/a)$

with $Y = mX + c$

it can be seen that the gradient of the graph should be -1 and the intercept on the $(1/y)$ axis gives the value of $1/a$

Note that in all graphical work the scales should be chosen to give the greatest possible accuracy, i.e. the range of values given in the table should have as much spread as possible. This sometimes means that the horizontal scale does not include zero and the value of c cannot then be read from the graph. In these circumstances, which arise in the next example, we find c by using the equation $Y = mX + c$ together with the measured value of m and the coordinates of any point P on the graph (*not* a pair of values from the table).

Examples 26b (continued)

3. In an experiment, values of a variable y were measured for selected values of a variable x

   The results are shown in the table below. It is believed that x and y are related by a law of the form $2y + 10 = ab^{(x-3)}$. Confirm this graphically and find approximate values for a and b

x	10	12	15	20	21
y	37.5	90	320	2440	3700

If $2y + 10 = ab^{(x-3)}$, taking logs of both sides gives

$$\log(2y + 10) = (x - 3)\log b + \log a$$

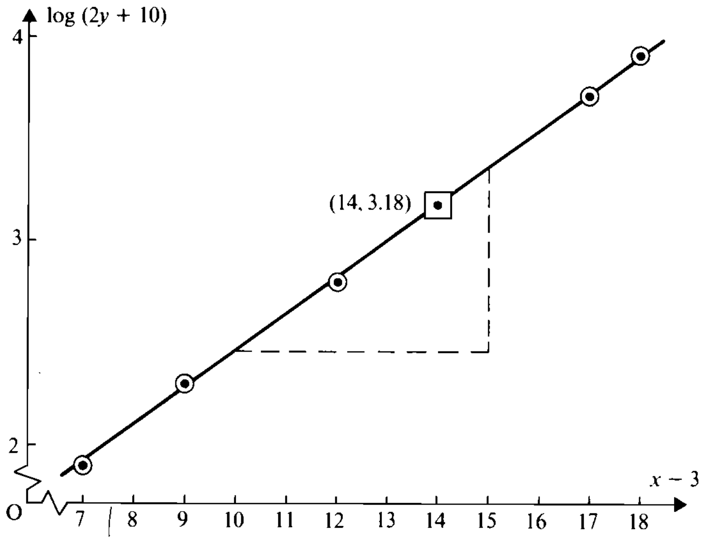
which is of the form $Y = mX + c$

where $Y = \log(2y + 10)$, $X = x - 3$ and $m = \log b$, $c = \log a$
i.e. $[\log(2y + 10)]$ and $[x - 3]$ obey a linear law.

So we need to tabulate corresponding values of $(x - 3)$ and $\log(2y + 10)$ from the given values of x and y

$x - 3$	7	9	12	17	18
$\log(2y + 10)$	1.9	2.3	2.8	3.7	3.9

Then plotting $\log(2y + 10)$ against $x - 3$ gives the graph below.



The straight line shows that there is a linear relationship between $\log(2y + 10)$ and $x - 3$, confirming that $2y + 10 = ab^{x-3}$

From the graph, the gradient is 0.175

$$\therefore \qquad \log b \approx 0.175 \quad \Rightarrow \quad b \approx 1.49$$

Using the point P(14, 3.18) and $m = 0.175$ then $Y = mX + c$ gives

$$3.18 = (0.175)(14) + c \quad \Rightarrow \quad c = 0.73$$

i.e. $$\log a \approx 0.73 \quad \Rightarrow \quad a \approx 5.37$$

When attempting to reduce a relationship between two variables to a form from which a straight line graph can be drawn, the given equation must be expressed in the form

$$Y = mX + c$$

where X and Y are variable terms, values for which must be calculable from the given data, i.e. X and Y must not contain unknown constants. On the other hand m and c must be constants, but may be unknown.

Now X and Y may be functions of one or both variables, as for example

$$f(xy) = m\,g(xy) + c$$

is a linear relationship between $f(xy)$ and $g(xy)$

So to reduce a non-linear relationship to a linear form we:

1) try to express it in a form containing three terms,

2) make one of those terms constant,

3) remove unknown constants from the coefficient of one of the variable terms.

These objectives will now be applied in the next worked example.

Examples 26b (continued)

4. It is known that two variables x and y are related by the law

(a) $ae^y = x^2 - bx$ (b) $y = \dfrac{1}{(x-a)(x-b)}$

In each case state how you would reduce the law to a linear form so that a straight line graph could be drawn from experimental data.

(a) $ae^y = x^2 - bx$

This equation has three terms, one of which becomes constant when we divide by x, giving $a\dfrac{e^y}{x} = x - b$

We also have a variable term (x) whose coefficient is a known constant. This equation may now be written as

$$x = a\frac{e^y}{x} + b$$

Comparing with $$Y = mX + c$$

we see that if values of x are plotted against corresponding values of e^y/x, a straight line will result whose gradient is a and whose intercept on the vertical axis is b

Note that the original equation can be arranged in linear form in a variety of ways

e.g. $$\left(\frac{e^y}{x^2}\right) = -\frac{b}{a}\left(\frac{1}{x}\right) + \frac{1}{a} \quad \text{or} \quad \left(\frac{x^2}{e^y}\right) = b\left(\frac{x}{e^y}\right) + a$$

(b) $y = \dfrac{1}{(x-a)(x-b)}$

Although this form suggests the use of partial fractions, this approach increases the number of times the unknown constants appear. It is better to invert the equation giving

$$\frac{1}{y} = (x-a)(x-b)$$

$$\Rightarrow \quad \frac{1}{y} = x^2 - x(a+b) + ab$$

$$\Rightarrow \quad x^2 - \frac{1}{y} = (a+b)x - ab$$

We can now compare this form with $Y = mX + c$

Thus plotting values of $(x^2 - 1/y)$ against corresponding values of x will give a straight line whose gradient is $a+b$ and whose intercept on the vertical axis is $-ab$

EXERCISE 26b

1. Reduce each of the given relationships to the form $Y = mX + c$. In each case give the functions equivalent to X and Y and the constants equivalent to m and c

(a) $\dfrac{1}{y} = ax + b$ (b) $y(y-b) = x - a$

(c) $ae^x = y(y-b)$ (d) $x = y^2 + y - k$

(e) $y = ax^{n+2}$ (f) $y^a = e^{x+k}$

In Questions 2–7, the table gives sets of values for the related variables and the law which relates the variables. By drawing a straight line graph find approximate values for a and b

2. $y = ax + ab$

x	3	5	7	10
y	−2	2	6	12

3. $s = ab^{-t}$

t	1	2	3	4
s	1.5	0.4	0.1	0.02

4. $r^2 = a\theta - b$

θ	1	4	10	25	40
r	1.6	2	2.6	3.8	4.7

5. $ay = b^x$

x	5	6	7	8
y	1.07	2.13	4.27	8.53

6. $\dfrac{a}{V} + \dfrac{b}{L} = 1$

V	2.5	3	5.5	7	12
L	2.5	1.5	0.79	0.7	0.6

7. $y = (x-a)(x-b)$

x	1	2	3	4	5	6
y	−6	−4	0	6	14	24

8. The variables x and y are believed to satisfy a relationship of the form $y = k(x+1)^n$. Show that the experimental values shown in the table do satisfy the relationship. Find approximate values for k and n

x	4	8	15	19	24
y	4.45	4.60	4.80	4.89	5.00

9. Two variables s and t are related by a law of the form $s = ke^{-nt}$ The values in the table were obtained from an experiment. Show graphically that these values do verify the relationship and use the graph to find approximate values of k and n

t	1	1.5	2	2.5	3
s	1230	590	260	140	60

10. Two variables x and t are related by a law of the form $x = \cos(\alpha t + \varepsilon)$. The table shows the values of x obtained experimentally for some different values of t. Show graphically that these values do NOT satisfy this relationship unless two of the values of x are assumed to be incorrect.
Estimate the correct values of x

t	0.5	1	1.5	1.75	2	2.25
x	−0.26	−0.90	−0.40	−0.90	−0.71	−0.49

CHAPTER 27

COORDINATE GEOMETRY AND CURVES

LOCI

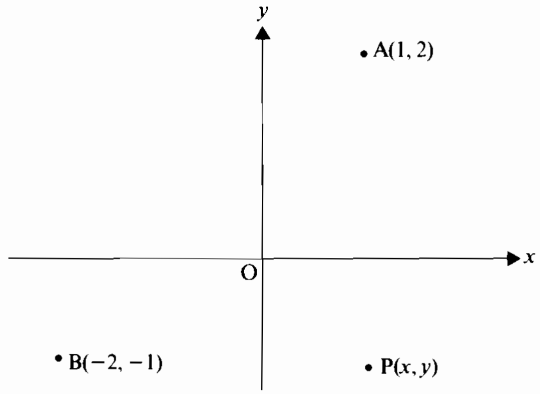
In general, within a plane a point P can be anywhere and, further, if (x, y) are the coordinates of P, then x and y can take any values independently of each other.

However, when the possible positions of P are restricted by some condition to a line (curved or straight), the set of points satisfying this condition is called the *locus* of P

Further, the relationship between x and y which applies only to the locus of P defines that locus and is called the *Cartesian equation* of P

Examples 27a

1. A point, P, is restricted so that it is equidistant from the points A(1, 2) and B(−2, −1). Find the Cartesian equation of P



P is restricted to those positions where PA = PB

Translating this condition into a relationship between x and y gives the equation of the locus of P

$$PA^2 = (x-1)^2 + (y-2)^2 \quad \text{and} \quad PB^2 = (x+2)^2 + (y+1)^2$$

If $PA = PB$ then $PA^2 = PB^2$

Using the given condition in this form avoids introducing square roots.

$$\therefore \quad PA = PB \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = (x+2)^2 + (y+1)^2$$

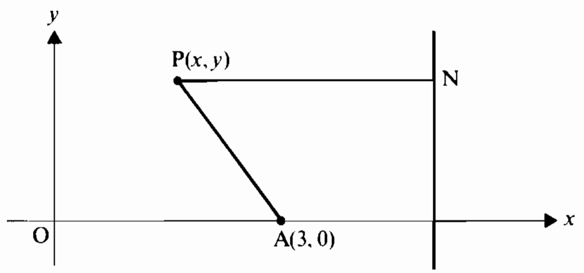
$$\Rightarrow \quad 0 = 6x + 6y$$

Therefore $x + y = 0$ is the equation of the locus of P

Note that the line $x + y = 0$ is the perpendicular bisector of AB

---

2. A point $P(x, y)$ is twice as far from the point $A(3, 0)$ as it is from the line $x = 5$. Find the equation of the locus of P.



The restriction on P is that $PA = 2PN$

Now $\quad PA = 2PN \quad \Rightarrow \quad PA^2 = 4PN^2$

But $\quad PA^2 = y^2 + (x-3)^2$ and $PN^2 = (5-x)^2$

$\therefore$ P satisfies the given condition $\iff y^2 + (x-3)^2 = 4(5-x)^2$

i.e. the equation of the locus of P is $y^2 - 3x^2 + 34x = 91$

EXERCISE 27a

Find the Cartesian equation of the locus of the set of points P in each of the following cases.

1. P is equidistant from the point $(4, 1)$ and the line $x = -2$

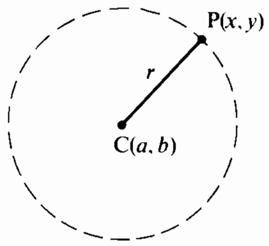
2. P is equidistant from $(3, 5)$ and $(-1, 1)$
3. P is three times as far from the line $x = 8$ as from the point $(2, 0)$
4. P is equidistant from the lines $3x + 4y + 5 = 0$ and $12x - 5y + 13 = 0$
5. P is at a constant distance of two units from the point $(3, 5)$
6. P is at a constant distance of five units from the line $4x - 3y = 1$
7. A is the point $(-1, 0)$, B is the point $(1, 0)$ and angle APB is a right angle.

CIRCLES

If a point P is at a constant distance, r, from a fixed point C then the locus of P is a circle whose centre is C and whose radius is r. In this section we look at a variety of methods for dealing with coordinate geometry problems involving circles.

The Equation of a Circle

A point $P(x, y)$ is at a constant distance, r, from the point $C(a, b)$



P is on the circle if and only if $CP = r$, i.e. $CP^2 = r^2$

Now $\quad CP^2 = (x-a)^2 + (y-b)^2$

$\therefore \quad P(x, y)$ is on the circle $\iff (x-a)^2 + (y-b)^2 = r^2$

$$(x-a)^2 + (y-b)^2 = r^2$$

is the equation of a circle with centre (a, b) and radius r

For example, the equation of a circle with centre $(-2, 3)$ and radius 1 is

$$[x-(-2)]^2+[y-3]^2 = 1$$

$$\Rightarrow \quad x^2+y^2+4x-6y+12 = 0$$

As well as being able to write down the equation of a circle given its centre and radius, it is equally important to be able to recognise an equation as that of a circle. Expanding and simplifying the equation of a circle with centre (a, b) and radius r gives

$$x^2+y^2-2ax-2ay+(a^2+b^2-r^2) = 0$$

which can be expressed as

$$x^2+y^2+2gx+2fy+c = 0$$

where g, f and c are constants.

Comparing coefficients gives

$$g = -a, \quad f = -b, \quad c = a^2+b^2-r^2 \quad \Rightarrow \quad r^2 = f^2+g^2-c$$

So

$x^2+y^2+2gx+2fy+c = 0$ is the general equation of a circle provided that $g^2+f^2-c > 0$

The centre of the circle is $(-g, -f)$ and the radius is $\sqrt{(g^2+f^2-c)}$

Note that the coefficients of x^2 and y^2 are equal and that no xy term is present.

Examples 27b

1. Find the centre and radius of the circle whose equation is

$$x^2+y^2+8x-2y+13 = 0$$

There are two ways of finding the centre and radius of this circle. The first method involves forming perfect squares so that we can compare the given equation with $(x-a)^2+(y-b)^2 = r^2$

$$x^2+8x+16+y^2-2y+1 = 16+1-13$$

$$\Rightarrow \quad (x+4)^2+(y-1)^2 = 4$$

$\therefore$ the centre is $(-4, 1)$ and the radius is 2

Alternatively we can compare the given equation with the general equation of a circle giving

$$2g = 8 \Rightarrow g = 4, \quad 2f = -2 \Rightarrow f = -1 \quad \text{and} \quad c = 13$$

The centre, $(-g, -f)$, is $(-4, 1)$ and the radius, $\sqrt{(g^2+f^2-c)}$, is 2

2. Show that $2x^2+2y^2-6x+10y = 1$ is the equation of a circle and find its centre and radius.

Before we can compare this equation with the general form for the equation of a circle, we must divide the given equation by 2

$$2x^2+2y^2-6x+10y-1 = 0 \quad \Rightarrow \quad x^2+y^2-3x+5y-\tfrac{1}{2} = 0$$

Comparing with $\quad x^2+y^2+2gx+2fy+c = 0$

shows that $2g = -3, \; 2f = 5, \; c = -\frac{1}{2}$

$\Rightarrow \quad (g^2+f^2-c) = 9$ which is greater than 0

Therefore the equation does represent a circle.

The centre is $(-\frac{3}{2}, \frac{5}{2})$ and the radius is 3

EXERCISE 27b

1. Write down the equation of the circle with
(a) centre $(1, 2)$, radius 3
(b) centre $(0, 4)$, radius 1
(c) centre $(-3, -7)$, radius 2
(d) centre $(4, 5)$, radius 3

2. Find the centre and radius of the circle whose equation is
(a) $x^2+y^2+8x-2y-8 = 0$
(b) $x^2+y^2+x+3y-2 = 0$
(c) $x^2+y^2+6x-5 = 0$
(d) $2x^2+2y^2-3x+2y+1 = 0$
(e) $x^2+y^2 = 4$
(f) $(x-2)^2+(y+3)^2 = 9$
(g) $2x+6y-x^2-y^2 = 1$
(h) $3x^2+3y^2+6x-3y-2 = 0$

3. Determine which of the following equations represent circles.
(a) $x^2+y^2 = 8$
(b) $2x^2+y^2+3x-4 = 0$
(c) $x^2-y^2 = 8$
(d) $x^2+y^2+4x-2y+20 = 0$
(e) $x^2+y^2+8 = 0$
(f) $x^2+y^2+4x-2y-20 = 0$

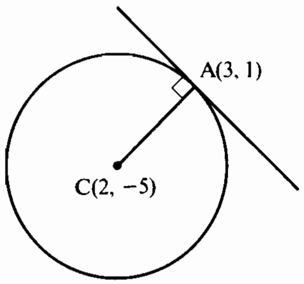
Tangents to Circles and Other Problems

The following worked examples illustrate how a variety of problems concerning circles can be solved easily with the aid of a diagram and the use of the simple geometric properties of a circle.

It is unnecessary to use calculus methods to find the equation of a tangent to a circle.

Examples 27c

1. Find the equation of the tangent at the point (3, 1) on the circle $x^2 + y^2 - 4x + 10y - 8 = 0$. What is the angle between this tangent and the positive direction of the x-axis?



The centre of the circle is C(2, −5)

The tangent at A is perpendicular to the radius CA

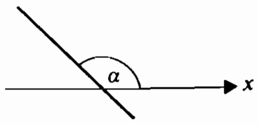
The gradient of CA is $\dfrac{1-(-5)}{3-2} = 6$

Therefore the gradient of the tangent at A is $-\frac{1}{6}$ and the tangent goes through A(3, 1)

So its equation is $\quad y - 1 = -\frac{1}{6}(x-3)$

i.e. $\quad 6y + x = 9$

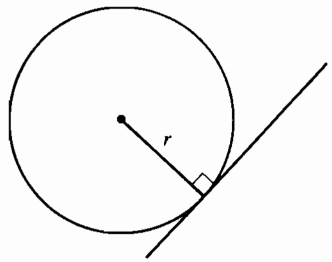
If α is the angle between the tangent and the positive direction of the x-axis,



then $\quad \tan\alpha = -\frac{1}{6}$

$\Rightarrow \quad \alpha = 170.5°$

2. Determine whether the lines $5y = 12x - 33$ and $3x + 4y = 9$ are tangents to the circle $x^2 + y^2 + 2x - 8y = 8$



A line is a tangent to a circle if and only if the distance from the centre of the circle to the line is equal to the radius.

Writing the equation of the circle as

$$(x+1)^2 + (y-4)^2 = 8 + 1 + 16 = 25$$

shows that C(−1, 4) is the centre and the radius is 5

For the line $5y = 12x - 33$, i.e. $12x - 5y - 33 = 0$, the distance d_1 from the centre (−1, 4) is given by

$$d_1 = \left|\frac{12(-1) - 5(4) - 33}{\sqrt{[12^2 + (-5)^2]}}\right| = \left|-\frac{65}{13}\right| = 5$$

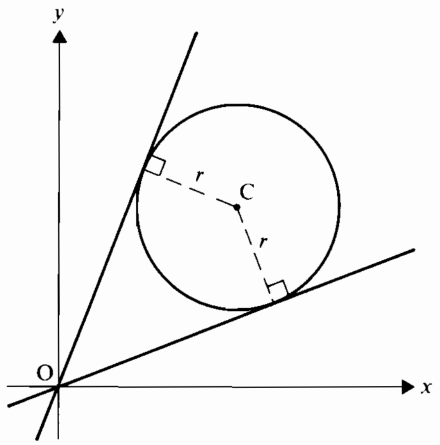
i.e. $d_1 = r$. Thus $12x - 5y - 33 = 0$ *is* a tangent.

For the line $3x + 4y = 9$, i.e. $3x + 4y - 9 = 0$, the distance d_2 from (−1, 4) is given by

$$d_2 = \left|\frac{3(-1) + 4(4) - 9}{\sqrt{[3^2 + 4^2]}}\right| = \frac{4}{5}$$

i.e. $d_2 \neq 5$. Thus $3x + 4y - 9 = 0$ *is not* a tangent.

3. Find the equations of the tangents from the origin to the circle $x^2 + y^2 - 5x - 5y + 10 = 0$



The given circle has centre $(\frac{5}{2}, \frac{5}{2})$

and radius $$\sqrt{\left[\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 10\right]} = \frac{\sqrt{10}}{2}$$

Any line through the origin has equation $y = mx$, i.e. $mx - y = 0$

If the line is a tangent to the circle, the distance from the centre, $(\frac{5}{2}, \frac{5}{2})$, to the line is equal to the radius

i.e. $$\left|\frac{m(\frac{5}{2}) - (\frac{5}{2})}{\sqrt{[(m)^2 + (-1)^2]}}\right| = \frac{\sqrt{10}}{2}$$

$$\Rightarrow \quad \left(\frac{5}{2}m - \frac{5}{2}\right)^2 = \frac{10}{4}(m^2 + 1)$$

$$\Rightarrow \quad 3m^2 - 10m + 3 = 0$$

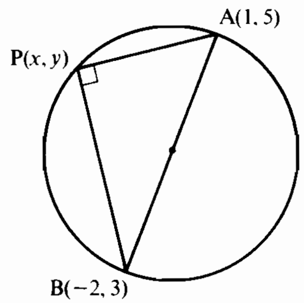
$$\Rightarrow \quad (3m - 1)(m - 3) = 0$$

$$\Rightarrow \quad m = \tfrac{1}{3} \text{ or } 3$$

So the two tangents from the origin to the given circle are

$$y = 3x \quad \text{and} \quad 3y = x$$

4. Find the equation of the circle whose diameter is the line joining the points A(1, 5) and B(−2, 3)



We can use the fact that the angle in a semicircle is 90°

P(x, y) is a point on the circle if and only if

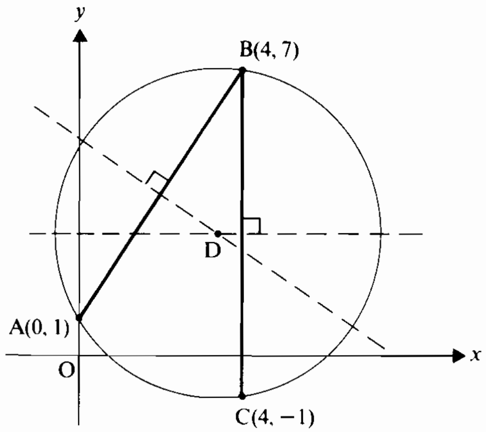
$$(\text{gradient AP}) \times (\text{gradient BP}) = -1$$

The gradient of AP is $\dfrac{y-5}{x-1}$ and the gradient of PB is $\dfrac{y-3}{x+2}$

$$\therefore \quad P(x, y) \text{ is on the circle} \iff \left(\frac{y-5}{x-1}\right)\left(\frac{y-3}{x+2}\right) = -1$$

$\therefore$ the equation of the circle is $x^2 + y^2 + x - 8y + 13 = 0$

5. Find the equation of the circle that goes through the points A(0, 1), B(4, 7) and C(4, −1)



We can use the fact that the centre of a circle lies on the perpendicular bisector of a chord.

The midpoint of AB is the point $(2, 4)$ and the gradient of AB is $\frac{3}{2}$

$\therefore$ the perpendicular bisector of AB is the line

$$2x + 3y = 16 \qquad [1]$$

The midpoint of BC is the point $(4, 3)$ and BC is vertical

$\therefore$ the perpendicular bisector of BC is horizontal and its equation is

$$y = 3 \qquad [2]$$

Solving equations [1] and [2] gives $x = \frac{7}{2}$ and $y = 3$

$\therefore$ D is the point $(\frac{7}{2}, 3)$

The radius, r, is the length of DA (or DC or DB), i.e.

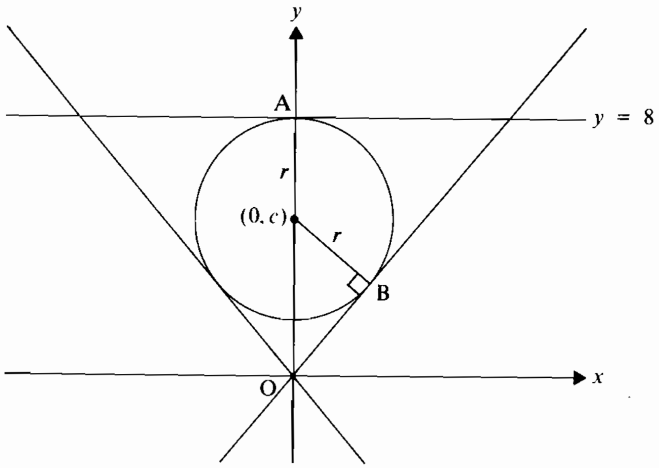
$$r^2 = (3-1)^2 + (\tfrac{7}{2} - 0)^2 = \tfrac{65}{4}$$

Therefore the equation of the circle is

$$(x - \tfrac{7}{2})^2 + (y-3)^2 = \tfrac{65}{4} \quad \Rightarrow \quad x^2 + y^2 - 7x - 6y + 5 = 0$$

---

6. The lines $3y = 4x$, $4x + 3y = 0$ and $y = 8$ are tangents to a circle. Find the equation of the circle.



The centre, C, of the circle is on the line that bisects the angle between the tangents $3y = 4x$ and $4x + 3y = 0$. These two tangents are equally inclined to the y-axis so, from symmetry, C lies on the y-axis and its x-coordinate is 0

There is no information from which the y-coordinate of C can be found directly, so we will call it c. We can then use the fact that the distance from C to each tangent is equal to the radius, r, to form an equation in c.

$$r = CA = |8 - c| \quad \text{and} \quad r = CB = \pm(0 - 3c)/5$$

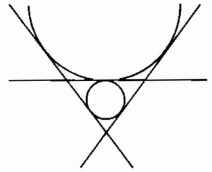
$$\therefore \quad 5(8 - c) = \pm(-3c) \quad \Rightarrow \quad c = 20 \quad \text{or} \quad c = 5$$

Therefore there are two circles that satisfy the given conditions.

From $r = |8 - c|$, the corresponding values of the radii are 12 and 3

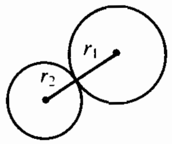
The equation of the circle is either

$$x^2 + (y - 20)^2 = 144 \quad \text{or} \quad x^2 + (y-5)^2 = 9$$

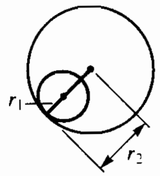


Touching Circles

If two circles touch externally then the distance between their centres is equal to the sum of their radii.



If the circles touch internally then the distance between their centres is equal to the difference of their radii.



---

7. Show that the circles $(x-8)^2 + (y-6)^2 = 25$ and $(5x - 16)^2 + (5y - 12)^2 = 25$ touch each other.

The circle $(x-8)^2 + (y-6)^2 = 25$ has centre $(8, 6)$ and radius 5

The equation $(5x-16)^2 + (5y-12)^2 = 25$ can be divided by 25, giving

$(x - \frac{16}{5})^2 + (y - \frac{12}{5})^2 = 1$ so the centre is $(\frac{16}{5}, \frac{12}{5})$ and the radius is 1

The distance between the centres is $\sqrt{[(8 - \frac{16}{5})^2 + (6 - \frac{12}{5})^2]} = 6$

The sum of the radii of the circles is $5 + 1 = 6$

Therefore the circles touch externally.

EXERCISE 27c

1. Determine whether the given line is a tangent to the given circle in each of the following cases.
   (a) $3x - 4y + 14 = 0$; $x^2 + y^2 + 4x + 6y - 3 = 0$
   (b) $5x + 12y = 4$; $x^2 + y^2 - 2x - 2y + 1 = 0$
   (c) $x + 2y + 6 = 0$; $x^2 + y^2 - 6x - 4y + 8 = 0$
   (d) $x + 2y + 6 = 0$; $x^2 + y^2 - 6x + 4y + 8 = 0$

2. Write down the equation of the tangent to the given circle at the given point.
   (a) $x^2 + y^2 - 2x + 4y - 20 = 0$; $(5, 1)$
   (b) $x^2 + y^2 - 10x - 22y + 129 = 0$; $(6, 7)$
   (c) $x^2 + y^2 - 8y + 3 = 0$; $(-2, 7)$

Find the equations of the following circles (in some cases more than one circle is possible).

3. A circle passes through the points $(1, 4)$, $(7, 5)$ and $(1, 8)$

4. A circle has its centre on the line $x + y = 1$ and passes through the origin and the point $(4, 2)$

5. The line joining $(2, 1)$ to $(6, 5)$ is a diameter of a circle.

6. A circle with centre $(2, 7)$ passes through the point $(-3, -5)$

7. A circle intersects the y-axis at the origin and at the point $(0, 6)$ and also touches the x-axis.

8. A circle touches the negative x and y axes and also the line $7x + 24y + 12 = 0$

Find the equations of the tangents specified in Questions 9 to 11.

9. Tangents *from* the origin to the circle $x^2 + y^2 - 10x - 6y + 25 = 0$

10. The tangent *at* the origin to the circle $x^2 + y^2 + 2x + 4y = 0$

11. Tangents to the circle $x^2 + y^2 - 4x + 6y - 7 = 0$ which are parallel to the line $2x + y = 3$

12. Show that if the point P is twice as far from the point $(4, -2)$ as it is from the origin then P lies on a circle. Find the centre and radius of this circle.

13. Determine which of the following pairs of circles touch.
   (a) $x^2 + y^2 + 2x - 4y + 1 = 0$; $x^2 + y^2 - 6x - 10y + 25 = 0$
   (b) $x^2 + y^2 + 8x + 2y - 8 = 0$; $x^2 + y^2 - 16x - 8y = 64$
   (c) $x^2 + y^2 + 6x = 0$; $x^2 + y^2 + 6x - 4y + 12 = 0$
   (d) $x^2 + y^2 + 2x - 8y + 1 = 0$; $x^2 + y^2 - 6y = 0$
   (e) $x^2 + y^2 + 2x = 3$; $x^2 + y^2 - 6x - 3 = 0$

14. If $y = 2x + c$ is a tangent to the circle $x^2 + y^2 + 4x - 10y - 7 = 0$ find the value(s) of c

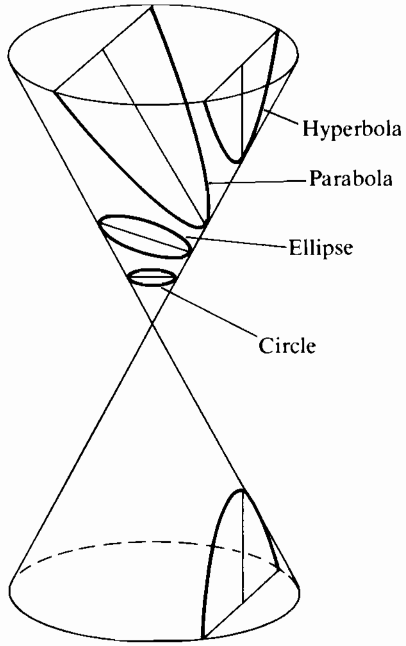
15. Find the condition that m and c satisfy if the line $y = mx + c$ touches the circle $x^2 + y^2 - 2ax = 0$

THREE MORE CLASSIC CURVES

The geometry of the circle, together with that of three more curves, the parabola, the ellipse and the hyperbola, has been part of mathematical investigation since classical times.

These curves were first defined and studied by Apollonius of Perga (*c*.250–200 BC). He defined them as curves traced out on the surface of a cone when it is cut by a plane; hence these curves are also known as conic sections.

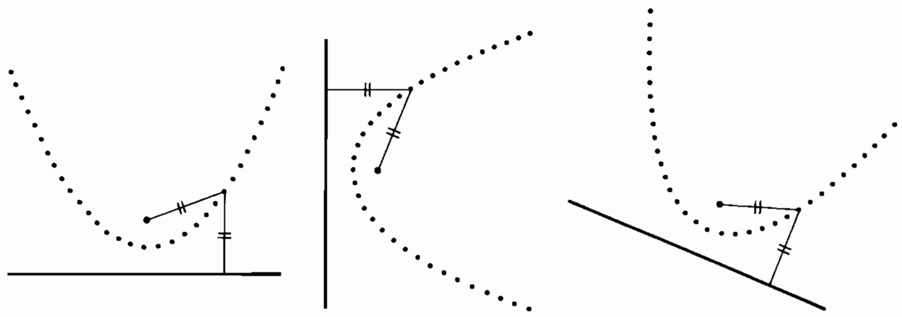
Each of these curves can also be defined as a locus and its properties analysed using coordinate geometry.



The general form of the Cartesian equation of the circle is now known, and in this section we will look briefly at the equations of the other curves.

The Parabola

If a point, P, is constrained so that its distance from a fixed line is equal to its distance from a fixed point, then the locus of P is a parabola. The curve given by this definition can be seen by plotting some of the possible positions of P.

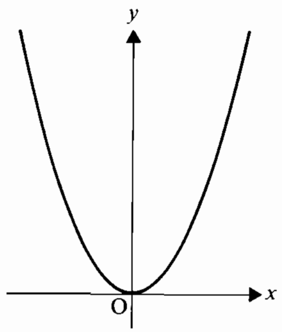


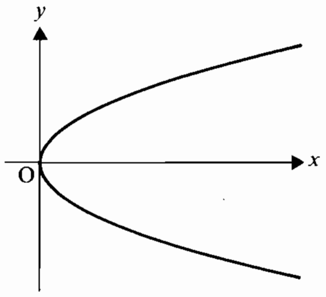
This is a familiar curve; it has an axis of symmetry and, in Chapter 12, we saw that when the axis of symmetry is parallel to Oy, the Cartesian equation of a parabola has the form $y = ax^2 + bx + c$

Similarly, when the axis of symmetry is parallel to Ox, the equation has the form $x = ay^2 + by + c$

The simplest forms of these equations are

$y = x^2$ $\qquad$ $x = y^2$, (i.e. $y^2 = x$)

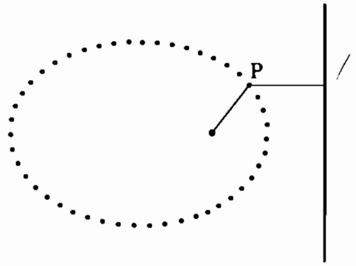




The classical position of the parabola for Cartesian analysis is with its axis parallel to Ox

The Ellipse

The locus of a point, P, is an ellipse when P moves so that its distance from a fixed point and its distance from a fixed line are in a constant ratio which is less than 1. This can be verified by plotting some positions of P for a value of the constant, say $\frac{1}{2}$

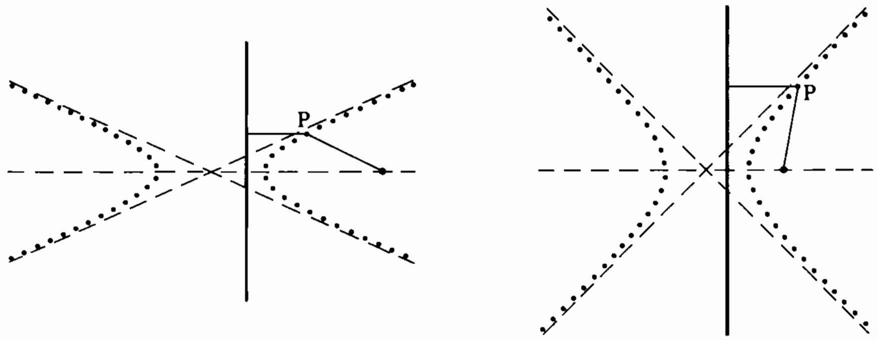


The Cartesian equation of a particular ellipse can be found from the locus definition using the method adopted earlier in this chapter.

The standard equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

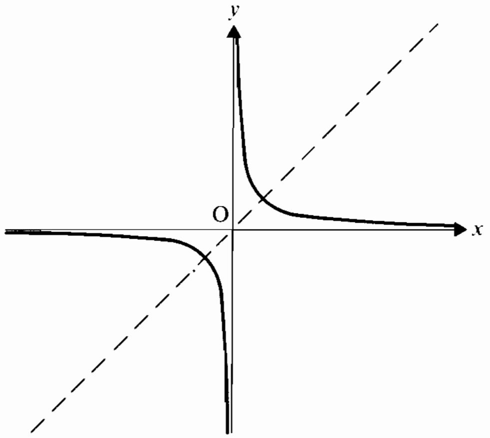
The Hyperbola

The locus definition of a hyperbola is very similar to that of the ellipse. A hyperbola is the locus of P when the ratio of the distance of P from a fixed point to the distance of P from a fixed line is constant and greater than 1. The shape of the hyperbola can be seen when some positions of P are plotted.



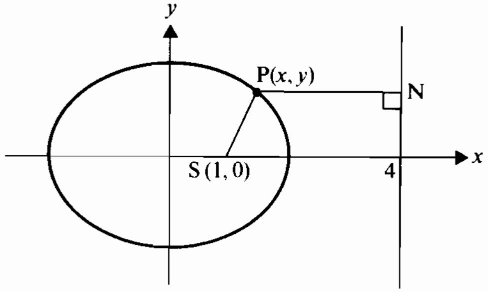
The standard equation of a hyperbola in this position is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

All hyperbolas have a line of symmetry and a pair of asymptotes. When these asymptotes are perpendicular, the curve has a familiar shape; it is called a rectangular hyperbola. Further, when the asymptotes are the x and y axes, the equation of the curve has the form $y = c^2/x$ or $xy = c^2$



Example 27d

$P(x, y)$ is constrained so that the ratio of its distance from $(1, 0)$ to its distance from the line $x = 4$ is equal to $\frac{1}{2}$. Find the equation of the locus of P.



$$\frac{PS}{PN} = \frac{1}{2} \quad \Rightarrow \quad \frac{PS^2}{PN^2} = \frac{1}{4}$$

Now $\quad PS^2 = (x-1)^2 + y^2$ and $PN^2 = (x-4)^2$

$\therefore$ $P(x, y)$ is on the curve $\iff \dfrac{(x-1)^2 + y^2}{(x-4)^2} = \dfrac{1}{4}$

$\therefore$ the equation of the locus of P is $3x^2 + 4y^2 = 12$

EXERCISE 27d

Find the equation of the locus of P and name the curve in each of the following cases.

1. P is twice as far from the point $(2, 0)$ as it is from the line $x = 8$
2. P is equidistant from the point $(0, 4)$ and the line $y = -4$
3. P is half as far from the point $(2, 0)$ as it is from the line $x = 8$
4. Without plotting them, name each of the following curves.
   (a) $y = x^2 - 3$ (b) $y^2 + x^2 = 9$ (c) $xy = 1$
   (d) $x^2 - 3y^2 = 9$ (e) $y^2 = x + 2$ (f) $4x^2 + y^2 = 16$
5. The equation of a curve is $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Find the coordinates of the points where the curve cuts the x and y axes. Sketch the curve.
6. Sketch the curve whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
7. On the same set of axes sketch the curves
   $$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \text{and} \quad \frac{x^2}{16} - \frac{y^2}{4} = 1$$
   (a) Give the coordinates of the points where curves cross the coordinate axes.
   (b) At how many points do the curves intersect?

PARAMETERS

The relationship between the x and y coordinates of a point on any standard conic involves at least one term of order 2 (i.e. x^2, y^2, xy). This relationship can often be expressed more simply in the form of two equations, i.e.

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \quad \text{where } t \text{ is a parameter}$$

The use of parametric equations to plot curves, find gradients and hence tangents and normals to curves is covered in Chapter 23. In this chapter we look at other ways in which parametric equations can be used, and in particular at the parametric equations for conic sections.

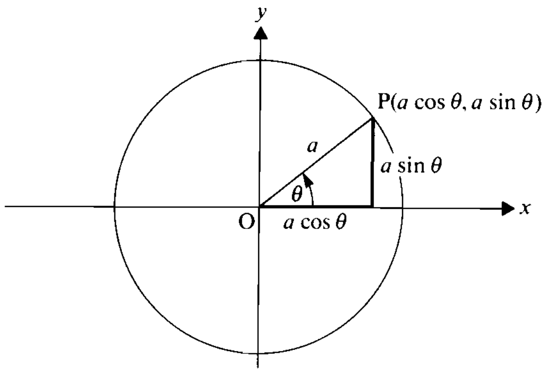
Parametric Equations for a Circle

Consider the circle whose centre is the origin and whose radius is a.

The Cartesian equation of this circle is $x^2 + y^2 = a^2$ [1]

The parametric equations of this circle are $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$ [2]

(Using the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to eliminate θ from the parametric equations verifies that equations [2] are equivalent to equation [1].) The parameter, θ, has graphical significance in this case, as can be seen in the diagram below.

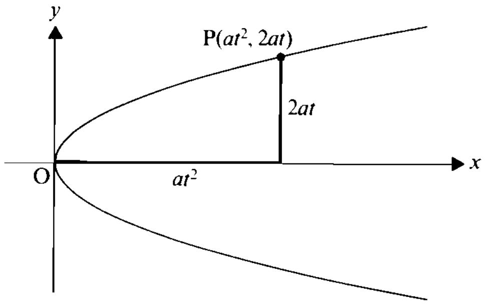


Parametric Equations for a Parabola

Consider the parabola whose vertex is the origin and whose axis is Ox.

If the Cartesian equation of this parabola is $y^2 = 4ax$ then

the parametric equations are $\begin{cases} x = at^2 \\ y = 2at \end{cases}$



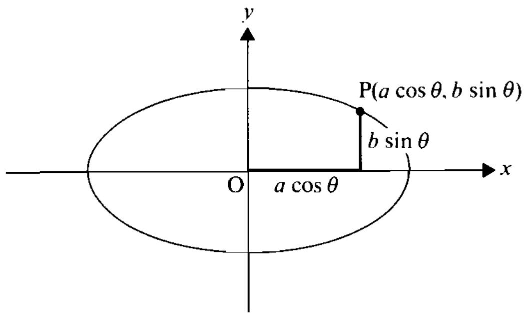
In this case t has no geometrical meaning.

Parametric Equations for an Ellipse

The standard equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

The parametric equations of this ellipse are $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$

There is no *obvious* geometrical significance of θ in this case.

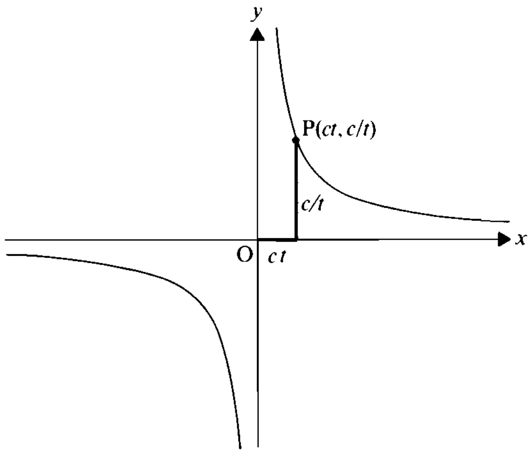


Parametric Equations for a Rectangular Hyperbola

The Cartesian equation of the rectangular hyperbola is $xy = c^2$

The parametric equations of this curve are $\begin{cases} x = ct \\ y = c/t \end{cases}$

Again, t has no *obvious* significance.



CURVE SKETCHING

Algebraic analysis is usually easier when the parametric equations of a curve are used. However sketching curves, particularly circles, is easier using Cartesian equations which show clearly the important features.

Examples 27e

1. On the same set of axes, sketch the curves

$$\begin{cases} x = 2\cos\theta + 1 \\ y = 2\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} x = t^2 \\ y = 4t \end{cases}$$

Hence determine the number of points in which the curves cut.

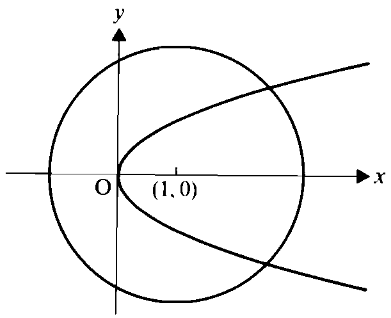
The first pair of equations can be converted to a Cartesian equation by finding $\cos\theta$ in terms of x and $\sin\theta$ in terms of y and then using the identity $\cos^2\theta + \sin^2\theta \equiv 1$

$$\left.\begin{array}{l} x = 2\cos\theta + 1 \\ y = 2\sin\theta \end{array}\right\} \Rightarrow \quad (x-1)^2 + y^2 = 4$$

This is a circle, centre $(1, 0)$ radius 2

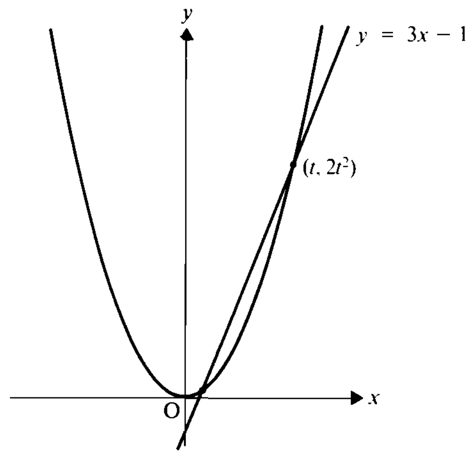
$$\left.\begin{array}{l} x = t^2 \\ y = 4t \end{array}\right\} \Rightarrow \quad y^2 = 16x$$

This is a parabola, vertex $(0, 0)$ and axis Ox.



The sketch shows that there are two points of intersection.

2. Find the coordinates of the points where the line $y = 3x - 1$ cuts the curve whose parametric equations are $x = t$, $y = 2t^2$



The coordinates of any point on the curve are $(t, 2t^2)$

The line cuts the curve where the coordinates of a point on the curve satisfy the equation of the line, i.e. where

$$2t^2 = 3t - 1$$

$$\Rightarrow \quad 2t^2 - 3t + 1 = 0 \quad \Rightarrow \quad (t-1)(2t-1) = 0$$

This is a quadratic equation in t, giving two values of t and therefore giving two distinct points on the curve. This is expected from the sketch which indicates that there are two points of intersection.

So $\quad t = 1$ or $\frac{1}{2}$

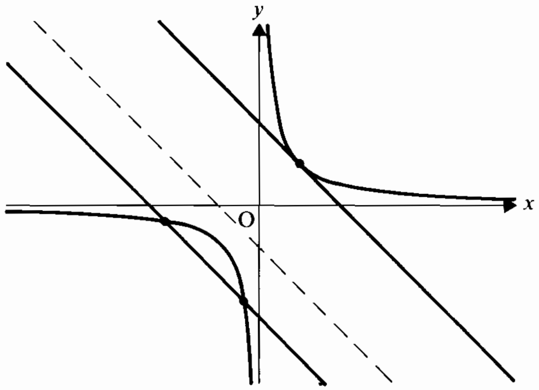
Therefore the points of intersection are $(1, 2)$ and $(\frac{1}{2}, \frac{1}{2})$

3. Form an equation which gives the values of t at points where the line $y = mx + c$ crosses the curve whose parametric equations are $x = 2t$, $y = 2/t$

Find the condition that must be satisfied by m and c if

(a) the line cuts the curve in two places

(b) the line is a tangent to the curve.



The point $(2t, 2/t)$ is any point on the curve, and the line $y = mx + c$ cuts the curve where $(2t, 2/t)$ is also a point on the line, i.e. where

$$\frac{2}{t} = m(2t) + c \quad \Rightarrow \quad 2mt^2 + ct - 2 = 0 \qquad [1]$$

This equation is quadratic in t, so it can have either two distinct real roots, a repeated root or no real roots.

(a) $y = mx + c$ cuts the curve in two places if equation [1] has real distinct roots,

i.e. if $\quad$ '$b^2 - 4ac > 0$' $\Rightarrow$ $c^2 - 4(2m)(-2) > 0$

$\Rightarrow$ $c^2 + 16m > 0$

(b) $y = mx + c$ is a tangent to the curve if equation [1] has a repeated root,

i.e. if $\quad$ '$b^2 - 4ac = 0$' $\Rightarrow$ $c^2 + 16m = 0$

(If equation [1] has no real roots then the line $y = mx + c$ does not meet the curve.)

EXERCISE 27e

1. Sketch the curves given parametrically by

(a) $x = t^2, y = 2t$ (b) $x = 3\cos\theta, y = 3\sin\theta$
(c) $x = t^2 + 1, y = t$ (d) $x = 3t, y = 3/t$
(e) $x = 4t, y = t^2$ (f) $x = 3\cos\theta, y = 4\sin\theta$
(g) $x = 3t + 2, y = t^2 - 1$ (h) $x = \cos\theta + 2, y = \sin\theta - 1$

2. On the same set of axes sketch the curves defined parametrically by

(a) $x = t^2, y = t$ and $x = \cos\theta, y = \sin\theta$
(b) $x = 2/t, y = 2t$ and $x = 4\cos\theta, y = 6\sin\theta$
(c) $x = t, y = 3t - 1$ and $x = 4t^2, y = 2t$

3. On the same set of axes sketch the curves given by $x = 2t, y = t^2$ and $x = 4\cos\theta, y = 4\sin\theta$
Find the coordinates of the points of intersection of these two curves.

4. Determine whether the line $y = 2x + 1$ cuts, touches or misses each of the following curves.

(a) $x = t^2, y = 4t$ (b) $x = t^2, y = t$
(c) $x = 2t^2, y = 4t$ (d) $x = \cos\theta, y = \sin\theta + 1$

5. A curve has parametric equations $x = 2t^2, y = 4t$. Find

(a) the Cartesian equation of the curve,
(b) the equation of the tangent at the point where $y = 8$
(c) the equation of the chord joining the points on the curve where $t = p$ and $t = q$
(d) the coordinates of the points where $y = x - 6$ cuts the curve,
(e) the value of k for which $y = x + k$ is a tangent to the curve,
(f) the coordinates of the point(s) of intersection of the curve and the circle $x^2 + y^2 - 2x = 16$
(g) the coordinates of the point(s) of intersection of the curve and the curve given parametrically by $x = 8s, y = 8/s$

CHAPTER 28

BASIC INTEGRATION

DIFFERENTIATION REVERSED

When x^2 is differentiated with respect to x the derivative is $2x$

Conversely, if the derivative of an unknown function is $2x$ then it is clear that the unknown function could be x^2

This process of finding a function from its derivative, which reverses the operation of differentiating, is called *integration*.

The Constant of Integration

As seen above, $2x$ is the derivative of x^2,
but it is also the derivative of $x^2 + 3$, $x^2 - 9$, and, in fact,
the derivative of x^2 + any constant.

Therefore the result of integrating $2x$, which is called *the integral of $2x$*, is not a unique function but is of the form

$$x^2 + K \quad \text{where } K \text{ is any constant}$$

K is called *the constant of integration*.

This is written

$$\int 2x\,dx = x^2 + K$$

where $\int \ldots dx$ means the integral of... w.r.t. x

Integrating *any* function reverses the process of differentiating so, for any function $f(x)$ we have

$$\int \frac{d}{dx} f(x)\,dx = f(x) + K$$

e.g. because differentiating x^3 w.r.t. x gives $3x^2$ we have

$$\int 3x^2\,dx = x^3 + K$$

and it follows that

$$\int x^2\,dx = \tfrac{1}{3}x^3 + K$$

Note that it is not necessary to write $\frac{1}{3}K$ in the second form, as K represents *any* constant in either expression.

In general, the derivative of x^{n+1} is $(n+1)x^n$ so

$$\int x^n\,dx = \frac{1}{(n+1)}x^{n+1} + K$$

i.e. to integrate a power of x,

increase the power by 1 and *divide* by the new power.

This rule can be used to integrate any power of x *except* -1, *which is considered later.*

Integrating a Sum or Difference of Functions

We saw in Chapter 22 that a function can be differentiated term by term. Therefore, as integration reverses differentiation, integration also can be done term by term.

Example 28a

Find the integral of $x^7 + \dfrac{1}{x^2} - \sqrt{x}$

$$\begin{aligned}\int \left(x^7 + \frac{1}{x^2} - \sqrt{x}\right) dx &= \int (x^7 + x^{-2} - x^{1/2})\,dx \\ &= \int x^7\,dx + \int x^{-2}\,dx - \int x^{1/2}\,dx \\ &= \frac{1}{8}x^8 + \frac{1}{-1}x^{-1} - \frac{1}{\frac{3}{2}}x^{3/2} + K \\ &= \frac{1}{8}x^8 - \frac{1}{x} - \frac{2}{3}x^{3/2} + K\end{aligned}$$

EXERCISE 28a

Integrate with respect to x,

1. x^5
2. $\dfrac{1}{x^5}$
3. $\sqrt[4]{x}$
4. x^{-3}
5. $\dfrac{1}{x^{5/2}}$
6. $x^{-1/2}$
7. x^1
8. $\dfrac{1}{\sqrt[3]{x}}$

Integrating $(ax + b)^n$

First consider the function $f(x) = (2x + 3)^4$

To differentiate $f(x)$ we make a substitution

i.e. $u = 2x + 3 \quad \Rightarrow \quad f(x) = u^4$

giving $$\frac{d}{dx}(2x + 3)^4 = (4)(2)(2x + 3)^3$$

Hence $$\int (4)(2)(2x + 3)^3\,dx = (2x + 3)^4 + K$$

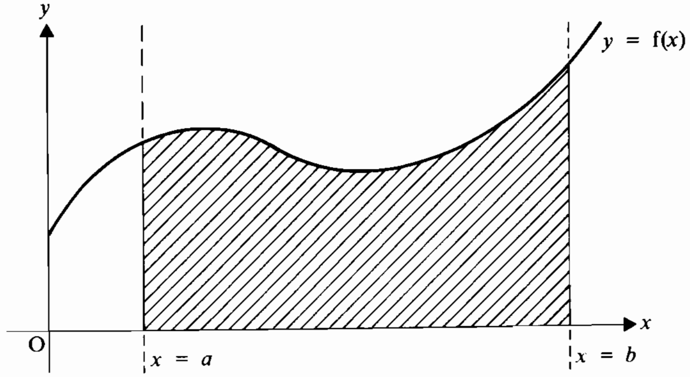
or $$\int (2x + 3)^3\,dx = \frac{1}{(2)(4)}(2x + 3)^4 + K$$

Considering $f(x) = (ax + b)^{n+1}$ in a similar way gives the general result

$$\int (ax + b)^n\,dx = \frac{1}{(a)(n + 1)}(ax + b)^{n+1} + K$$

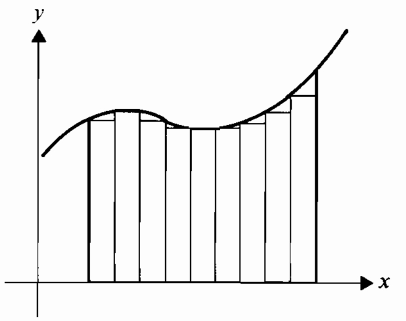
USING INTEGRATION TO FIND AN AREA

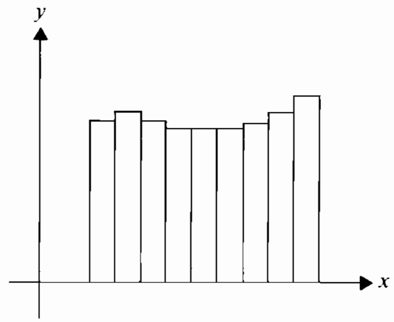
The area shown in the diagram is bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$



118

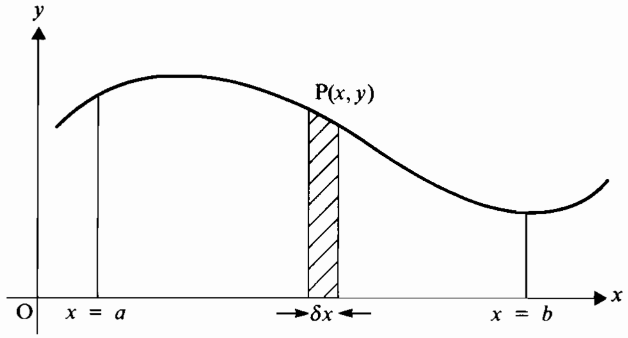
There are several elementary ways in which this area can be estimated, e.g. by counting squares on graph paper. A better method is to divide the area into thin vertical strips and treat each strip, or *element*, as being approximately rectangular.





The sum of the areas of the rectangular strips then gives an approximate value for the required area. The thinner the strips are, the better is the approximation.

Note that every strip has one end on the x-axis, one end on the curve and two vertical sides, i.e., they all have the same type of boundaries.



Now, considering a typical element bounded on the left by the ordinate through a general point $P(x, y)$, we see that

the width of the element represents a small increase in the value of x and so can be called δx

Also, if A represents the part of the area up to the ordinate through P, then

the area of the element represents a small increase in the value of A and so can be called δA

The shape of a typical strip is approximately a rectangle of height y and width δx

Therefore, for any element

$$\delta A \approx y\delta x \qquad [1]$$

The required area can now be found by adding the areas of all the strips from $x = a$ to $x = b$

The notation for a summation of this kind is $\sum_{x=a}^{x=b} \delta A$

so, $$\text{total area} = \sum_{x=a}^{x=b} \delta A$$

$$\Rightarrow \quad \text{total area} \approx \sum_{x=a}^{x=b} y\,\delta x$$

As δx gets smaller the accuracy of the results increases until, in the limiting case,

$$\text{total area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x$$

Equation [1] above can also be written in the alternative form

$$\frac{\delta A}{\delta x} \approx y$$

This form too becomes more accurate as δx gets smaller giving, in the limiting case,

$$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y$$

But $$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \text{ is } \frac{dA}{dx} \text{ so } \frac{dA}{dx} = y$$

Hence $$A = \int y\,dx$$

The boundary values of x defining the total area are $x = a$ and $x = b$ and we indicate this by writing

$$\text{total area} = \int_a^b y\,dx$$

The total area can therefore be found in two ways, either as the limit of a sum or by integration

i.e. $$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$$

and we conclude that integration is a process of summation.

The application of integration to problems involving summation continues in Chapter 33. In this chapter we will use integration only to find areas bounded by straight lines and a curve, but first we must investigate the meaning of $\int_a^b y\,dx$

DEFINITE INTEGRATION

Suppose that we wish to find the area bounded by the x-axis, the lines $x = a$ and $x = b$ and the curve $y = 3x^2$

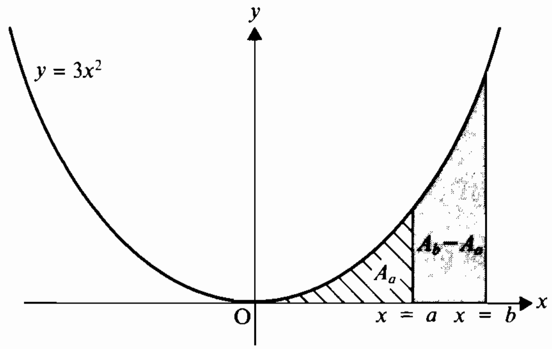
Using the method above we find that $A = \int 3x^2\,dx$

$$\text{i.e. } A = x^3 + K$$

From this area function we can find the value of A corresponding to a particular value of x.

Hence using $x = a$ gives $A_a = a^3 + K$

and using $x = b$ gives $A_b = b^3 + K$



Then the area between $x = a$ and $x = b$ is given by $A_b - A_a$ where $A_b - A_a = (b^3 + K) - (a^3 + K) = b^3 - a^3$

Now $A_b - A_a$ is referred to as

the definite integral from a to b of $3x^2$

and is denoted by $\int_a^b 3x^2\,dx$

i.e. $\int_a^b 3x^2\,dx = (x^3)_{x=b} - (x^3)_{x=a}$

The RHS of this equation is usually written in the form $\left[x^3\right]_a^b$ where a and b are called the *boundary values* or *limits of integration*; b is the *upper limit* and a is the *lower limit.*

Whenever a definite integral is calculated, the constant of integration disappears.

Note. A definite integral can be found in this way only if the function to be integrated is defined for every value of x from a to b, e.g. $\int_{-1}^{1} \frac{1}{x^2}\,dx$ cannot be found directly as $\frac{1}{x^2}$ is undefined when $x = 0$

Example 28b

Evaluate $\int_1^4 \frac{1}{(x+3)^2}\,dx$

$$\int_1^4 \frac{1}{(x+3)^2}\,dx \equiv \int_1^4 (x+3)^{-2}\,dx$$

$$= \left[-(x+3)^{-1}\right]_1^4$$

$$= \{-(4+3)^{-1}\} - \{-(1+3)^{-1}\}$$

$$= -\tfrac{1}{7} + \tfrac{1}{4} = \tfrac{3}{28}$$

EXERCISE 28b

Evaluate each of the following definite integrals.

1. $\int_0^2 x^3\,dx$
2. $\int_1^2 \sqrt{x^5}\,dx$
3. $\int_2^4 (x^2+4)\,dx$
4. $\int_3^8 \sqrt{(1+x)}\,dx$

5. $\int_0^3 (x^2+2x-1)\,dx$
6. $\int_0^2 (x^3-3x)\,dx$
7. $\int_{-1}^0 \frac{1}{(1-x)^2}\,dx$
8. $\int_{-1}^2 \frac{3}{\sqrt{(x+2)}}\,dx$
9. $\int_{-1}^0 (2+3x)^6\,dx$
10. $\int_{1/2}^7 (4x-1)^{1/3}\,dx$

FINDING AREA BY DEFINITE INTEGRATION

119 120

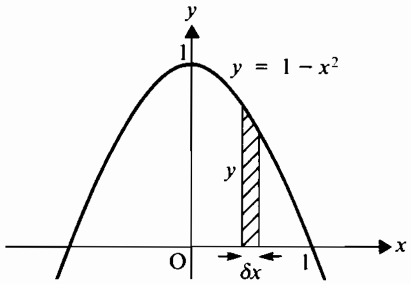
As we have seen, the area bounded by a curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis, can be found from the definite integral

$$\int_a^b f(x)\,dx$$

It is recommended, however, that this is not regarded as a *formula* but that the required area is first considered as the summation of the areas of elements, a typical element being shown in a diagram.

Example 28c

Find the area in the first quadrant bounded by the x and y axes and the curve $y = 1 - x^2$



The required area starts at the y-axis, i.e. at $x = 0$ and ends where the curve crosses the x-axis, i.e. where $x = 1$. So it is given by

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=1} y\,\delta x = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1$$

$$= (1 - \tfrac{1}{3}) - (0 - 0) = \tfrac{2}{3}$$

The required area is $\frac{2}{3}$ of a square unit.

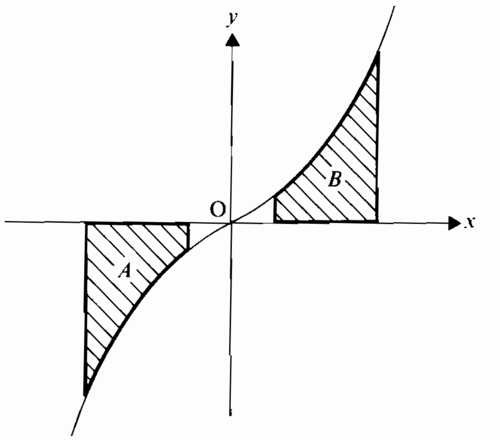
EXERCISE 28c

In each question find the area with the given boundaries.

1. The x-axis, the curve $y = x^2 + 3$ and the lines $x = 1$, $x = 2$
2. The curve $y = \sqrt{x}$, the x-axis and the lines $x = 4$, $x = 9$
3. The x-axis, the lines $x = -1$, $x = 1$, and the curve $x^2 + 1$
4. The curve $y = x^2 + x$, the x-axis and the line $x = 3$
5. The positive x and y axes and the curve $y = 4 - x^2$
6. The lines $x = 2$, $x = 4$, the x-axis and the curve $y = x^3$
7. The curve $y = 4 - x^2$, the positive y-axis and the negative x-axis.
8. The x-axis, the lines $x = 1$ and $x = 2$, and the curve $y = \frac{1}{2}x^3 + 2x$
9. The x-axis and the lines $x = 1$, $x = 5$, and $y = 2x$
   Check the result by sketching the required area and finding it by mensuration

The Meaning of a Negative Result

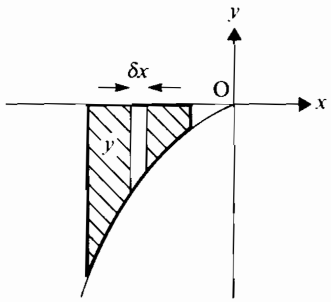
Consider the area bounded by $y = 4x^3$ and the x-axis if the other boundaries are the lines
(a) $x = -2$ and $x = -1$ (b) $x = 1$ and $x = 2$



This curve is symmetrical about the origin so the two shaded areas are equal.

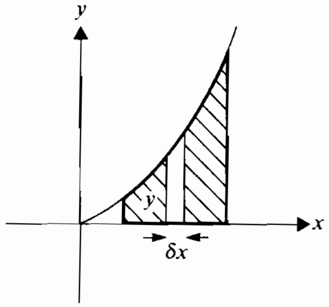
(a) Considering A

$$\lim_{\delta x \to 0} \sum_{x=-2}^{x=-1} y\,\delta x = \int_{-2}^{-1} y\,dx$$
$$= \int_{-2}^{-1} 4x^3\,dx$$
$$= \Big[x^4\Big]_{-2}^{-1}$$
$$= 1 - 16 = -15$$



(b) Considering B

$$\lim_{\delta x \to 0} \sum_{x=1}^{x=2} y\,\delta x = \int_{1}^{2} y\,dx$$
$$= \int_{1}^{2} 4x^3\,dx$$
$$= \Big[x^4\Big]_{1}^{2}$$
$$= 16 - 1 = 15$$

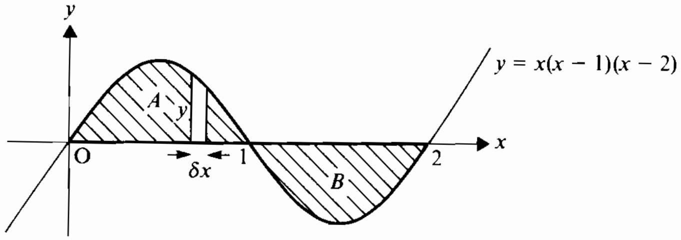


So we see that, while the magnitudes of the two areas are equal, the result for the area of A, which is below the x-axis, is negative. This is explained by the fact that the length of a strip in A was taken as y, which is negative for the part of the curve bounding A.

Note. Care must be taken with problems involving a curve that crosses the x-axis between the boundary values.

Example 28d

Find the area enclosed between the curve $y = x(x-1)(x-2)$ and the x-axis.



The area enclosed between the curve and the x-axis is the sum of the areas A and B.

For A we use
$$\int_0^1 y\,dx = \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$
$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1$$
$$= \tfrac{1}{4}$$

For B we use
$$\int_1^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$$
$$= (4 - 8 + 4) - (\tfrac{1}{4} - 1 + 1)$$
$$= -\tfrac{1}{4}$$

The minus sign refers only to the *position* of area B relative to the x-axis.
The actual area is $\frac{1}{4}$ of a square unit.
So the total shaded area is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ of a square unit.

EXERCISE 28d

In each Question from 1 to 5 find the specified area.

1. The area below the x-axis and above the curve $y = x^2 - 1$

2. The area bounded by the curve $y = 1 - x^3$, the x-axis and the lines $x = 2$, $x = 3$

3. The area between the x and y axes and the curve $y = (x-1)^2$

4. Sketch the curve $y = x(x^2 - 1)$, showing where it crosses the x-axis. Find
   (a) the area enclosed above the x-axis and below the curve
   (b) the area enclosed below the x-axis and above the curve
   (c) the total area between the curve and the x-axis

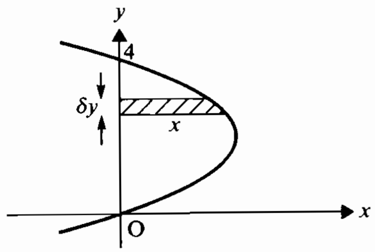
5. Repeat Question 4 for the curve $y = x(4 - x^2)$

6. Evaluate
   (a) $\int_0^2 (x-2)\,dx$ (b) $\int_2^4 (x-2)\,dx$ (c) $\int_0^4 (x-2)\,dx$

   Interpret your results by means of a sketch.

USING HORIZONTAL ELEMENTS

Suppose that area between the curve $x = y(4-y)$ and the y-axis is required.



The curve crosses the y-axis where $y = 0$ and $y = 4$ as shown.

A vertical element is not suitable in this case because it has *both ends on the curve* and its length is therefore not easily found.

However it is easy to find the approximate area of a *horizontal* strip, by treating it as a rectangle with length x and width δy

i.e. area of element $\approx x\,\delta y$ and the required area is therefore given by

$$\lim_{\delta y \to 0} \sum_{y=0}^{y=4} x\,\delta y = \int_0^4 x\,dy = \int_0^4 y(y-4)\,dy$$

EXERCISE 28e

1. Evaluate
   (a) $\int_3^6 (y^2 - y)\,dy$ (b) $\int_2^{10} \frac{1}{\sqrt{(2y+5)}}\,dy$ (c) $\int_{-4}^3 (4-y)^{-2/3}\,dy$

In Questions 2 to 5 find the area specified by the given boundaries.

2. The y-axis and the curve $x = 9 - y^2$

3. The curve $x = y^2$, the y-axis and the lines $y = 1$, $y = 2$

4. The y-axis, the curve $x = \sqrt{y}$ and the line $y = 4$

5. The y-axis and the curve $x = (y-2)(y+1)$

6. Find the area in the first quadrant bounded by the x and y axes and the curve $y = 16 - x^2$
   (a) by using vertical elements
   (b) by using horizontal elements and the equation of the curve in the form $x = \sqrt{(16-y)}$

7. If $y = x^2$, show by means of sketch graphs and *not* by evaluating the integrals, that $\int_0^1 y\,dx = 1 - \int_0^1 x\,dy$

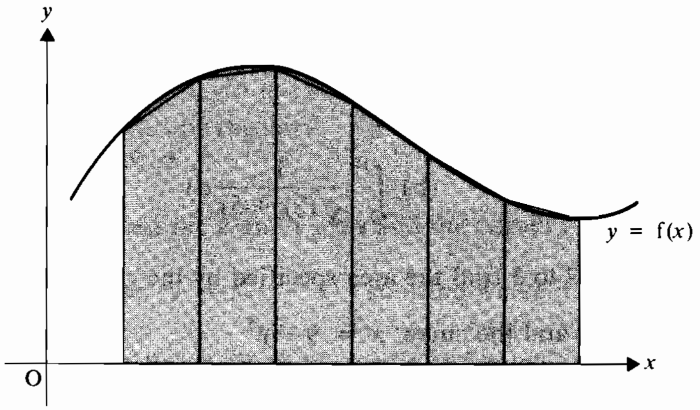
THE APPROXIMATE VALUE OF A DEFINITE INTEGRAL

We know that the definite integral $\int_a^b f(x)\,dx$ can be used to evaluate the area between the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$. It is not always possible, however, to find a function whose derivative is $f(x)$. In such cases the definite integral, and hence the exact value of the specified area, cannot be found.

If, on the other hand, we divide the area into a *finite* number of strips then the sum of their areas gives an approximate value for the required area and hence an approximate value of the definite integral. When using this approach it is convenient to choose strips whose widths are all the same.

The Trapezium Rule

121



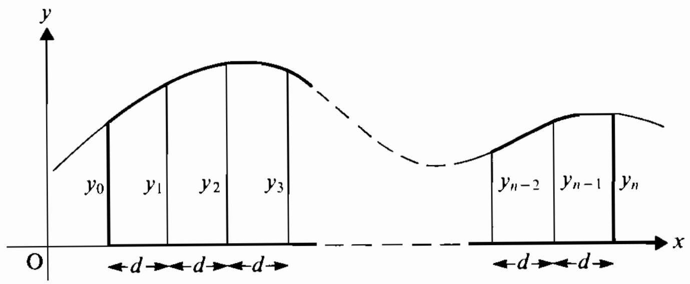
When the area shown in the diagram is divided into vertical strips, each strip is approximately a trapezium.

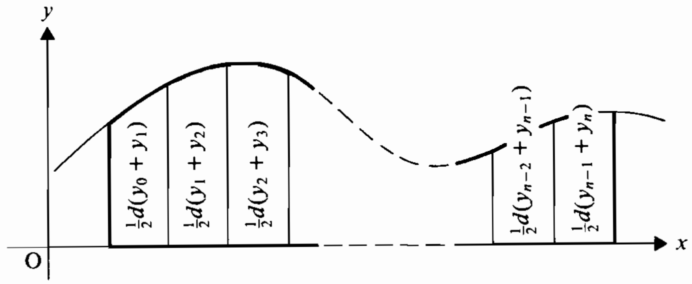
If the width of the strip and its two vertical sides are known, the area of the strip can be found using the formula

$$\text{area} = \tfrac{1}{2}(\text{sum of } \parallel \text{ sides}) \times \text{width}$$

The sum of the areas of all the strips then gives an approximate value for the area under the curve.

Now suppose that there are n strips, *all with the same width*, d say, and that the vertical edges of the strips (i.e. the ordinates) are labelled $y_0, y_1, y_2, \dots, y_{n-1}, y_n$





The sum of the areas of all the strips can be written down as follows:

$$\tfrac{1}{2}(y_0 + y_1)(d) + \tfrac{1}{2}(y_1 + y_2)(d) + \tfrac{1}{2}(y_2 + y_3)(d) + \dots$$
$$\dots + \tfrac{1}{2}(y_{n-2} + y_{n-1})(d) + \tfrac{1}{2}(y_{n-1} + y_n)(d)$$

Therefore the area, A, under the curve is given approximately by

$$A \approx \tfrac{1}{2}(d)\left[y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n\right]$$

This formula is known as the *Trapezium Rule*

An easy way to remember the formula in terms of ordinates is

half width of strip × (first + last + twice all the others)

Simpson's Rule

122

A formula which gives a better approximation than that obtained from the trapezium rule is known as *Simpson's Rule*.
Using the same notation as before it states that

$$A \approx \tfrac{1}{3}(d)\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots 2y_{n-2} + 4y_{n-1} + y_n\right]$$

i.e. $A \approx \tfrac{1}{3}d\left[\{y_0 + y_n\} + 4\{y_1 + y_3 + \dots\} + 2\{y_2 + y_4 + \dots\}\right]$

This formula, given here without proof, is based on dividing the required area into equal width strips and, for each *pair* of strips, finding a parabola which passes through the top of the three ordinates bounding the two strips.

Because this formula is based on pairs of strips it follows that it can be used only when the number of strips is even,
i.e. when the number of ordinates is odd.

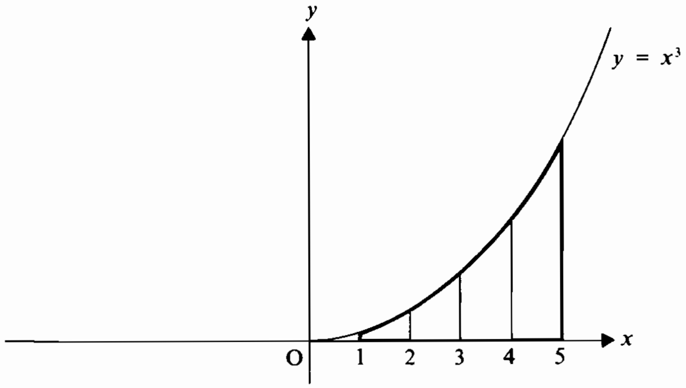
When the number of strips is large the calculation involved in either the trapezium rule or Simpson's rule is tedious, but a computer can deal with it very simply. So only a few simple questions are set in the next exercise.

Example 28f

Find an approximate value for the definite integral $\int_1^5 x^3\,dx$ using

(a) the trapezium rule (b) Simpson's rule with five ordinates.

The given definite integral represents the area bounded by the x-axis, the lines $x = 1$ and $x = 5$, and the curve $y = x^3$



When five ordinates are used there are four strips and their widths must all be the same. From $x = 1$ to $x = 5$ there are four units so the width of each strip must be 1 unit. Hence the five ordinates are where $x = 1$, $x = 2$, $x = 3$, $x = 4$ and $x = 5$

$\therefore\ y_0 = 1^3 = 1,\ y_1 = 2^3 = 8,\ y_2 = 3^3 = 27,\ y_3 = 64,\ y_4 = 125$

(a) Using the trapezium rule, the required area, A, is given by

$$A \approx \tfrac{1}{2}(1)\left[1 + 125 + 2\{8 + 27 + 64\}\right] = 162$$

The required area is approximately 162 square units.

(b) There is an odd number of ordinates so Simpson's rule can be used.

A is given by

$$A \approx \tfrac{1}{3}(1)\left[\{1 + 125\} + 4\{8 + 64\} + 2\{27\}\right] = 156$$

The required area is approximately 156 square units.

The degree of accuracy of an answer given by either of these rules clearly depends upon the number of strips into which the required area is divided, because the narrower the strip, (a) the nearer its shape becomes to a trapezium, (b) the more nearly does the top of two strips fit a parabola.

EXERCISE 28f

In Questions 1 to 6 estimate the value of each definite integral, using
(a) the trapezium rule (b) Simpson's rule
each with (i) 3 ordinates (ii) 5 ordinates.

1. $\int_0^4 x^2\,dx$ **2.** $\int_1^3 \frac{1}{x^2}\,dx$

3. $\int_0^{2\pi/3} \sqrt{\sin x}\,dx$ **4.** $\int_1^3 \ln x\,dx$

5. Find the true value of the definite integrals given in Questions 1 and 2 Complete the table below and note the comparative accuracy of the results given by the two rules.

Value using	$\int_0^4 x^2\,dx$	$\int_1^3 \frac{1}{x^2}\,dx$
Trapezium rule with 5 ordinates		
Simpson's rule with 5 ordinates		
Definite integration		

MIXED EXERCISE 28

Integrate with respect to x

1. $x^2 - 1/x^2$ 2. $\sqrt{(3x+7)}$ 3. $\sqrt{x} + 1/\sqrt{x}$

4. $\dfrac{1}{(4x-3)}$ 5. $x^3 - \dfrac{1}{(1-x)^3}$ 6. $\dfrac{x^2-1}{\sqrt{x}}$

Evaluate

7. $\displaystyle\int_5^6 (6-x)^4\,dx$ 8. $\displaystyle\int_{-1}^{12} \frac{3}{\sqrt[3]{(2y+3)}}\,dy$ 9. $\displaystyle\int_1^{32} \left(\sqrt[5]{x} - \frac{1}{\sqrt[5]{x}}\right) dx$

Find the areas specified in Questions 10 to 12

10. Bounded by the x and y axes and the curve $y = 1 - x^3$

11. Bounded by the curve $x = y^2 - 4$ and the y-axis.

12. The *total* area between the curve $y = (x-1)(x-2)(x-3)$ and the x-axis.

13. (a) Find an approximate value for the area between the x-axis and the curve $y = (x-1)(x-4)$, using the trapezium rule with 4 ordinates.

 (b) Evaluate $\displaystyle\int_1^4 (x-1)(x-4)\,dx$

14. (a) Use the trapezium rule with 3 ordinates to estimate the value of $\displaystyle\int_0^5 (3+x)\,dx$

 (b) Find the value of $\displaystyle\int_0^5 (3+x)\,dx$

 (c) Explain the connection between the results of (a) and (b).

CHAPTER 29

FURTHER INTEGRATION 1

STANDARD INTEGRALS

Whenever a function $f(x)$ is *recognised* as the derivative of a function $f(x)$ then

$$\frac{d}{dx}f(x) = f'(x) \quad \Rightarrow \quad \int f'(x)\,dx = f(x) + K$$

Thus any function whose derivative is known can be established as a standard integral.

INTEGRATING EXPONENTIAL FUNCTIONS

It is already known that $\dfrac{d}{dx}e^x = e^x$

hence $\displaystyle\int e^x\,dx = e^x + K$

Further, we have $\dfrac{d}{dx}(ce^x) = ce^x$

and $\dfrac{d}{dx}e^{(ax+b)} = ae^{(ax+b)}$

Hence $\displaystyle\int ce^x\,dx = ce^x + K$

and $\displaystyle\int e^{(ax+b)} = \frac{1}{a}e^{(ax+b)} + K$

e.g. $\displaystyle\int 2e^x\,dx = 2e^x + K$ and $\displaystyle\int 4e^{(1-3x)}\,dx = (4)(-\tfrac{1}{3})e^{(1-3x)} + K$

To integrate an exponential function where the given base is not e but is some other constant, a say, the base must first be changed to e as follows.

Using $a^x = e^z$ and taking logs to the base e we have

$$x \ln a = z$$

Hence $\quad a^x = e^{x\ln a} \quad \Rightarrow \quad \int a^x \, dx = \int e^{x \ln a} \, dx$

$$\Rightarrow \quad \int a^x \, dx = \frac{1}{\ln a} e^{x \ln a} + K$$

i.e. $\quad \int a^x \, dx = \frac{1}{\ln a} a^x + K$

Alternatively, this result can be obtained directly if it is remembered that

$$\frac{d}{dx}(a^x) = (\ln a) a^x$$

Example 29a

Write down the integral of e^{3x} w.r.t. x and hence evaluate $\int_0^1 e^{3x} \, dx$

$$\int e^{3x} \, dx = \tfrac{1}{3} e^{3x} + K$$

The constant of integration disappears when a definite integral is calculated, hence

$$\int_0^1 e^{3x} \, dx = \left[\tfrac{1}{3} e^{3x}\right]_0^1 = \tfrac{1}{3} e^3 - \tfrac{1}{3} e^0$$

i.e. $\quad \int_0^1 e^{3x} \, dx = \tfrac{1}{3}(e^3 - 1)$

EXERCISE 29a

Integrate each function w.r.t. x

1. e^{4x}	**2.** $4e^{-x}$	**3.** $e^{(3x-2)}$	**4.** $2e^{(1-5x)}$
5. $6e^{-2x}$	**6.** $5e^{(x-3)}$	**7.** $e^{(2+x/2)}$	**8.** 2^x
9. $4^{(2+x)}$	**10.** $e^{2x} + \dfrac{1}{e^{2x}}$	**11.** $a^{(1-2x)}$	**12.** $2^x + x^2$

Evaluate the following definite integrals.

13. $\int_0^2 e^{2x} \, dx$	**14.** $\int_{-1}^1 2e^{(x+1)} \, dx$
15. $\int_2^3 e^{(2-x)} \, dx$	**16.** $\int_0^2 -e^x \, dx$

FUNCTIONS WHOSE INTEGRALS ARE LOGARITHMIC

To Integrate $\frac{1}{x}$

At first sight it looks as though we can write $\frac{1}{x} = x^{-1}$ and integrate by using the rule $\int x^n \, dx = \frac{1}{n+1} x^{(n+1)} + K$

However, this method fails when $n = -1$ because the resulting integral is meaningless.

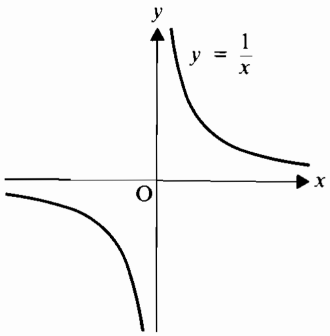
Taking a second look at $\frac{1}{x}$ it can be *recognised* as the derivative of $\ln x$.

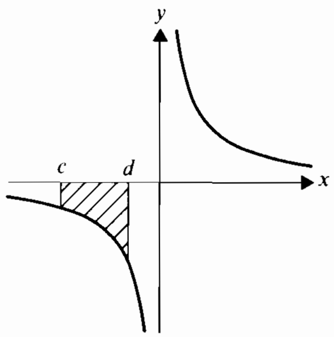
It must be remembered, however, that $\ln x$ is defined only when $x > 0$. Hence, provided that $x > 0$ we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \Leftrightarrow \quad \int \frac{1}{x} \, dx = \ln x + K$$

Now if $x < 0$ the statement $\int \frac{1}{x} \, dx = \ln x$ is not valid because the log of a negative number does not exist.

However, $\frac{1}{x}$ exists for negative values of x, as the graph of $y = \frac{1}{x}$ shows.





Also, the definite integral $\int_c^d \frac{1}{x}\,dx$, which is represented by the shaded area, clearly exists. It must, therefore, be possible to integrate $\frac{1}{x}$ when x is negative and we see below how to deal with the problem.

If $x < 0$ then $-x > 0$

i.e. $$\int \frac{1}{x}\,dx = \int \frac{-1}{(-x)}\,dx = \ln(-x) + K$$

Thus, when $x > 0$, $\int \frac{1}{x}\,dx = \ln x + K$

and when $x < 0$, $\int \frac{1}{x}\,dx = \ln(-x) + K$

These two results can be combined so that, for both positive and negative values of x, we have

$$\int \frac{1}{x}\,dx = \ln|x| + K$$

where $|x|$ denotes the numerical value of x regardless of sign

e.g. $|-1| = 1$ and $|-4| = 4$

The expression $\ln|x| + K$ can be simplified if K is replaced by $\ln A$, where A is a positive constant, giving

$$\int \frac{1}{x}\,dx = \ln|x| + \ln A = \ln A|x|$$

Further $\frac{d}{dx}(\ln x^c) = \frac{d}{dx}(c\ln x) = \frac{c}{x}$

$$\therefore \qquad \int \frac{c}{x}\,dx = c\ln|x| + K \quad \text{or} \quad c\ln A|x|$$

e.g. $$\int \frac{4}{x}\,dx = 4\ln|x| + K \quad \text{or} \quad 4\ln A|x|$$

Also $\frac{d}{dx}\ln(ax+b) = \frac{a}{ax+b}$

$$\therefore \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + K = \frac{1}{a}\ln A|ax+b|$$

e.g. $$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5| \quad \text{or} \quad \frac{1}{2}\ln A|2x+5|$$

and $$\int \frac{1}{4-3x}\,dx = -\frac{1}{3}\ln|4-3x| + K \quad \text{or} \quad -\frac{1}{3}\ln A|4-3x|$$

$$= \frac{1}{3}\ln\frac{A}{|4-3x|}$$

EXERCISE 29b

Integrate w.r.t. x giving each answer in a form which
(a) uses K (b) uses $\ln A$ and is simplified.

1. $\frac{1}{2x}$ **2.** $\frac{4}{x}$ **3.** $\frac{1}{3x+1}$ **4.** $\frac{3}{1-2x}$

5. $\frac{6}{2+3x}$ **6.** $\frac{3}{4-2x}$ **7.** $\frac{4}{1-x}$ **8.** $\frac{5}{6-7x}$

Evaluate

9. $\int_1^2 \frac{3}{x}\,dx$ **10.** $\int_1^2 \frac{1}{2x}\,dx$ **11.** $\int_4^5 \frac{2}{x-3}\,dx$ **12.** $\int_0^1 \frac{1}{2-x}\,dx$

INTEGRATING TRIGONOMETRIC FUNCTIONS

Knowing the derivatives of the six trig functions, we can recognise the following integrals.

$$\frac{d}{dx}(\sin x) = \cos x \iff \int \cos x\,dx = \sin x + K$$

$$\frac{d}{dx}(\cos x) = -\sin x \iff \int \sin x\,dx = -\cos x + K$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \iff \int \sec^2 x\,dx = \tan x + K$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x \iff \int \sec x\tan x\,dx = \sec x + K$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x \iff \int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + K$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \iff \int \operatorname{cosec}^2 x\,dx = -\cot x + K$$

Remembering the derivatives of some variations of the basic trig functions we also have

$$\int c\cos x\,dx = c\sin x + K$$

and $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + K$$

with similar results for the remaining trig integrals,

e.g. $$\int 3\sec^2 x\,dx = 3\tan x + K$$

$$\int \sin 4\theta\,d\theta = -\tfrac{1}{4}\cos 4\theta + K$$

$$\int \operatorname{cosec}^2(2x+\tfrac{3}{4}\pi)\,dx = -\tfrac{1}{2}\cot(2x+\tfrac{3}{4}\pi) + K$$

$$\int \operatorname{cosec} 5\theta\cot 5\theta\,d\theta = -\tfrac{1}{5}\operatorname{cosec} 5\theta + K$$

$$\int \sec(\tfrac{1}{2}\pi - 6x)\tan(\tfrac{1}{2}\pi - 6x)\,dx = -\tfrac{1}{6}\sec(\tfrac{1}{2}\pi - 6x) + K$$

Note that there is no need to *learn* these standard integrals. Knowledge of the standard derivatives is sufficient.

EXERCISE 29c

Integrate each function w.r.t. x

1. $\sin 2x$	**2.** $\cos 7x$	**3.** $\sec^2 4x$
4. $\sin(\frac{1}{4}\pi + x)$	**5.** $3\cos(4x - \frac{1}{2}\pi)$	**6.** $\sec^2(\frac{1}{3}\pi + 2x)$
7. $\operatorname{cosec}^2 4x$	**8.** $2\sin(3x - \alpha)$	**9.** $5\cos(\alpha - \frac{1}{2}x)$
10. $5\sec 4x\tan 4x$	**11.** $\cos 3x - \cos x$	**12.** $\sec^2 2x - \operatorname{cosec}^2 4x$

Evaluate

13. $\int_0^{\pi/6} \sin 3x\,dx$ **14.** $\int_{\pi/4}^{\pi/6} \cos(2x - \frac{1}{2}\pi)\,dx$

15. $\int_0^{\pi/2} 2\sin(2x - \frac{1}{2}\pi)\,dx$ **16.** $\int_0^{\pi/8} \sec^2 2x\,dx$

FUNCTIONS WHOSE INTEGRALS ARE INVERSE TRIG FUNCTIONS

We know that $y = \arcsin x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1-x^2)}}$

Therefore $$\int \frac{1}{\sqrt{(1-x^2)}}\,dx = \arcsin x + K$$

Similarly it can be seen that

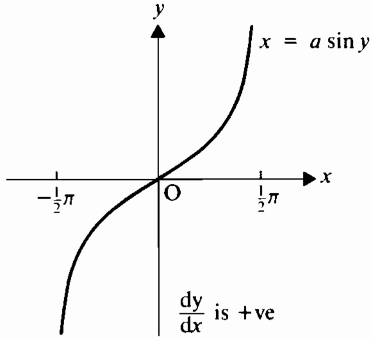
$$\int \frac{1}{1+x^2}\,dx = \arctan x + K$$

Now consider $y = \arcsin \dfrac{x}{a} \Rightarrow x = a\sin y$

Hence $$\frac{dx}{dy} = a\cos y$$

$$= a\left(\frac{\sqrt{(a^2-x^2)}}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$$



Therefore $$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \arcsin\frac{x}{a} + K$$

Similar working shows that if $y = \arctan \dfrac{x}{a} \Rightarrow x = a\tan y$

then $$\frac{dx}{dy} = a\sec^2 y = a\left(\frac{\sqrt{(a^2-x^2)}}{a}\right)^2$$

Therefore $$\int \frac{a}{a^2+x^2}\,dx = \arctan\frac{x}{a} + K$$

EXERCISE 29d

Write down the integral w.r.t. x of each function.

1. $\dfrac{1}{\sqrt{(1-x^2)}}$ 2. $\dfrac{2}{x^2+4}$ 3. $\dfrac{1}{\sqrt{(4-x^2)}}$ 4. $\dfrac{3}{1+x^2}$

5. $\dfrac{5}{x^2+9}$ 6. $\dfrac{1}{\sqrt{(9-x^2)}}$ 7. $\dfrac{1}{16+x^2}$ 8. $\dfrac{5}{\sqrt{(2-x^2)}}$

EXERCISE 29e

This exercise contains a variety of functions, including those dealt with in Chapter 28.
The reader is advised always to check that an integral is correct by differentiating it mentally.

Integrate w.r.t. x

1. $\sin(\frac{1}{2}\pi - 2x)$ 2. $e^{(4x-1)}$ 3. $\sec^2 7x$

4. $\dfrac{1}{2x-3}$ 5. $\dfrac{1}{\sqrt{(2x-3)}}$ 6. $\dfrac{1}{(3x-2)^2}$

7. 5^x 8. $\operatorname{cosec}\frac{1}{2}x \cot\frac{1}{2}x$ 9. $(3x-5)^2$

10. $e^{(4x-5)}$ 11. $\sqrt{(4x-5)}$ 12. $\operatorname{cosec}^2 3x$

13. $\dfrac{3}{2(1-x)}$ 14. $10^{(x+1)}$ 15. $\cos(3x - \frac{1}{3}\pi)$

Evaluate

16. $\displaystyle\int_{-1/2}^{1/2} \sqrt{(1-2x)}\,dx$ 17. $\displaystyle\int_0^2 e^{(x/2+1)}\,dx$ 18. $\displaystyle\int_{\pi/4}^{\pi/2} \sin 4x\,dx$

THE RECOGNITION ASPECT OF INTEGRATION

We have already seen the importance of the recognition aspect of integration in compiling a set of standard integrals.
Recognition is equally important when it is used to avoid serious errors in integration.

Consider, for instance, the derivative of the product $x^2 \sin x$
Using the product formula gives

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x$$

Clearly the derivative is not a simple product, therefore

the integral of a product is not itself a product

i.e. integration is not distributive when applied to a product

On the other hand, when we differentiate the function of a function $(1+x^2)^3$ we get

$$\frac{d}{dx}(1+x^2)^3 = 6x(1+x^2)^2$$

This time the derivative *is* a product so clearly the integral of a product *may* be a function of a function.

INTEGRATING PRODUCTS

First consider the function e^u where u is a function of x.
Differentiating as a function of a function gives

$$\frac{d}{dx}(e^u) = \left(\frac{du}{dx}\right)(e^u)$$

Thus any product of the form $\left(\dfrac{du}{dx}\right)e^u$ can be integrated by recognition, since

$$\int \left(\frac{du}{dx}\right) e^u\,dx = e^u + K$$

e.g.

$$\int 2x\,e^{x^2}\,dx = e^{x^2} + K \quad (u = x^2)$$

$$\int \cos x\, e^{\sin x}\,dx = e^{\sin x} + K \quad (u = \sin x)$$

$$\int x^2 e^{x^3}\,dx = \tfrac{1}{3}\int 3x^2 e^{x^3}\,dx = \tfrac{1}{3}e^{x^3} + K \quad (u = x^3)$$

In these simple cases the substitution of u for $f(x)$ is done mentally.
All the results can be checked by differentiating them mentally.

Similar, but slightly less simple functions, can also be integrated by changing the variable but for these the substitution is written down.

Changing the Variable

Consider a general function $g(u)$ where u is a function of x

$$\frac{d}{dx}g(u) = \frac{du}{dx}g'(u) \quad \text{or} \quad g'(u)\frac{du}{dx}$$

Therefore $$\int g'(u)\frac{du}{dx}\,dx = g(u) + K \qquad [1]$$

We also know that $$\int g'(u)\,du = g(u) + K \qquad [2]$$

Comparing [1] and [2] gives

$$\int g'(u)\frac{du}{dx}\,dx = \int g'(u)\,du$$

Replacing $g'(u)$ by $f(u)$ gives

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

i.e. $$\ldots\frac{du}{dx}dx \equiv \ldots du \qquad [3]$$

Thus integrating (a function of u) $\frac{du}{dx}$ w.r.t. x is *equivalent to* integrating (the same function of u) w.r.t. u

i.e. the relationship in [3] is neither an equation nor an identity but is a pair of equivalent operations.

Suppose, for example, that we want to find $\int 2x(x^2+1)^5\,dx$

Writing the integral in the form $\int (x^2+1)^5\,2x\,dx$ and making the substitution $u = x^2 + 1$ gives

$$\int (x^2+1)^5\,2x\,dx = \int u^5(2x)\,dx$$

But $\frac{du}{dx} = 2x$ and $\ldots\frac{du}{dx}dx \equiv \ldots du$

Therefore $$\ldots 2x\,dx \equiv \ldots du$$

i.e. $$\int (x^2+1)^5\,2x\,dx = \int u^5\,du$$
$$= \tfrac{1}{6}u^6 + K = \tfrac{1}{6}(x^2+1)^6 + K$$

In practice we can go direct from $\frac{du}{dx} = 2x$ to the equivalent operators $\ldots 2x\,dx \equiv \ldots du$ by 'separating the variables'.

Products which can be integrated by this method are those in which one factor is basically the derivative of the function in the other factor.

Examples 29f

1. Integrate $x^2\sqrt{(x^3+5)}$ w.r.t. x

In this product x^2 is basically the derivative of x^3+5 so we choose the substitution $u = x^3 + 5$

If $u = x^3 + 5$ then $\frac{du}{dx} = 3x^2$

$\Rightarrow$ $$\ldots du \equiv \ldots 3x^2\,dx$$

Hence $$\int x^2\sqrt{(x^3+5)}\,dx = \tfrac{1}{3}\int (x^3+5)^{1/2}(3x^2\,dx) = \tfrac{1}{3}\int u^{1/2}\,du$$
$$= (\tfrac{1}{3})(\tfrac{2}{3})u^{3/2} + K$$

i.e. $$\int x^2\sqrt{(x^3+5)}\,dx = \tfrac{2}{9}(x^3+5)^{3/2} + K$$

---

2. Find $\int \cos x \sin^3 x\,dx$

Writing the given integral in the form $\cos x(\sin x)^3$ shows that a suitable substitution is $u = \sin x$

If $u = \sin x$ then $\ldots du \equiv \ldots \cos x\,dx$

$\therefore$ $$\int \cos x\sin^3 x\,dx = \int (\sin x)^3\cos x\,dx = \int u^3\,du$$
$$= \tfrac{1}{4}u^4 + K$$

i.e. $$\int \cos x\sin^3 x\,dx = \tfrac{1}{4}\sin^4 x + K$$

Applied generally, the method used above shows that

$$\int \cos x \sin^n x \, dx = \frac{1}{n+1}\sin^{(n+1)}x + K$$

and similarly that

$$\int \sin x \cos^n x \, dx = \frac{-1}{n+1}\cos^{(n+1)}x + K$$

---

3. Find $\int \frac{\ln x}{x} dx$

Initially this looks like a fraction but once it is recognised as the product of $\frac{1}{x}$ and $\ln x$, it is clear that $\frac{1}{x} = \frac{d}{dx}(\ln x)$ and that we can make the substitution $u = \ln x$

If $u = \ln x$ then $\dots du \equiv \dots \frac{1}{x} dx$

Hence $\int \frac{1}{x} \ln x \, dx = \int u \, du = \frac{1}{2}u^2 + K$

i.e. $\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + K$

Note that $(\ln x)^2$ is *not* the same as $\ln x^2$

EXERCISE 29f

Integrate the following expressions w.r.t. x

1. $4x^3 e^{x^4}$

2. $\sin x \, e^{\cos x}$

3. $\sec^2 x \, e^{\tan x}$

4. $(2x + 1)e^{(x^2 + x)}$

5. $\operatorname{cosec}^2 x \, e^{(1 - \cot x)}$

6. $(1 + \cos x) e^{(x + \sin x)}$

7. $2x \, e^{(1 + x^2)}$

8. $(3x^2 - 2) e^{(x^3 - 2x)}$

Find the following integrals by making the substitution suggested.

9. $\int x(x^2 - 3)^4 dx$ $\quad u = x^2 - 3$

10. $\int x\sqrt{(1 - x^2)} \, dx$ $\quad u = 1 - x^2$

11. $\int \cos 2x(\sin 2x + 3)^2 dx$ $\quad u = \sin 2x + 3$

12. $\int x^2(1 - x^3) \, dx$ $\quad u = 1 - x^3$

13. $\int e^x\sqrt{(1 + e^x)} \, dx$ $\quad u = 1 + e^x$

14. $\int \cos x \sin^4 x \, dx$ $\quad u = \sin x$

15. $\int \sec^2 x \tan^3 x \, dx$ $\quad u = \tan x$

16. $\int x^n(1 + x^{n+1})^2 dx$ $\quad u = 1 + x^{n+1}$

17. $\int \operatorname{cosec}^2 x \cot^2 x \, dx$ $\quad u = \cot x$

18. $\int \sqrt{x}\sqrt{(1 + x^{3/2})} \, dx$ $\quad u = 1 + x^{3/2}$

By using a suitable substitution, or by integrating at sight, find

19. $\int x^3(x^4 + 4)^2 dx$

20. $\int e^x(1 - e^x)^3 dx$

21. $\int \sin\theta\sqrt{(1 - \cos\theta)} \, d\theta$

22. $\int (x + 1)\sqrt{(x^2 + 2x + 3)} \, dx$

23. $\int x \, e^{x^2 + 1} dx$

24. $\int \sec^2 x (1 + \tan x) \, dx$

DEFINITE INTEGRATION WITH A CHANGE OF VARIABLE

A definite integral can be evaluated only after the appropriate integration has been performed. Should this require a change of variable, e.g. from x to u, it is usually most convenient to change the limits of integration also from x values to u values.

Example 29g

By using the substitution $u = x^3 + 1$, evaluate $\int_0^1 x^2\sqrt{(x^3+1)}\,dx$

If $u = x^3 + 1$ then $\quad \dots du \equiv \dots 3x^2\,dx$

and $\begin{cases} x = 0 \Rightarrow u = 1 \\ x = 1 \Rightarrow u = 2 \end{cases}$

Hence
$$\int_0^1 x^2\sqrt{(x^3+1)}\,dx = \tfrac{1}{3}\int_1^2 \sqrt{u}\,du$$
$$= \tfrac{1}{3}\left[\tfrac{2}{3}u^{3/2}\right]_1^2$$
$$= \tfrac{2}{9}(2\sqrt{2} - 1)$$

EXERCISE 29g

Evaluate

1. $\int_0^1 x\,e^{x^2}\,dx$
2. $\int_0^{\pi/2} \cos x \sin^4 x\,dx$
3. $\int_1^2 \frac{1}{x}\ln x\,dx$
4. $\int_1^2 x^2(x^3-1)^4\,dx$
5. $\int_0^{\pi/4} \sec^2 x\,e^{\tan x}\,dx$
6. $\int_1^2 x(1+2x^2)\,dx$
7. $\int_2^3 (x-1)e^{(x^2-2x)}\,dx$
8. $\int_0^{\pi/6} \cos x(1+\sin^2 x)\,dx$
9. $\int_1^3 \frac{1}{x}(\ln x)^2\,dx$
10. $\int_0^{\sqrt{3}} x\sqrt{(1+x^2)}\,dx$

INTEGRATION BY PARTS

It is not always possible to express a product in the form $f(u)\frac{du}{dx}$ so an alternative approach is needed.

Looking again at the differentiation of a product uv where u and v are both functions of x we have

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \quad \Rightarrow \quad v\frac{du}{dx} = \frac{d}{dx}(uv) - u\frac{dv}{dx}$$

Now $v\frac{du}{dx}$ can be taken to represent a product which is to be integrated w.r.t. x

Thus
$$\int v\frac{du}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int u\frac{dv}{dx}\,dx$$

i.e.
$$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$$

At this stage it may appear that the RHS is more complicated than the original product on the LHS.

However, by careful choice of the factor to be replaced by v we can ensure that $u\frac{dv}{dx}$ is easier to integrate than $v\frac{du}{dx}$

The factor chosen to be replaced by v is usually the one whose derivative is a simpler function. It must also be remembered, however, that the other factor is replaced by $\frac{du}{dx}$ and therefore it must be possible to integrate it.

This method for integrating a product is called *integrating by parts*.

Examples 29h

1. Integrate $x\,e^x$ w.r.t. x

Taking $\quad v = x$ and $\frac{du}{dx} = e^x$

gives $\quad \frac{dv}{dx} = 1$ and $u = e^x$

Then $$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$$

gives $$\int x\,e^x\,dx = (e^x)(x) - \int (e^x)(1)\,dx$$
$$= x\,e^x - e^x + K$$

---

2. Find $\int x^2 \sin x\,dx$

Taking $v = x^2$ and $\dfrac{du}{dx} = \sin x$

gives $\dfrac{dv}{dx} = 2x$ and $u = -\cos x$

Then $$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx$$

gives $$\int x^2 \sin x\,dx = (-\cos x)(x^2) - \int (-\cos x)(2x)\,dx$$
$$= -x^2\cos x + 2\int x\cos x\,dx \qquad [1]$$

At this stage the integral on the RHS cannot be found without *repeating* the process of integrating by parts on the term $\int x\cos x\,dx$ as follows.

Taking $v = x$ and $\dfrac{du}{dx} = \cos x$

gives $\dfrac{dv}{dx} = 1$ and $u = \sin x$

Then $$\int x\cos x\,dx = (\sin x)(x) - \int (\sin x)(1)\,dx$$
$$= x\sin x + \cos x + K$$

Hence equation [1] becomes
$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + K$$

---

3. Find $\int x^4 \ln x\,dx$

Because $\ln x$ can be differentiated but *not integrated*, we must use $v = \ln x$

Taking $v = \ln x$ and $\dfrac{du}{dx} = x^4$

gives $\dfrac{dv}{dx} = \dfrac{1}{x}$ and $u = \frac{1}{5}x^5$

The formula for integrating by parts then gives
$$\int x^4\ln x\,dx = (\tfrac{1}{5}x^5)(\ln x) - \int (\tfrac{1}{5}x^5)\left(\frac{1}{x}\right)dx$$
$$= \tfrac{1}{5}x^5\ln x - \tfrac{1}{5}\int x^4\,dx$$

$$\Rightarrow \quad \int x^4\ln x\,dx = \tfrac{1}{5}x^5\ln x - \tfrac{1}{25}x^5 + K$$

Special Cases of Integration by Parts

An interesting situation arises when an attempt is made to integrate $e^x\cos x$ or $e^x\sin x$

---

4. Find $\int e^x\cos x\,dx$

Taking $v = e^x$ and $\dfrac{du}{dx} = \cos x$

gives $\dfrac{dv}{dx} = e^x$ and $u = \sin x$

Hence $$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx \qquad [1]$$

But since $\int e^x\sin x\,dx$ is very similar to $\int e^x\cos x\,dx$ it seems that we have made no progress. However, if we now apply integration by parts to $\int e^x\sin x\,dx$ an interesting situation emerges.

Taking $\quad v = e^x \quad \text{and} \quad \dfrac{du}{dx} = \sin x$

gives $\quad \dfrac{dv}{dx} = e^x \quad \text{and} \quad u = -\cos x$

so that $\quad \displaystyle\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

or $\quad \displaystyle\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \qquad [2]$

Adding [1] and [2] gives

$$2\int e^x \cos x \, dx = e^x(\sin x + \cos x) + K$$

Clearly the same two equations can be used to give

$$2\int e^x \sin x \, dx = e^x(\sin x - \cos x) + K$$

Note that neither of the equations [1] and [2] contains a completed integration process, so the constant of integration is introduced only when these two equations have been combined.

Note also that the same choice of function for v must be made in both applications of integration by parts, i.e. we chose $v = e^x$ each time.

Integration of $\ln x$

So far we have found no way to integrate $\ln x$. Now, however, if $\ln x$ is regarded as the product of 1 and $\ln x$ we can apply integration by parts as follows.

Examples 29h (continued)

5. Find $\displaystyle\int \ln x \, dx$

Taking $\quad v = \ln x \quad \text{and} \quad \dfrac{du}{dx} = 1$

gives $\quad \dfrac{dv}{dx} = \dfrac{1}{x} \quad \text{and} \quad u = x$

Then $\quad \displaystyle\int v\frac{du}{dx}dx = uv - \int u\frac{dv}{dx}dx$

becomes $\quad \displaystyle\int \ln x \, dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$

$$= x\ln x - x + K$$

i.e. $\quad \displaystyle\int \ln x \, dx = x(\ln x - 1) + K$

This 'trick' of multiplying by 1 to create a product can also be used to integrate $\arcsin x$ and other inverse trig functions as will be shown in Examples 29i.

EXERCISE 29h

Integrate the following functions w.r.t. x

1. $x\cos x$ **2.** $x^2 e^x$ **3.** $x^3 \ln 3x$

4. $x e^{-x}$ **5.** $3x \sin x$ **6.** $e^x \sin 2x$

7. $e^{2x}\cos x$ **8.** $x^2 e^{4x}$ **9.** $e^{-x}\sin x$

10. $\ln 2x$ **11.** $e^x(x+1)$ **12.** $x(1+x)^7$

13. $x\sin(x+\frac{1}{6}\pi)$ **14.** $x\cos nx$ **15.** $x^n \ln x$

16. $3x\cos 2x$ **17.** $2e^x \sin x \cos x$ **18.** $x^2 \sin x$

19. $e^{ax}\sin bx$

20. By writing $\cos^3\theta$ as $(\cos^2\theta)(\cos\theta)$ use integration by parts to find $\displaystyle\int \cos^3\theta \, d\theta$.

Each of the following products can be integrated either:

(a) by immediate recognition, or

(b) by a suitable change of variable, or

(c) by parts.

Choose the best method in each case and hence integrate each function.

21. $(x-1)e^{x^2-2x+4}$ **22.** $(x+1)^2 e^x$ **23.** $\sin x(4+\cos x)^3$

24. $\cos x\, e^{\sin x}$ **25.** $x^4\sqrt{(1+x^5)}$ **26.** $e^x(e^x+2)^4$

27. $x\, e^{2x^2-1}$ **28.** $x(1-x^2)^9$ **29.** $\cos x \sin^5 x$

DEFINITE INTEGRATION BY PARTS

When using the formula

$$\int v\frac{du}{dx}dx = uv - \int u\frac{dv}{dx}dx$$

it must be appreciated that the term uv on the RHS is fully integrated. Consequently in a definite integration, uv must be *evaluated between the appropriate boundaries*

i.e. $$\int_a^b v\frac{du}{dx}dx = \Big[uv\Big]_a^b - \int_a^b u\frac{dv}{dx}dx$$

Examples 29i

1. Evaluate $\int_0^1 x\,e^x\,dx$

$$\int x\,e^x\,dx = \int v\frac{du}{dx}dx$$

where $\quad v = x \quad$ and $\quad \frac{du}{dx} = e^x$

Hence $$\int_0^1 x\,e^x\,dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx$$

$$= \Big[xe^x\Big]_0^1 - \Big[e^x\Big]_0^1$$

$$= (e^1 - 0) - (e^1 - e^0)$$

$$= e - e + 1$$

i.e. $$\int_0^1 x\,e^x\,dx = 1$$

2. Find the value of $\int_0^{1/2} \arccos x\,dx$

We regard $\arccos x$ as a product, i.e. $(1)(\arccos x)$ and integrate by parts. As $\arccos x$ cannot yet be integrated, it must be replaced by v

Taking $\quad v = \arccos x \quad$ and $\quad \frac{du}{dx} = 1$

gives $\quad \frac{dv}{dx} = \frac{-1}{\sqrt{(1-x^2)}} \quad$ and $\quad u = x$

Then $$\int_0^{1/2}(1)(\arccos x)\,dx = \Big[x\arccos x\Big]_0^{1/2} - \int_0^{1/2}\frac{-x}{\sqrt{(1-x^2)}}dx$$

$$= \{(\tfrac{1}{2})(\tfrac{1}{3}\pi) - 0\} + \int_0^{1/2}\frac{x}{\sqrt{(1-x^2)}}dx$$

Now $$\int_0^{1/2}\frac{x}{\sqrt{(1-x^2)}}dx = \int_1^{3/4}\frac{1}{\sqrt{u}}(-\tfrac{1}{2}dt)$$

where $t = 1 - x^2$ and

x	0	$\frac{1}{2}$
t	1	$\frac{3}{4}$

$$= \Big[-\sqrt{t}\Big]_1^{3/4}$$

$$= -\tfrac{1}{2}\sqrt{3} + 1$$

$\therefore$ $$\int \arccos x\,dx = \tfrac{1}{6}\pi - \tfrac{1}{2}\sqrt{3} + 1$$

EXERCISE 29i

Evaluate

1. $\int_0^{\pi/2} x\sin x\,dx$ **2.** $\int_1^2 x^5\ln x\,dx$ **3.** $\int_0^1 (x+1)e^x\,dx$

4. $\int_0^{\pi} e^x\cos x\,dx$ **5.** $\int_1^2 x\sqrt{(x-1)}\,dx$ **6.** $\int_0^{\pi/2} x^2\cos x\,dx$

7. $\int_0^1 \arcsin x\,dx$ **8.** $\int_0^1 x^2e^x\,dx$ **9.** $\int_1^{\sqrt{3}} \arctan x\,dx$

MIXED EXERCISE 29

Integrate the following functions, taking care to choose the best method in each case.

1. x^2e^{2x}
2. $2x\exp(x^2)$
3. $\sec^2x(3\tan x - 4)$
4. $(x+1)\ln(x+1)$
5. $\sec^2x\tan^3x$
6. $x^2\cos x$
7. $\sin x\, e^{\cos x}$
8. $x(2x+3)^7$
9. $(1-x)\exp(1-x)^2$
10. $xe^{(2x-1)}$
11. $\cos x\sin^5x$
12. $\sin x(4+\cos x)^3$
13. $(x-1)e^{(x^2-2x+3)}$
14. $x^2(1-x^3)^9$

Evaluate each definite integral.

15. $\int_1^3 e^{3x}\,dx$
16. $\int_0^{\pi/8}\cos 4x\,dx$
17. $\int_0^1 \frac{1}{x-2}\,dx$
18. $\int_{\pi/4}^{\pi/2}\operatorname{cosec}^2x\,dx$
19. $\int_0^1 x^2e^{3x^3}\,dx$
20. $\int_0^{\pi/4}x\cos 2x\,dx$
21. $\int_0^1 \frac{1}{2-x}\ln(2-x)\,dx$
22. $\int_1^2 x^2\ln x\,dx$

CHAPTER 30

ALGEBRA 2

RELATIONSHIPS BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

The general quadratic equation $ax^2+bx+c=0$ can be written

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad [1]$$

If the roots of the equation are α and β then the equation can also be written in the form

$$(x-\alpha)(x-\beta)=0$$

$$\Rightarrow \quad x^2-(\alpha+\beta)x+\alpha\beta=0 \qquad [2]$$

Comparing the terms in equations [1] and [2] shows that

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

e.g. if the equation $2x^2-3x+6=0$ has roots α and β, then

the sum of its roots, i.e. $\alpha+\beta$, is $-(-\frac{3}{2})=\frac{3}{2}$

and the product of its roots, i.e. $\alpha\beta$, is $\frac{6}{2}=3$

It also follows from the relationships above that any quadratic equation can be expressed in the form

$$x^2-(\text{sum of roots})x+(\text{product of roots})=0$$

e.g. the quadratic equation with roots whose sum is 7 and whose product is 10, is $x^2-7x+10=0$

Examples 30a

1. The roots of the equation $2x^2 - 7x + 4 = 0$ are α and β
   Find the values of $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\frac{1}{\alpha\beta}$ and hence write down the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

From $2x^2 - 7x + 4 = 0$ we have

$$\alpha + \beta = -(-\tfrac{7}{2}) = \tfrac{7}{2} \quad \text{and} \quad \alpha\beta = \tfrac{4}{2} = 2$$

To evaluate $\frac{1}{\alpha} + \frac{1}{\beta}$ we must first express it in terms of $\alpha + \beta$ and $\alpha\beta$ as these have known values.

Expressing $\frac{1}{\alpha} + \frac{1}{\beta}$ as a single fraction gives

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{7}{2}}{2} = \frac{7}{4}$$

and $$\frac{1}{\alpha\beta} = \frac{1}{2}$$

The required equation has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ therefore

the sum of its roots is $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{7}{4}$

and the product of its roots is $\alpha\beta = \frac{1}{2}$

Hence the required equation is $x^2 - \frac{7}{4}x + \frac{1}{2} = 0$

i.e. $$4x^2 - 7x + 2 = 0$$

Alternatively the following method can be used.

For the given equation, $2x^2 - 7x + 4 = 0$, $x = \alpha$ and β

and for the required equation, $aX^2 + bX + c = 0$, $X = \frac{1}{\alpha}$ and $\frac{1}{\beta}$

Therefore $$X = \frac{1}{x} \quad \Rightarrow \quad x = \frac{1}{X}$$

Substituting $\frac{1}{X}$ for x in the given equation we get

$$2\left(\frac{1}{X}\right)^2 - 7\left(\frac{1}{X}\right) + 4 = 0$$

i.e. $$4X^2 - 7X + 2 = 0$$

and this is the required equation.

The alternative method can be used only if each new root depends in the same way on each original root. For example, if the given equation has roots α, β and the required equation has roots α^2, β^2 it can be used, but if the required equation has roots $\alpha + \beta$, $\alpha - \beta$ it cannot.

---

2. If α and β are the roots of $x^2 + 3x - 2 = 0$ find the values of $\alpha^3 + \beta^3$ and $\alpha^3\beta^3$. Write down the equation whose roots are α^3 and β^3

From $x^2 + 3x - 2 = 0$ we see that $\alpha + \beta = -3$

and $\alpha\beta = -2$

To express $\alpha^3 + \beta^3$ in terms of $\alpha + \beta$ and $\alpha\beta$ we can use

$$(\alpha + \beta)^3 \equiv \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$
$$\equiv \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$$

therefore
$$\alpha^3 + \beta^3 \equiv (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$
$$= (-3)^3 - 3(-2)(-3)$$
$$= -45$$
$$\alpha^3\beta^3 \equiv (\alpha\beta)^3 = (-2)^3 = -8$$

As the required equation has roots α^3 and β^3, the sum of its roots is $\alpha^3 + \beta^3 = -45$ and the product of its roots is $\alpha^3\beta^3 = -8$

Therefore the required equation is $x^2 - (-45)x + (-8) = 0$

i.e. $x^2 + 45x - 8 = 0$

Note that, although the alternative method could be used in this example, as $X = x^3$, it is not recommended because the resulting equation $(\sqrt[3]{X})^2 + 3(\sqrt[3]{X}) + 4 = 0$ is not easy to simplify.

---

3. Find the range of values of k for which the roots of the equation $x^2 - 2x - k = 0$ are real. If the roots of this equation differ by 1, find the value of k

If $x^2 - 2x - k = 0$ has real roots then $b^2 - 4ac \geqslant 0$

i.e. $(-2)^2 - 4(1)(-k) \geqslant 0$

$\Rightarrow \quad 4 + 4k \geqslant 0$

$\therefore \quad k \geqslant -1$

If one root of the equation is α, then the other is $\alpha + 1$

The sum of the roots is $2\alpha + 1 = -(-2)$

$\Rightarrow \quad 2\alpha = 1$

$\therefore \quad \alpha = \frac{1}{2}$

The product of the roots is $\alpha(\alpha + 1) = -k$

Therefore $k = -\frac{3}{4}$

EXERCISE 30a

1. Write down the sums and products of the roots of the following equations.

   (a) $x^2 - 3x + 2 = 0$ (b) $4x^2 + 7x - 3 = 0$

   (c) $x(x - 3) = x + 4$ (d) $\dfrac{x-1}{2} = \dfrac{3}{x+2}$

   (e) $x^2 - kx + k^2 = 0$ (f) $ax^2 - x(a+2) - a = 0$

2. Write down the equation, the sum and product of whose roots are

   (a) $3, 4$ (b) $-2, \frac{1}{2}$ (c) $\frac{1}{3}, -\frac{2}{5}$ (d) $-\frac{1}{4}, 0$

   (e) a, a^2 (f) $-(k+1), k^2 - 3$ (g) $\dfrac{b}{a}, \dfrac{c^2}{b}$

3. The roots of the equation $2x^2 - 4x + 5 = 0$ are α and β. Find the value of

   (a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (b) $(\alpha + 1)(\beta + 1)$ (c) $\alpha^2 + \beta^2$

   (d) $\alpha^2\beta + \alpha\beta^2$ (e) $(\alpha - \beta)^2$ (f) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

4. The roots of $x^2 - 2x + 3 = 0$ are α and β. Find the equation whose roots are

   (a) $\alpha + 2, \beta + 2$ (b) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (c) α^2, β^2 (d) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

DIVISION OF ONE POLYNOMIAL BY ANOTHER POLYNOMIAL

Long division can be used to divide say $x^3 + 4x^2 - 7$ by $x^2 - 3$ ($x^3 + 4x^2 - 7$ is called the *dividend* and $x^2 - 3$ is called the *divisor*.)

$$\begin{array}{r|l} & x + 4 \\ \hline x^2 - 3 & x^3 + 4x^2 \qquad - 7 \\ & x^3 \qquad - 3x \\ \hline & 4x^2 + 3x - 7 \\ & 4x^2 \qquad - 12 \\ \hline & + 3x + 5 \end{array}$$

Divide x^3 by x^2: it goes in x times.
Multiply the divisor by x and then subtract it from the dividend.
The result is the new dividend; repeat the process until the dividend is not divisible by x^2

The number over the division line is the *quotient*, and what is left is called the *remainder*.

The relationship between the divisor, the dividend, the quotient and the remainder can be expressed as

$$x^3 + 4x^2 - 7 \equiv (x + 4)(x^2 - 3) + 3x + 5$$

IMPROPER FRACTIONS

When the highest power of x in the numerator of a fraction is greater than or *equal* to the highest power of x in the denominator, the fraction is called *improper*.

For example, $\dfrac{x^3 + 4x^2 - 7}{x^2 - 3}$ and $\dfrac{x^2 + 7}{x^2 - 2}$ are improper fractions.

Improper fractions can be expressed in a form in which any fractions are proper, by dividing the numerator by the denominator,

i.e. $\dfrac{x^3 + 4x^2 - 7}{x^2 - 3}$ can be written as $x + 4 + \dfrac{3x + 5}{x^2 - 3}$

Long division is not always necessary; it is often simpler to rearrange the numerator by adding and subtracting appropriate terms. This is illustrated in the following worked example.

Example 30b

Express $\dfrac{x^3+5x^2-3x}{x^3+1}$ in a form without an improper fraction.

$$\frac{x^3+5x^2-3x}{x^3+1} \equiv \frac{x^3+1+5x^2-3x-1}{x^3+1} \equiv \frac{x^3+1}{x^3+1}+\frac{5x^2-3x-1}{x^3+1}$$
$$\equiv 1+\frac{5x^2-3x-1}{x^3+1}$$

Notice that we add 1 to x^3 so that the expression can be split into two fractions, one of which divides out exactly. It is important to realise that, having added 1 to the numerator we also have to subtract 1 so that the value of the numerator is not altered.
After some practice, it is possible to do the intermediate steps mentally.

EXERCISE 30b

1. Find the quotient and the remainder for each of the following divisions.
   (a) $(x^3+x^2-3x+6)\div(x^2+3)$
   (b) $(x^4-5x^2+2)\div(x+1)$
   (c) $(2x^3-4x^2+3x-1)\div(x^2-1)$
   (d) $(3x^3-5)\div(x-2)$
   (e) $(x^5-5x^2+1)\div(x^3+1)$

2. Express the following fractions as the sum of a polynomial and a proper fraction.
   (a) $\dfrac{x+4}{x+1}$ (b) $\dfrac{2x}{x-2}$ (c) $\dfrac{x^2+3}{x^2-1}$
   (d) $\dfrac{x^2}{x-2}$ (e) $\dfrac{x^2+3x}{x-4}$ (f) $\dfrac{x^2-4}{x(x+1)}$

PARTIAL FRACTIONS

It was seen in Chapter 2 that we can express a proper fraction whose denominator consists of linear factors as a number of separate fractions,

e.g. $\dfrac{x+1}{(x-2)(x+2)}$ can be expressed as $\dfrac{3}{4(x-2)}+\dfrac{1}{4(x+2)}$

One method for finding the numerators of the separate fractions is given in Chapter 2 but there is a quicker way which we will now look at.

The Cover-up Method

Consider $f(x) = \dfrac{x}{(x-2)(x-3)} = \dfrac{A}{x-2}+\dfrac{B}{x-3}$

$$= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow$ $x \equiv A(x-3)+B(x-2)$

When $x=2$, $A=\dfrac{2}{2-3}=-2$,

which is the value of $\dfrac{x}{(x-3)}$ when $x=2$

i.e. $A=f(2)$ with the factor $(x-2)$ 'covered up'.

Similarly when $x=3$, $B=\dfrac{3}{3-2}=3$

which is the value of $\dfrac{x}{(x-2)}$ when $x=3$

i.e. $B=f(3)$ with the factor $(x-3)$ covered up.

Hence $$\frac{x}{(x-2)(x-3)} \equiv \frac{-2}{x-2}+\frac{3}{x-3}$$

Note that this method can be used only for linear factors.

Examples 30c

1. Express $\dfrac{1}{(2x-1)(x+3)}$ in partial fractions.

$$\frac{1}{(2x-1)(x+3)} = \frac{f(\frac{1}{2}) \text{ with } (2x-1) \text{ covered up}}{(2x-1)}+\frac{f(-3) \text{ with } (x+3) \text{ covered up}}{(x+3)}$$
$$= \frac{2}{7(2x-1)}-\frac{1}{7(x+3)}$$

Note that the intermediate step does not need to be written down.

Quadratic Factors in the Denominator

It is also possible to decompose fractions with quadratic or higher degree factors.

Consider $\dfrac{x^2+1}{(x^2+2)(x-1)}$

This is a proper fraction, so its partial fractions are also proper, i.e.

$\dfrac{x^2+1}{(x^2+2)(x-1)}$ can be expressed in the form $\dfrac{Ax+B}{x^2+2}+\dfrac{C}{x-1}$

Using the cover-up method gives $C = \frac{2}{3}$, but to find A and B the partial fraction form must be expressed as a single fraction giving

$$x^2+1 \equiv (Ax+B)(x-1)+\tfrac{2}{3}(x^2+2)$$

The values of A and B can then be found by substituting any suitable values for x

We will choose $x = 0$ and $x = -1$ as these are simple values to handle. (We do not choose $x = 1$ as it eliminates A and B and it was used to find C.)

$x = 0$ gives $\quad 1 = B(-1)+\frac{2}{3}(2) \quad \Rightarrow \quad B = \frac{1}{3}$

$x = -1$ gives $\quad 2 = (-A+\frac{1}{3})(-2)+\frac{2}{3}(3) \quad \Rightarrow \quad A = \frac{1}{3}$

$$\therefore \quad \frac{x^2+1}{(x^2+2)(x-1)} \equiv \frac{x+1}{3(x^2+2)}+\frac{2}{3(x-1)}$$

A Repeated Factor in the Denominator

Consider the fraction $\dfrac{2x-1}{(x-2)^2}$

This is a proper fraction, and it is possible to express this as two fractions with numerical numerators as we can see if we adjust numerator,

i.e. $$\frac{2x-1}{(x-2)^2} \equiv \frac{2(x-2)-1+4}{(x-2)^2} \equiv \frac{2}{x-2}+\frac{3}{(x-2)^2}$$

Any fraction whose denominator is a repeated linear factor can be expressed as separate fractions with numerical numerators, for example,

$\dfrac{2x^2-3x+4}{(x-1)^3}$ can be expressed as $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$

In the general case the values of the numerators can be found using the method in the next worked example.

To summarise, a proper fraction can be decomposed into partial fractions and the form of the partial fractions depends on the form of the factors in the denominator where

a linear factor gives a partial fraction of the form $\dfrac{A}{ax+b}$

a quadratic factor gives a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$

a repeated factor gives two partial fractions of the form

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

Examples 30c (continued)

2. Express $\dfrac{x-1}{(x+1)(x-2)^2}$ in partial fractions.

$$\frac{x-1}{(x+1)(x-2)^2} \equiv \frac{-\frac{2}{9}}{x+1}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$$

$\Rightarrow \quad x-1 \equiv (-\frac{2}{9})(x-2)^2+B(x+1)(x-2)+C(x+1)$

$x = 2$ gives $C = \frac{1}{3}$

Comparing coefficients of x^2 gives $0 = -\frac{2}{9}+B \quad \Rightarrow \quad B = \frac{2}{9}$

$$\therefore \quad \frac{x-1}{(x+1)(x-2)^2} \equiv -\frac{2}{9(x+1)}+\frac{2}{9(x-2)}+\frac{1}{3(x-2)^2}$$

Note that C can be found by the cover-up method, but B cannot.

3. Express $\dfrac{x^3}{(x+1)(x-3)}$ in partial fractions.

This fraction is improper and it must be divided out to obtain a mixed fraction before it can be expressed in partial fractions.

$$\begin{array}{r l}
 & x+2 \\
x^2-2x-3 \,\big)\!\! & x^3 \\
 & x^3-2x^2-3x \\
 & \quad 2x^2+3x \\
 & \quad 2x^2-4x-6 \\
 & \qquad 7x+6
\end{array}$$

$$\therefore \quad \frac{x^3}{(x+1)(x-3)} \equiv x+2+\frac{7x+6}{(x+1)(x-3)}$$

$$\equiv x+2+\frac{1}{4(x+1)}+\frac{27}{4(x-3)}$$

EXERCISE 30c

1. Use the cover-up method to express in partial fractions,

(a) $\dfrac{2}{(x+1)(x-1)}$ (b) $\dfrac{3}{(x-2)(x+1)}$ (c) $\dfrac{1}{x(x-3)}$

(d) $\dfrac{4}{(x-1)(x+3)}$ (e) $\dfrac{1}{(x^2-1)}$ (f) $\dfrac{2}{(2x+1)(2x-1)}$

2. Express in partial fractions,

(a) $\dfrac{2}{(x-1)(x^2+1)}$ (b) $\dfrac{x^2+1}{x(2x^2+1)}$

(c) $\dfrac{x^2+3}{x(x^2+2)}$ (d) $\dfrac{2x^2+x+1}{(x-3)(2x^2+1)}$

(e) $\dfrac{x^3-1}{(x+2)(2x+1)(x^2+1)}$ (f) $\dfrac{x^2+1}{x(2x^2-1)(x-1)}$

(g) $\dfrac{x}{(x-1)(x-2)^2}$ (h) $\dfrac{x^2-1}{x^2(2x+1)}$

(i) $\dfrac{3}{x(3x-1)^2}$ (j) $\dfrac{x^2}{(x+1)(x-1)}$

(k) $\dfrac{x^2-2}{(x+3)(x-1)}$ (l) $\dfrac{x^3+3}{(x-1)(x+1)}$

THE REMAINDER THEOREM

When $f(x) = x^3-7x^2+6x-2$ is divided by $x-2$, we get a quotient and a remainder. The relationship between these quantities can be written as

$$f(x) = x^3-7x^2+6x-2 \equiv (\text{quotient})(x-2)+\text{remainder}$$

Now substituting 2 for x eliminates the term containing the quotient, giving

$$f(2) = \text{remainder}$$

This is a particular illustration of the more general case, namely if a polynomial $f(x)$, is divided by $(x-a)$ then

$$f(x) \equiv (\text{quotient})(x-a)+\text{remainder}$$
$$\Rightarrow \quad f(a) = \text{remainder}$$

This result is called the *remainder theorem* and can be summarised as

when a polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$

Examples 30d

1. Find the remainder when
 (a) x^3-2x^2+6 is divided by $x+3$
 (b) $6x^2-7x+2$ is divided by $2x-1$

(a) When $f(x) = x^3-2x^2+6$ is divided by $x+3$, the remainder is

$$f(-3) = (-3)^3-2(-3)^2+6 = -39$$

(b) If $f(x) = 6x^2-7x+2$, then
$f(x) = (2x-1)(\text{quotient})+\text{remainder}$

$$\Rightarrow \quad \text{remainder} = f(\tfrac{1}{2}) = 0$$

Note that as the remainder is zero, $2x-1$ is a factor of $f(x)$

The Factor Theorem

This is a special case of the remainder theorem because if $x-a$ is a factor of a polynomial $f(x)$ then there is no remainder when $f(x)$ is divided by $x-a$,

i.e. $$f(a) = 0$$

This result, which is called the factor theorem, states that

**if, for a polynomial $f(x)$, $f(a) = 0$
then $x-a$ is a factor of $f(x)$**

The factor theorem is very helpful when factorising cubics or higher degree polynomials.

Examples 30d (continued)

2. Factorise $x^4 - 3x^3 + 4x^2 - 8$

$$f(x) \equiv x^4 - 3x^3 + 4x^2 - 8$$

We will test for factors of the form $x-a$ by finding $f(a)$ for various values of a. Note that, as the factors of 8 are 1, 2, 4 and 8, the values we choose for a must belong to the set $\{\pm1, \pm2, \pm4, \pm8\}$

$f(1) = 1-3+4-8 \neq 0$, so $x-1$ is not a factor of $f(x)$

$f(-1) = 1+3+4-8 = 0$, therefore $x+1$ is a factor of $f(x)$

Now that a factor has been found, it should be taken out; this can be done by inspection or by long division.

$$x^4 - 3x^3 + 4x^2 - 8 \equiv (x+1)(x^3 - 4x^2 + 8x - 8)$$

If $g(x) = x^3 - 4x^2 + 8x - 8$,

then $g(-1) = -1-4-8-8 \neq 0$ so $x+1$ is not a factor of $g(x)$

Note that, having taken $x-1$ out of $f(x)$, we tried it again as a possible factor of $g(x)$. This should always be done as repeated factors are common.

$g(2) = 8-16+16-8 = 0$, therefore $x-2$ is a factor of $g(x)$

By inspection, $x^3 - 4x^2 + 8x - 8 \equiv (x-2)(x^2 - 2x + 4)$ and $x^2 - 2x + 4$ has no linear factors.

Therefore $\quad x^4 - 3x^3 + 4x^2 - 8 \equiv (x+1)(x-2)(x^2 - 2x + 4)$

The Factors of $a^3 - b^3$ and $a^3 + b^3$

$a^3 - b^3 = 0$ when $a = b$, hence $a-b$ is a factor of $a^3 - b^3$

Therefore $\quad a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)$

in particular $\quad x^3 - 1 \equiv (x-1)(x^2 + x + 1)$

Also, $a^3 + b^3 = 0$ when $a = -b$, so $a+b$ is a factor of $a^3 + b^3$

Therefore $\quad a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2)$

in particular $\quad x^3 + 1 \equiv (x+1)(x^2 - x + 1)$

EXERCISE 30d

1. Find the remainder when the following functions are divided by the linear factors indicated.

(a) $x^3 - 2x + 4, \quad x - 1$ (b) $x^3 + 3x^2 - 6x + 2, \quad x + 2$
(c) $2x^3 - x^2 + 2, \quad x - 3$ (d) $x^4 - 3x^3 + 5x, \quad 2x - 1$
(e) $9x^5 - 5x^2, \quad 3x + 1$ (f) $x^3 - 2x^2 + 6, \quad x - a$
(g) $x^2 + ax + b, \quad x + c$ (h) $x^4 - 2x + 1, \quad ax - 1$

2. Determine whether the following linear functions are factors of the given polynomials.

(a) $x^3 - 7x + 6, \quad x - 1$ (b) $2x^2 + 3x - 4, \quad x + 1$
(c) $x^3 - 6x^2 + 6x - 2, \quad x - 2$ (d) $x^3 - 27, \quad x - 3$
(e) $2x^4 - x^3 - 1, \quad 2x - 1$ (f) $x^3 + ax^2 - a^2x - a^3, \quad x + a$

3. Factorise the following functions as far as possible.

(a) $x^3 + 2x^2 - x - 2$ (b) $x^3 - x^2 - x - 2$
(c) $x^4 - 1$ (d) $x^4 + x^3 - 3x^2 - 4x - 4$
(e) $2x^3 - x^2 + 2x - 1$ (f) $27x^3 - 1$
(g) $x^3 + a^3$ (h) $x^3 - y^3$

4. If $x^2 - 7x + a$ has a remainder 1 when divided by $x + 1$, find a

5. If $x - 2$ is a factor of $ax^2 - 12x + 4$ find a

6. One solution of the equation $x^2 + ax + 2 = 0$ is $x = 1$, find a

7. One root of the equation $x^2 - 3x + a = 0$ is 2. Find the other root.

SOLUTION OF POLYNOMIAL EQUATIONS AND INEQUALITIES

The factor theorem is very useful when solving cubic or higher degree equations or inequalities.

Examples 30e

1. Find the values of x for which $x^3 - 2x^2 - x + 2 < 0$

Problems involving inequalities are often easier to solve if dealt with graphically.

Consider the curve $y = x^3 - 2x^2 - x + 2$

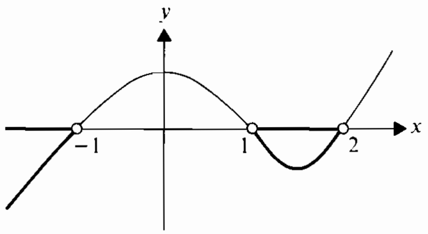
This is a cubic curve and it crosses the x-axis where $x^3 - 2x^2 - x + 2 = 0$ so we will try to factorise $x^3 - 2x^2 - x + 2$ using the factor theorem.

$$f(x) = x^3 - 2x^2 - x + 2$$

$f(1) = 0$, $\therefore$ $x - 1$ is a factor of $f(x)$

$\Rightarrow$ $x^3 - 2x^2 - x + 2 \equiv (x-1)(x^2 - x - 2) \equiv (x-1)(x+1)(x-2)$

$\therefore$ $y = f(x)$ cuts the x-axis at $x = -1, 1, 2$ $\Rightarrow$



From the sketch,

$x^3 - 2x^2 - x + 2 < 0$ when $x < -1$ and $1 < x < 2$

2. The equation $f(x) = 0$ has a repeated root, where $f(x) = 4x^2 + px + q$ When $f(x)$ is divided by $x + 1$ the remainder is 1. Find the values of p and q.

$$f(-1) = 4 - p + q = 1 \quad \Rightarrow \quad p = q + 3 \qquad [1]$$

If $4x^2 + px + q = 0$ has a repeated root then '$b^2 - 4ac$' $= 0$

i.e. $$p^2 - 16q = 0 \qquad [2]$$

Solving equations [1] and [2] simultaneously gives

$$(q+3)^2 - 16q = 0 \quad \Rightarrow \quad q^2 - 10q + 9 = 0$$

$$\Rightarrow \quad (q-9)(q-1) = 0$$

$\therefore$ either $q = 9$ and $p = 12$ or $q = 1$ and $p = 4$

EXERCISE 30e

1. Factorise $2x^3 - x^2 - 2x + 1$. Hence find the values of x for which $2x^3 - x^2 - 2x + 1 > 0$

2. Given that $f(x) = x^3 - x^2 - x - 2$ show that $y = f(x)$ cuts the x-axis once only. Find the values of x for which $x^3 - x^2 - x < 2$

3. Find the value of p for which $x = \frac{1}{2}$ is a solution of the equation $4x^2 - px + 3 = 0$

4. Show that the x coordinates of the points of intersection of the curves $y = \dfrac{1}{x}$ and $x^2 + 4y^2 = 5$ satisfy the equation $x^4 - 5x^2 + 4 = 0$
   Solve this equation.

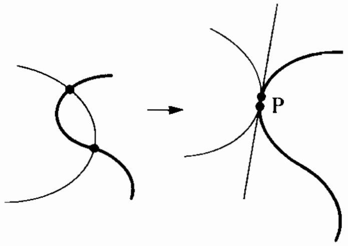
5. Factorise $x^4 - 5x^3 + 5x^2 + 5x - 6$. Hence sketch the curve $y = x^4 - 5x^3 + 5x^2 + 5x - 6$

6. A function f is defined by
   $$f(x) = 5x^3 - px^2 + x - q$$
   When $f(x)$ is divided by $x - 2$, the remainder is 3. Given that $(x - 1)$ is a factor of $f(x)$
   (a) find p and q
   (b) find the number of real roots of the equation
   $$5x^3 - px^2 + x - q = 0$$

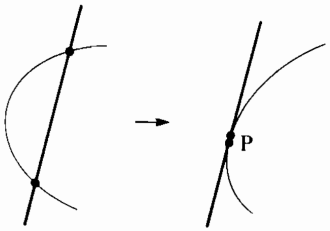
INTERSECTION OF CURVES

The points of intersection of any two curves (and/or lines) can be found by solving their equations simultaneously. Each real root then gives a point of intersection. Some roots may be repeated and we will now look at the significance of this situation.

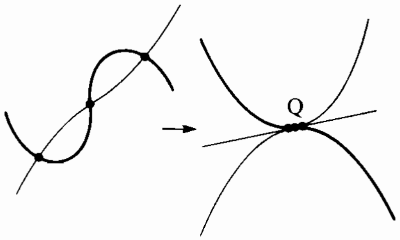
If there are two equal roots the curves meet twice at the same point P, i.e. they touch at P and have a common tangent at P.



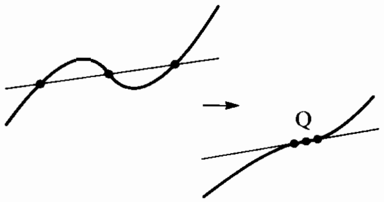
In particular, when a line and a curve meet twice at the same point P, the line is a tangent to the curve at P.



If there are three equal roots the curves meet three times at the same point Q. The curves have a common tangent at Q but this time each curve crosses, at Q, to the opposite side of the common tangent.

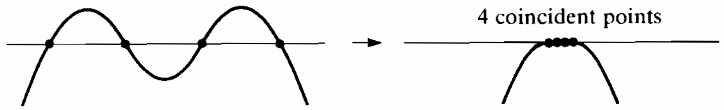


In particular, when a line and a curve meet at three coincident points Q, the line is a tangent to the curve at Q and the curve has a point of inflexion at Q.

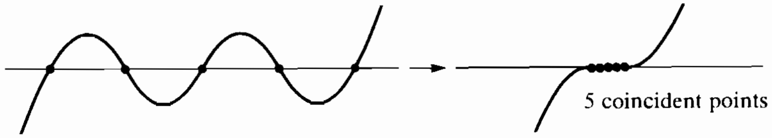


Taking this argument further it becomes clear that,

(a) when the number of coincident points of intersection of a line and a curve is even, the curve touches the line and remains on the same side of the line;



(b) when the number of coincident points of intersection is odd, the curve touches the line and crosses it. Thus the curve has a point of inflexion.



so if the solution of $\begin{cases} y = mx + c \\ y = f(x) \end{cases}$

has $\begin{cases} \text{distinct roots, the curve crosses the line at distinct points.} \\ \text{repeated roots, the line touches the curve.} \end{cases}$

Examples 30f

1. Find the equations of the tangents with gradient 1 to the curve $x^2 + 2y^2 = 6$ and find the coordinates of their points of contact.

Any line with gradient 1 has equation $y = x + c$

This line meets the curve $x^2 + 2y^2 = 6$ where $x^2 + 2(x + c)^2 = 6$

i.e. where $3x^2 + 4cx + (2c^2 - 6) = 0$ [1]

For the line to touch the curve, this equation must have equal roots.

So $(4c)^2 - 12(2c^2 - 6) = 0 \Rightarrow c = \pm 3$

$\therefore$ the equations of the tangents are $y = x + 3$ and $y = x - 3$

To find the coordinates of the points of contact we have to go back to equation [1].

When $c = 3$, [1] becomes $x^2 + 4x + 4 = 0 \Rightarrow x = -2$

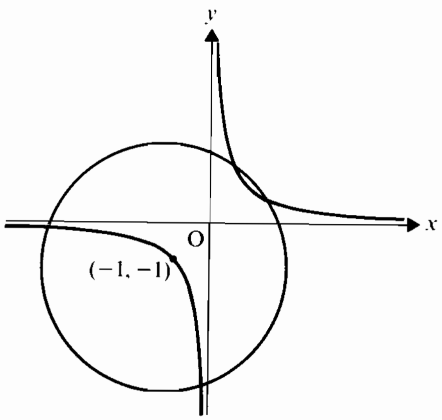
When $c = -3$, [1] becomes $x^2 - 4x + 4 = 0 \Rightarrow x = 2$

The corresponding values of y are found from the equations of the tangents.

The points of contact of the two tangents are $(2, -1)$ and $(-2, 1)$

2. Find the points of intersection of the curves $xy = 1$ and $x^2 + y^2 + 2x + 2y - 6 = 0$

The curve $xy = 1$ is a rectangular hyperbola and the other curve is a circle. Rearranging the equation of the circle as $(x+1)^2 + (y+1)^2 = 8$ tells us that its centre is $(-1, -1)$ and its radius is $2\sqrt{2}$



From the sketch, we expect 4 distinct points of intersection. However it is worth noting that the points in the first quadrant appear to be close together, and as this is only a sketch, it may be that these points are coincident or even do not exist.

Now points of intersection satisfy both equations simultaneously. Solving the equations by eliminating y gives

$$x^2 + \left(\frac{1}{x}\right)^2 + 2x + 2\left(\frac{1}{x}\right)^2 - 6 = 0 \quad \Rightarrow \quad x^4 + 2x^3 - 6x^2 + 2x + 1 = 0$$

Using the factor theorem gives $(x-1)(x-1)(x^2+4x+1) = 0$

Hence $x = 1$ (twice) and $x = -2 \pm \sqrt{3}$

i.e. there are two equal roots and two distinct roots.

Therefore (calculating the corresponding values of y from the equation $xy = 1$)

the curves *touch* at $(1, 1)$

and *cut* at $\left(-2-\sqrt{3}, \dfrac{1}{-2-\sqrt{3}}\right)$ and $\left(-2+\sqrt{3}, \dfrac{1}{-2+\sqrt{3}}\right)$

3. Find the points of intersection of the line $y = x + 2$ and the curve $y = x^4 - 2x^3 + 3x + 1$, showing that one of them is a point of inflexion on the curve.

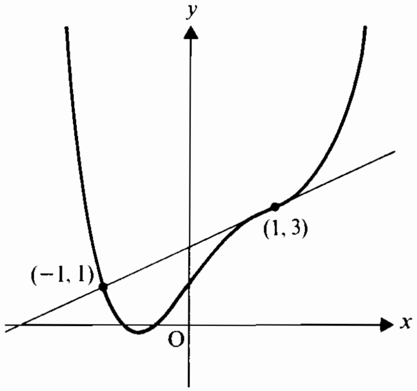
The line and the curve meet at points whose x coordinates are given by

$$x^4 - 2x^3 + 3x + 1 = x + 2 \quad \Rightarrow \quad x^4 - 2x^3 + 2x - 1 = 0$$

Using the factor theorem gives $(x-1)^3(x+1) = 0$

Thus there are three coincident points where $x = 1$ and one point at $x = -1$

Therefore the line cuts the curve at a point of inflexion $(1, 3)$ and cuts it again at $(-1, 1)$



EXERCISE 30f

Investigate the possible intersection of the following lines and curves giving the coordinates of all common points. State clearly those cases where the line touches the curve.

1. $y = x + 1;\quad y^2 = 4x$
2. $2y + x = 3;\quad x^2 - y^2 - 3y + 3 = 0$
3. $y = x - 5;\quad x^2 + 2y^2 = 7$
4. $2y - x = 4;\quad x^2 + y^2 - 4x = 4$
5. $y = 0;\quad y = x^2 - 3x + 2$

6. $y = 0;\quad y = x^3 + 5x^2 + 6x$

7. $y = 0;\quad y = (x-1)^2(x-2)^2$

8. $y = 0;\quad y = (x+3)^3(x+2)$

9. $x = 0;\quad x = y^4$

Find the value of k such that the given line shall touch the given curve.

10. $y = x + 2;\quad y^2 = kx$

11. $y = kx + 3;\quad xy + 9 = 0$

12. $y = 3x - k;\quad x^2 + 2y^2 = 8$

Find the points of intersection or points of contact (if any) of the following pairs of curves. Illustrate your results by drawing diagrams.

13. $y^2 = 8x;\quad xy = 1$

14. $x^2 + y^2 + 2x - 7 = 0;\quad y^2 = 4x$

15. $xy = 2;\quad 2x^2 + 2y^2 - 6x + 3y - 10 = 0$

16. $9x^2 = 2y;\quad y^2 = 6x$

17. Find the value(s) or ranges of values of λ for which the line $y = 2x + \lambda$
    (a) touches
    (b) cuts in real points
    (c) does not meet, the curve $y^2 + 2x^2 = 4$

18. Sketch the curves $y = 3x^4$, $y = 4(2-x)^5$, $y = 2(x+3)^7$ $y = -5x^6$

19. Find the equation(s) of the tangent(s):
    (i) from the point $(1, 0)$ (ii) with gradient $-\frac{1}{2}$
    to each of the following curves,
    (a) $y^2 + 4x = 0$ (b) $xy = 9$ (c) $x^2 = 6y$

CHAPTER 31

FURTHER INTEGRATION 2

INTEGRATING FRACTIONS

Some expressions have an integral that is a function but there are many others for which an exact integral cannot be found. In this book we are concerned mainly with expressions which *can* be integrated but even with this proviso the reader should be aware that, while the methods suggested usually work, they are not infallible.

There are several different methods for integrating fractions, the appropriate method in a particular case depending upon the form of the fraction. Consequently it is very important that each fraction be categorised carefully to avoid embarking on unnecessary and lengthy working.

Method 1 Using Recognition

Consider the function $\ln u$ where $u = f(x)$

Differentiating with respect to x gives

$$\frac{d}{dx}\ln u = \left(\frac{1}{u}\right)\left(\frac{du}{dx}\right) \quad \text{i.e.} \quad \frac{du/dx}{u}$$

i.e.
$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

Hence
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + K$$

Thus all fractions of the form $f'(x)/f(x)$ can be integrated *immediately* by recognition, e.g.

$$\int \frac{\cos x}{1+\sin x}\,dx = \ln|1+\sin x| + K \quad \text{as} \quad \frac{d}{dx}(1+\sin x) = \cos x$$

$$\int \frac{e^x}{e^x+4}\,dx = \ln|e^x+4| + K \quad \text{as} \quad \frac{d}{dx}(e^x+4) = e^x$$

Note, however, that $\int \frac{x}{\sqrt{(1+x)}}\,dx$ is *not* equal to $\ln|\sqrt{(1+x)}| + K$

because $\frac{d}{dx}\sqrt{(1+x)}$ is not equal to x

Method 1 applies only to an integral whose numerator is basically the derivative of *the complete denominator*.
An integral whose numerator is the derivative, not of the complete denominator but of a function *within* the denominator, belongs to the next type.

Method 2 Using Substitution

Consider the integral $\int \frac{2x}{\sqrt{(x^2+1)}}\,dx$

Noting that $2x$ is the derivative of x^2+1 we make the substitution $u = x^2+1$, i.e.

$$\text{if } u = x^2+1 \text{ then } \ldots du \equiv \ldots 2x\,dx$$

By this change of variable the given integral is converted into the simple form $\int \frac{1}{\sqrt{u}}\,du$

Examples 31a

1. Find $\int \frac{x^2}{1+x^3}\,dx$

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx$$

This integral is of the form $\int \frac{f'(x)}{f(x)}\,dx$ so we use recognition

$$= \tfrac{1}{3}\ln|1+x^3| + K$$

---

2. By writing $\tan x$ as $\frac{\sin x}{\cos x}$ find $\int \tan x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{f'(x)}{f(x)}\,dx \quad \text{where} \quad f(x) = \cos x$$

so $$\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + K$$

$\therefore$ $$\int \tan x\,dx = K - \ln|\cos x| \quad \text{or} \quad K + \ln|\sec x|$$

Note that, similarly, $\int \cot x\,dx = \ln|\sin x| + K$

These results are quotable.

---

3. Find $\int \frac{e^x}{(1-e^x)^2}\,dx$

e^x is basically the derivative of $1-e^x$ but not of $(1-e^x)^2$ so we make the substitution $u = 1-e^x$

If $u = 1-e^x$ then $\ldots du \equiv \ldots -e^x\,dx$

So $$\int \frac{e^x}{(1-e^x)^2}\,dx = \int \frac{-1}{u^2}\,du = \frac{1}{u} + K$$

$\therefore$ $$\int \frac{e^x}{(1-e^x)^2}\,dx = \frac{1}{1-e^x} + K$$

---

4. Find $\int \frac{\sec^2 x}{\tan^3 x}\,dx$

$\sec^2 x$ is the derivative of $\tan x$ but not of $\tan^3 x$

Taking $u = \tan x$ gives $\ldots du \equiv \ldots \sec^2 x\,dx$

Then $$\int \frac{\sec^2 x}{\tan^3 x}\,dx = \int \frac{1}{u^3}\,du = -\tfrac{1}{2}u^{-2} + K$$

i.e. $$\int \frac{\sec^2 x}{\tan^3 x}\,dx = \frac{-1}{2\tan^2 x} + K$$

EXERCISE 31a

In Questions 1 to 18 integrate each function w.r.t. x

1. $\dfrac{\cos x}{4+\sin x}$ **2.** $\dfrac{e^x}{3e^x-1}$ **3.** $\dfrac{x}{(1-x^2)^3}$

4. $\dfrac{\sin x}{\cos^3 x}$ **5.** $\dfrac{x^3}{1+x^4}$ **6.** $\dfrac{2x+3}{x^2+3x-4}$

7. $\dfrac{x^2}{\sqrt{(2+x^3)}}$ **8.** $\dfrac{\cos x}{(\sin x-2)^2}$ **9.** $\dfrac{1}{x\ln x}$ i.e. $\dfrac{1/x}{\ln x}$

10. $\dfrac{\cos x}{\sin^6 x}$ **11.** $\dfrac{\operatorname{cosec}^2 x}{\cot^4 x}$ **12.** $\dfrac{e^x}{\sqrt{(1-e^x)}}$

13. $\dfrac{x-1}{3x^2-6x+1}$ **14.** $\dfrac{\cos x}{\sin^n x}$ **15.** $\dfrac{\sin x}{\cos^n x}$

16. $\dfrac{\sec x\tan x}{4+\sec x}$ **17.** $\dfrac{\sec^2 x}{(1-\tan x)^3}$ **18.** $\dfrac{\sin x}{(3+\cos x)^2}$

19. By writing $\sec x$ as $\dfrac{\sec x(\sec x+\tan x)}{(\tan x+\sec x)}$ find $\int \sec x\,dx$

20. By writing $\operatorname{cosec} x$ as $\dfrac{\operatorname{cosec} x(\operatorname{cosec} x+\cot x)}{(\cot x+\operatorname{cosec} x)}$ find $\int \operatorname{cosec} x\,dx$

Evaluate

21. $\displaystyle\int_1^2 \frac{2x+1}{x^2+x}\,dx$ **22.** $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$

23. $\displaystyle\int_2^3 \frac{2x}{(x^2-1)^3}\,dx$ **24.** $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

25. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin 2x}{\cos(2x-\pi)}\,dx$ **26.** $\displaystyle\int_2^4 \frac{1}{x(\ln x)^2}\,dx$

Quotable Results

Some of the integrals found in Exercise 31a are important enough to be regarded as standard and are listed here

$$\int \tan x\,dx = \ln|\sec x|+K$$
$$\int \cot x\,dx = \ln|\sin x|+K$$
$$\int \sec x\,dx = \ln|\sec x+\tan x|+K$$
$$\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x+\cot x|+K$$

USING PARTIAL FRACTIONS

If a fraction has not fallen into any of the previous categories, it may be that it is easy to integrate when expressed in partial fractions. Remember however that only proper fractions can be converted directly into partial fractions; an improper fraction must first be divided out until it comprises non-fractional terms and a proper fraction.

It is not very often that actual long division is needed. Usually a simple adjustment in the numerator is all that is required, as the following examples show. When such an adjustment is not obvious, however, long division can always be used.

Examples 31b

1. Integrate $\dfrac{2x-3}{(x-1)(x-2)}$ w.r.t. x

Using the cover-up method gives

$$\frac{2x-3}{(x-1)(x-2)} = \frac{1}{x-1}+\frac{1}{x-2}$$

$$\therefore \quad \int \frac{2x-3}{(x-1)(x-2)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{1}{x-2}\,dx$$
$$= \ln|x-1|+\ln|x-2|+\ln A$$
$$= \ln A|(x-1)(x-2)|$$

2. Find $\int \frac{x^2+1}{x^2-1}\,dx$

This fraction is improper so, before we can factorise the denominator and use partial fractions we must adjust the given fraction as follows.

$$\frac{x^2+1}{x^2-1} = \frac{(x^2-1)+2}{x^2-1} = 1+\frac{2}{x^2-1} = 1+\frac{2}{(x-1)(x+1)}$$

Then $\int \frac{x^2+1}{x^2-1}\,dx = \int 1\,dx + \int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx$

$$= x + \ln|x-1| - \ln|x+1| + \ln A$$

$$= x + \ln\frac{A|x-1|}{|x+1|}$$

Even when improper fractions do not need conversion into partial fractions, it is still essential to reduce to proper form before attempting to integrate, i.e.

$$\int \frac{2x+4}{x+1}\,dx = \int \frac{2(x+1)+2}{x+1}\,dx = \int 2\,dx + \int \frac{2}{x+1}\,dx$$

$$= 2x + 2\ln A|x+1|$$

The reader should not fall into the trap of thinking that, whenever the denominator of a fraction factorises, integration will involve partial fractions.
Careful scrutiny is vital, as fractions requiring quite different integration techniques often *look* very similar. The following example shows this clearly.

Examples 31b (continued)

3. Integrate w.r.t. x,

(a) $\frac{x+1}{x^2+2x-8}$ (b) $\frac{x+1}{(x^2+2x-8)^2}$ (c) $\frac{x+2}{x^2+2x-8}$

(a) This fraction is basically of the form $f'(x)/f(x)$

$$\int \frac{x+1}{x^2+2x-8}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x-8}\,dx$$

$$= \tfrac{1}{2}\ln A|x^2+2x-8|$$

(b) This time the numerator is basically the derivative of the function *within* the denominator so we use

$$u = x^2+2x-8 \quad\Rightarrow\quad \ldots du \equiv \ldots(2x+2)\,dx \equiv \ldots 2(x+1)\,dx$$

$$\therefore \quad \int \frac{x+1}{(x^2+2x-8)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u}+K$$

$$= K - \frac{1}{2(x^2+2x-8)}$$

(c) In this fraction the numerator is not related to the derivative of the denominator so, as the denominator factorises, we use partial fractions.

$$\int \frac{x+2}{x^2+2x-8}\,dx = \int \frac{\frac{1}{3}}{x+4}\,dx + \int \frac{\frac{2}{3}}{x-2}\,dx$$

$$= \tfrac{1}{3}\ln|x+4| + \tfrac{2}{3}\ln|x-2| + \ln A$$

$$= \ln A|(x+4)^{1/3}(x-2)^{2/3}|$$

EXERCISE 31b

Integrate each of the following functions w.r.t. x

1. $\frac{2}{x(x+1)}$
2. $\frac{4}{(x-2)(x+2)}$
3. $\frac{x}{(x-1)(x+1)}$
4. $\frac{x-1}{x(x+2)}$
5. $\frac{x-1}{(x-2)(x-3)}$
6. $\frac{1}{x(x-1)(x+1)}$
7. $\frac{x}{x+1}$
8. $\frac{x+4}{x}$
9. $\frac{x}{x+4}$
10. $\frac{3x-4}{x(1-x)}$
11. $\frac{x^2-2}{x^2-1}$
12. $\frac{x^2}{(x+1)(x+2)}$

Choose the best method to integrate each function.

13. $\frac{x}{x^2-1}$
14. $\frac{2x}{(x^2-1)^2}$
15. $\frac{2}{x^2-1}$
16. $\frac{2x-5}{x^2-5x+6}$
17. $\frac{2x}{x^2-5x+6}$
18. $\frac{2x-3}{x^2-5x+6}$

Evaluate

19. $\displaystyle\int_0^4 \frac{x+2}{x+1}\,dx$ 20. $\displaystyle\int_{-1}^1 \frac{5}{x^2+x-6}\,dx$ 21. $\displaystyle\int_1^2 \frac{x+2}{x(x+4)}\,dx$

22. $\displaystyle\int_0^1 \frac{2}{3+2x}\,dx$ 23. $\displaystyle\int_{1/2}^3 \frac{2}{(3+2x)^2}\,dx$ 24. $\displaystyle\int_1^2 \frac{2x}{3+2x}\,dx$

The Use of Partial Fractions in Differentiation

Rational functions with two or more factors in the denominator are often easier to differentiate if expressed in partial fractions. This method is an alternative to logarithmic differentiation which can also be used for functions of this type.

However, the use of partial fractions is of particular benefit when a second derivative is required.

Example 31c

Find the first and second derivatives of $\dfrac{x}{(x-1)(x+1)}$

Taking $$y = \frac{x}{(x-1)(x+1)} = \frac{\frac{1}{2}}{(x-1)} + \frac{\frac{1}{2}}{(x+1)}$$
$$= \tfrac{1}{2}(x-1)^{-1} + \tfrac{1}{2}(x+1)^{-1}$$

gives $$\frac{dy}{dx} = -\tfrac{1}{2}(x-1)^{-2} - \tfrac{1}{2}(x+1)^{-2}$$
$$= \frac{-1}{2(x-1)^2} - \frac{1}{2(x+1)^2}$$

and $$\frac{d^2y}{dx^2} = (-2)(-\tfrac{1}{2})(x-1)^{-3} - (-2)(\tfrac{1}{2})(x+1)^{-3}$$
$$= \frac{1}{(x-1)^3} + \frac{1}{(x+1)^3}$$

EXERCISE 31c

In each question express the given function in partial fractions and hence find its first and second derivatives.

1. $\dfrac{2}{(x-2)(x-1)}$ 2. $\dfrac{3x}{(2x-1)(x-3)}$ 3. $\dfrac{x}{(x+2)(x-4)}$

4. $\dfrac{5}{(x+2)(x-3)}$ 5. $\dfrac{x}{(2x+3)(x+1)}$ 6. $\dfrac{3}{(3x-1)(x-1)}$

SPECIAL TECHNIQUES FOR INTEGRATING SOME TRIGONOMETRIC FUNCTIONS

To Integrate a Function Containing an Odd Power of $\sin x$ or $\cos x$

When $\sin x$ or $\cos x$ appear to an odd power other than 1, the identity $\cos^2x + \sin^2x \equiv 1$ is often useful in converting the given function to an integrable form,

e.g. $\sin^3x$ is converted to $(\sin^2x)(\sin x) \Rightarrow (1-\cos^2x)(\sin x)$
$\Rightarrow \sin x - \cos^2x\sin x$

Examples 31d

1. Integrate w.r.t. x, (a) $\cos^5x$ (b) $\sin^3x\cos^2x$

(a) $$\cos^5x = (\cos^2x)^2\cos x$$
$$= (1-\sin^2x)^2\cos x$$
$$= (1-2\sin^2x+\sin^4x)\cos x$$

$$\therefore \int \cos^5x\,dx = \int \cos x\,dx - 2\int \sin^2x\cos x\,dx + \int \sin^4x\cos x\,dx$$

For any value of n we know that $\displaystyle\int \sin^nx\cos x\,dx = \frac{1}{n+1}\sin^{n+1}x + K$

$$\therefore \int \cos^5x\,dx = \sin x - 2(\tfrac{1}{3})\sin^3x + (\tfrac{1}{5})\sin^5x + K$$
$$= \sin x - \tfrac{2}{3}\sin^3x + \tfrac{1}{5}\sin^5x + K$$

(b) $$\sin^3x\cos^2x = \sin x(1-\cos^2x)\cos^2x$$
$$= \cos^2x\sin x - \cos^4x\sin x$$

$$\therefore \int \sin^3x\cos^2x\,dx = \int \cos^2x\sin x\,dx - \int \cos^4x\sin x\,dx$$
$$= -\tfrac{1}{3}\cos^3x + \tfrac{1}{5}\cos^5x + K$$

To Integrate a Function Containing only Even Powers of $\sin x$ or $\cos x$

This time the double angle identities are useful.

e.g. $\cos^4 x$ becomes $(\cos^2 x)^2 = \{\frac{1}{2}(1+\cos 2x)\}^2$

$$= \tfrac{1}{4}\{1 + 2\cos 2x + \cos^2 2x\}$$

then we can use a double angle identity again

$$= \tfrac{1}{4}(1 + 2\cos 2x) + \tfrac{1}{4}\{\tfrac{1}{2}(1+\cos 4x)\}$$

$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x$$

Now each of these terms can be integrated.

Examples 31d (continued)

2. Integrate w.r.t. x, **(a)** $\sin^2 x$ **(b)** $16\sin^4 x\cos^2 x$

(a)

$$\int \sin^2 dx = \int \tfrac{1}{2}(1-\cos 2x)\,dx$$

$$= \int \tfrac{1}{2}\,dx - \tfrac{1}{2}\int \cos 2x\,dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + K$$

(b)

$$16\int \sin^4 x\cos^2 x\,dx = 16\int \{\tfrac{1}{2}(1-\cos 2x)\}^2\{\tfrac{1}{2}(1+\cos 2x)\}\,dx$$

$$= 2\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\,dx$$

$$= 2x - \sin 2x - 2\int \cos^2 2x\,dx + 2\int \cos^3 2x\,dx$$

Now $2\int \cos^2 2x\,dx = \int (1+\cos 4x)\,dx = x + \tfrac{1}{4}\sin 4x$

and $2\int \cos^3 2x\,dx = 2\int \cos 2x(1-\sin^2 2x)\,dx = \sin 2x - \tfrac{1}{3}\sin^3 2x$

$$\therefore\quad 16\int \sin^4 x\cos^2 x\,dx = x - \tfrac{1}{4}\sin 4x - \tfrac{1}{3}\sin^3 2x + K$$

Note that for a product with an odd power in one term and an even power in the other, the method for an odd power is usually best (see Example 1(b) on page 527).

It is important to appreciate that the techniques used in these examples, although they are of most general use, are by no means exhaustive. Because there are so many trig identities there is always the possibility that a particular integral can be dealt with in several different ways,

e.g. to integrate $\sin^2 x\cos^2 x$ w.r.t. x the best conversion would use $2\sin x\cos x \equiv \sin 2x$, so that

$$\int \sin^2 x\cos^2 x\,dx = \int (\tfrac{1}{2}\sin 2x)^2\,dx = \tfrac{1}{4}\int \tfrac{1}{2}(1-\cos 4x)\,dx$$

So it is always advisable to look for the identity which will make the given function integrable as quickly and simply as possible.

Further, as mentioned earlier, it must be remembered that there are many expressions whose integrals cannot be found as a function at all. (In examination papers, of course, any integral asked for *can* be found!)

To Integrate any Power of $\tan x$

The identity $\tan^2 x = \sec^2 x - 1$ is useful here,

e.g. $\tan^3 x$ becomes $\tan x(\sec^2 x - 1) \Rightarrow \sec^2 x\tan x - \tan x$

and we know that $\int \sec^2 x\tan^n x\,dx = \dfrac{1}{n+1}\tan^{n+1} x + K$

Examples 31d (continued)

3. Integrate w.r.t. x, (a) $\tan^4 x$ (b) $\tan^5 x$

(a)

$$\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\tan^2 x\,dx - \int \tan^2 x\,dx$$

$$= \tfrac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx$$

$$= \tfrac{1}{3}\tan^3 x - \tan x + x + K$$

(b) $\int \tan^5 x\,dx = \int \tan^3 x(\sec^2 x - 1)\,dx$

$= \int \sec^2 x \tan^3 x - \int \tan x(\sec^2 x - 1)\,dx$

$= \int \sec^2 x \tan^3 x\,dx - \int \sec^2 x \tan x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln \sec x + K$

Note that, to integrate *any* power of $\tan x$, the identity $\tan^2 x = \sec^2 x - 1$ is used to convert *$\tan^2 x$ only, one step at a time*, i.e. converting $\tan^4 x$ to $(1 - \sec^2 x)^2$ does not help.

Integrals Involving Multiple Angles

To integrate a product such as $\sin 5x \cos 3x$, the appropriate factor formula can be used to express the product as a sum. In this case,

$$\int \sin 5x \cos 3x\,dx = \frac{1}{2}\int (\sin 8x + \sin 2x)\,dx$$

and the RHS consists of two standard integrals.

EXERCISE 31d

Integrate each function w.r.t. x

1. $\cos^2 x$ **2.** $\cos^3 x$ **3.** $\sin^5 x$ **4.** $\tan^2 x$

5. $\sin^4 x$ **6.** $\tan^3 x$ **7.** $\cos^4 x$ **8.** $\sin^3 x$

Find

9. $\int \sin^2\theta \cos^3\theta\,d\theta$ **10.** $\int \sin^{10}\theta \cos^3\theta\,d\theta$

11. $\int \sin^n\theta \cos^3\theta\,d\theta$ **12.** $\int \sin^2\theta \cos^2\theta\,d\theta$

13. $\int 4\sin^2\theta \cos 3\theta\,d\theta$ (*Hint.* Change $\sin^2\theta$ into the double angle.)

14. $\int \tan^2\theta \sec^4\theta\,d\theta$ (*Hint.* Change $\sec^2\theta$ into $\tan^2\theta + 1$.)

Integrate w.r.t. t

15. $2\sin 4t \cos 3t$ **16.** $2\cos 2t \cos 5t$

17. $\sin 2t \cos 6t$ **18.** $\sin t \sin 3t$

19. $2\sin nt \cos mt$ **20.** $\cos nt \cos mt$

Evaluate

21. $\int_0^{\pi/2} \sin^3 x \cos^3 x\,dx$ **22.** $\int_0^{\pi/4} \tan^6 x\,dx$

23. $\int_{\pi/6}^{\pi/3} 2\sin 3x \cos 2x\,dx$ **24.** $\int_0^{\pi/3} \cos x \cos 3x\,dx$

CHAPTER 32

SYSTEMATIC INTEGRATION

At this stage it is possible to classify most of the integrals which the reader is likely to meet.
Once correctly classified, a given expression can be integrated using the method best suited to its category.

The simplest category comprises the quotable results listed below.

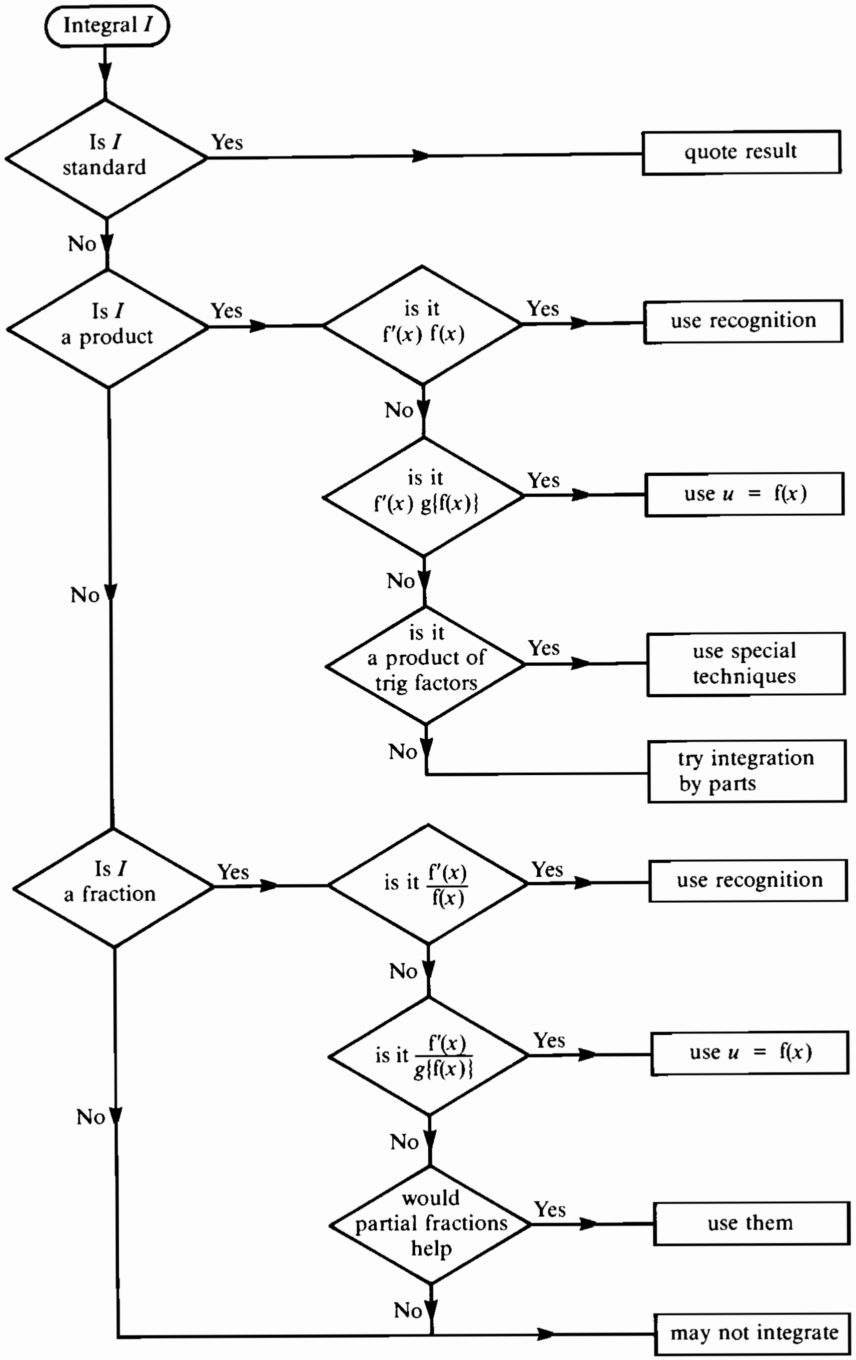
STANDARD INTEGRALS

Function	Integral	Function	Integral
x^n	$\frac{1}{n+1}x^{n+1}$ $(n \neq -1)$	$-\text{cosec}^2 x$	$\cot x$
e^x	e^x	$\tan x$	$\ln\|\sec x\|$
$\frac{1}{x}$	$\ln\|x\|$	$\sec x$	$\ln\|\sec x + \tan x\|$
$\cos x$	$\sin x$	$\frac{1}{1+x^2}$	$\arctan x$
$\sin x$	$-\cos x$	$\frac{1}{\sqrt{(1-x^2)}}$	$\arcsin x$
$\sec^2 x$	$\tan x$		

Each of these should be recognised equally readily when x is replaced by ax or $(ax + b)$, e.g.
for e^{ax+b} the standard integral is $\frac{1}{a}e^{ax+b}$ and for $\cos ax$ the standard integral is $\frac{1}{a}\sin ax$

CLASSIFICATION

When attempting to classify a particular function the following questions should be asked, *in order*, about the form of the integral.



Other Techniques

Although this systematic approach deals successfully with most integrals at this level, inevitably the reader will encounter some integrals for which no method is obvious. A fraction, for instance may be such that its *numerator* can be separated, thus producing *two* (or more) fractions of different types, e.g.

$$\int \frac{x+1}{\sqrt{(1-x^2)}}\,dx = \int \frac{x}{\sqrt{(1-x^2)}}\,dx + \int \frac{1}{\sqrt{(1-x^2)}}\,dx$$

Further, many expressions other than products and fractions can be integrated by making a suitable substitution. Because at this stage the reader cannot always be expected to 'spot' an appropriate change of variable, a substitution is suggested in all but the simplest of cases. The resulting integral must be converted so that is is expressed in terms of the original variable *except in the case of a definite integral* when it is usually much easier to change the limits (See Chapter 29).

The following examples illustrate some of the integrals which respond to a change of variable.

Examples 32a

1. Use the substitution $u = 1 + 2x$ to find $\int x(1+2x)^{11}\,dx$

$$u = 1 + 2x \quad \Rightarrow \quad \ldots du \equiv 2\,dx \quad \Rightarrow \quad \tfrac{1}{2}du \equiv \ldots dx$$

Hence $$\int x(1+2x)^{11}\,dx = \int \tfrac{1}{2}(u-1)(u^{11})(\tfrac{1}{2}\,du)$$

$$= \tfrac{1}{4}\int (u^{12} - u^{11})\,du$$

$$= \tfrac{1}{4}(\tfrac{1}{13}u^{13} - \tfrac{1}{12}u^{12}) + K$$

$$= \tfrac{1}{624}u^{12}(12u - 13) + K$$

i.e. $$\int x(1+2x)^{11}\,dx = \tfrac{1}{624}(1+2x)^{12}(24x-1) + K$$

---

2. Integrate $\dfrac{1}{\sqrt{(9-16x^2)}}$ w.r.t. x by using $x = \tfrac{3}{4}\sin\theta$

$$x = \tfrac{3}{4}\sin\theta \quad \Rightarrow \quad \ldots dx \equiv \ldots \tfrac{3}{4}\cos\theta\,d\theta$$

Hence $$\int \frac{1}{\sqrt{(9-16x^2)}}\,dx = \int \frac{1}{\sqrt{(9-9\sin^2\theta)}}(\tfrac{3}{4}\cos\theta)\,d\theta$$

$$= \frac{3}{4}\int \frac{\cos\theta}{3\sqrt{(1-\sin^2\theta)}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\cos\theta}\,d\theta = \tfrac{1}{4}\theta + K$$

i.e. $$\frac{1}{\sqrt{(9-16x^2)}}\,dx = \tfrac{1}{4}\arcsin\tfrac{4}{3}x + K$$

---

3. Complete the square in the denominator and then use $u = x + 1$ to find (a) $\displaystyle\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx$ (b) $\displaystyle\int_{-1}^{0} \frac{1}{\sqrt{(3-2x-x^2)}}\,dx$

$$u = x + 1 \quad \Rightarrow \quad \ldots du \equiv \ldots dx \quad \text{and} \quad \left\{\begin{array}{c|c|c} x & -1 & 0 \\ u & 0 & 1 \end{array}\right.$$

(a) First we deal with the denominator

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

Hence $$\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx = \int_{-1}^{0} \frac{1}{(x+1)^2+1}\,dx = \int_{0}^{1} \frac{1}{u^2+1}\,du$$

$$= \Big[\arctan u\Big]_0^1$$

i.e. $$\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx = \tfrac{1}{4}\pi$$

(b) $3 - 2x - x^2 = 4 - (x+1)^2$

Hence $$\int_{-1}^{0} \frac{1}{\sqrt{(3-2x-x^2)}}\,dx = \int_{0}^{1} \frac{1}{\sqrt{(4-u^2)}}\,du = \left[\arcsin\frac{u}{2}\right]_0^1$$

$$= \tfrac{1}{6}\pi$$

---

4. Integrate $\sqrt{(1-x^2)}$ w.r.t. x, using the substitution $x = \sin\theta$

$$x = \sin\theta \quad \Rightarrow \quad \ldots dx \equiv \ldots \cos\theta \, d\theta$$

Hence $$\int \sqrt{(1-x^2)}\,dx = \int \sqrt{(1-\sin^2\theta)}\cos\theta\,d\theta = \int \cos^2\theta\,d\theta$$

$$= \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta) + K$$

This expression must now be given in terms of x, so we use $\theta = \arcsin x$ and $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{(1-\sin^2\theta)} = 2x\sqrt{(1-x^2)}$

i.e. $$\int \sqrt{(1-x^2)}\,dx = \tfrac{1}{2}\{\arcsin x + x\sqrt{(1-x^2)}\} + K$$

The fourth example has been included specifically to demonstrate how to convert a term like $\sin 2\theta$ back to the variable x. It is suggested that the reader now re-work this example for the *definite* integral

$$\int_0^1 \sqrt{(1-x^2)}\,dx$$

EXERCISE 32a

Find the following integrals using the suggested substitution.

1. $\int (x+1)(x+3)^5\,dx$; $\quad x+3 = u$

2. $\int \dfrac{1}{4+x^2}\,dx$; $\quad x = 2\tan\theta$

3. $\int \dfrac{x}{\sqrt{(3-x)}}\,dx$; $\quad 3-x = u^2$

4. $\int x\sqrt{(x+1)}\,dx$; $\quad x+1 = u^2$

5. $\int \dfrac{2x+1}{(x-3)^6}\,dx$; $\quad x-3 = u$

6. $\int \dfrac{1}{\sqrt{(1+x^2)}}\,dx$; $\quad x = \tan\theta$

7. $\int 2x\sqrt{(3x-4)}\,dx$; $\quad 3x-4 = u^2$

8. $\int \dfrac{3}{25+4x^2}\,dx$; $\quad 2x = 5\tan\theta$

9. $\int \dfrac{1}{\sqrt{(x^2+4x+3)}}\,dx$; $\quad x+2 = \sec\theta$, after 'completing the square' in the denominator.

Devise a suitable substitution and hence find:

10. $\int 2x(1-x)^7\,dx$ **11.** $\int \dfrac{1}{\sqrt{(1-4x^2)}}\,dx$ **12.** $\int \dfrac{x+3}{(4-x)^5}\,dx$

EXERCISE 32b

Use the flow chart to classify each of the following integrals. Hence perform each integration using an appropriate method.

1. $\int e^{2x+3}\,dx$

2. $\int x\sqrt{(2x^2-5)}\,dx$

3. $\int \sin^2 3x\,dx$

4. $\int x\,e^{-x^2}\,dx$

5. $\int \sin 3\theta\cos\theta\,d\theta$

6. $\int u(u+7)^9\,du$

7. $\int \dfrac{x^2}{(x^3+9)^5}\,dx$

8. $\int \dfrac{\sin 2y}{1-\cos 2y}\,dy$

9. $\int \dfrac{1}{2x+7}\,dx$

10. $\int \dfrac{1}{\sqrt{(1-u^2)}}\,du$

11. $\int \sin 3x\sqrt{(1+\cos 3x)}\,dx$

12. $\int x\sin 4x\,dx$

13. $\int \dfrac{x+2}{x^2+4x-5}\,dx$

14. $\int \dfrac{x+1}{x^2+4x-5}\,dx$

15. $\int \dfrac{x+2}{(x^2+4x-5)^3}\,dx$

16. $\int 3y\sqrt{(9-y^2)}\,dy$

17. $\int e^{2x}\cos 3x\,dx$

18. $\int \ln 5x\,dx$

19. $\int \cos^3 2x\, dx$

20. $\int \operatorname{cosec}^2 x\, e^{\cot x}\, dx$

21. $\int \frac{\sin y}{\sqrt{(7 + \cos y)}}\, dy$

22. $\int x^2 e^x\, dx$

23. $\int \frac{x}{x^2 - 4}\, dx$

24. $\int \frac{x^2}{x^2 - 4}\, dx$

25. $\int \frac{1}{x^2 - 4}\, dx$

26. $\int \cos 4x \cos x\, dx$

27. $\int \sin^5 2\theta\, d\theta$

28. $\int \cos^2 u \sin^3 u\, du$

29. $\int \tan^2\theta\, d\theta$

30. $\int \frac{1 - 2x}{\sqrt{(1 - x^2)}}\, dx$

31. $\int y^2 \cos 3y\, dy$

32. $\int \frac{\sec^2 x}{1 - \tan x}\, dx$

33. $\int x\sqrt{(7 + x^2)}\, dx$

34. $\int \sin(5\theta - \pi/4)\, d\theta$

35. $\int \cos\theta \ln \sin\theta\, d\theta$

36. $\int \sec^2 u\, e^{\tan u}\, du$

DIFFERENTIAL EQUATIONS

An equation in which at least one term contains $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc., is called a *differential equation*. If it contains only $\frac{dy}{dx}$ it is of the first order whereas if it contains $\frac{d^2y}{dx^2}$ it is of the second order, and so on.

For example, $x + 2\frac{dy}{dx} = 3y$ is a first order differential equation and $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$ is a second order differential equation.

Each of these examples is a *linear* differential equation because none of the differential coefficients $\left(\text{i.e. } \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ is raised to a power higher than 1.

A differential equation represents a relationship between two variables. The same relationship can often be expressed in a form which does not contain a differential coefficient,

e.g. $\frac{dy}{dx} = 2x$ and $y = x^2 + K$ express the same relationship between x and y, but $\frac{dy}{dx} = 2x$ is a differential equation whereas $y = x^2 + K$ is not.

Converting a differential equation into a direct one is called *solving the differential equation*. This clearly involves some form of integration. There are many different types of differential equation, each requiring a specific technique for its solution. At this stage however we are going to deal with only one simple type, i.e. linear differential equations of the first order.

FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

Consider the differential equation $3y\frac{dy}{dx} = 5x^2$ [1]

Integrating both sides of the equation gives

$$\int 3y\frac{dy}{dx}\, dx = \int 5x^2\, dx$$

We saw in Chapter 29 that $\ldots\frac{dy}{dx}\, dx \equiv \ldots dy$

so $$\int 3y\, dy = \int 5x^2\, dx \quad [2]$$

Temporarily removing the integral signs from this equation gives

$$3y\, dy = 5x^2\, dx \quad [3]$$

This can be obtained direct from equation [1] by *separating the variables*, i.e. by *separating* dy *from* dx *and collecting on one side all the terms involving* y *together with* dy, *while all the* x *terms are collected, along with* dx *on the other side.*

It is vital to appreciate that what is shown in [3] above does not, in itself, have any meaning, and it *should not be written down as a step in the solution.* It simply provides a way of making a quick mental conversion from the differential equation [1] to the form [2] which is ready for two separate integrations.

Now returning to equation [2] and integrating each side we have

$$\tfrac{3}{2}y^2 = \tfrac{5}{3}x^3 + A$$

Note that it is unnecessary to introduce a constant of integration on both sides. It is sufficient to have a constant on one side only.

When solving differential equations, the constant of integration is usually denoted by A, B, etc. and is called the *arbitrary constant.*

The solution of a differential equation including the arbitrary constant is called *the general solution*, or, very occasionally, *the complete primitive.* It represents a family of straight lines or curves, each member of the family corresponding to one value of A.

Example 32c

Find the general solution of the differential equation

$$\frac{1}{x}\frac{dy}{dx} = \frac{2y}{x^2+1}$$

$$\frac{1}{x}\frac{dy}{dx} = \frac{2y}{x^2+1} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1}$$

So, after separating the variables we have,

$$\int \frac{1}{y}\,dy = \int \frac{2x}{x^2+1}\,dx$$

$$\Rightarrow \qquad \ln|y| = \ln|x^2+1| + A$$

Note that whenever we solve a differential equation some integration has to be done, so the systematic classification of each integral involved is an essential part of solving differential equations.

EXERCISE 32c

Find the general solution of each differential equation

1. $y\dfrac{dy}{dx} = \sin x$
2. $x^2\dfrac{dy}{dx} = y^2$
3. $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{1}{y^2-2}$
4. $\tan y\dfrac{dy}{dx} = \dfrac{1}{x}$

5. $\dfrac{dy}{dx} = y^2$
6. $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{1}{1-x^2}$
7. $(x-3)\dfrac{dy}{dx} = y$
8. $\tan y\dfrac{dx}{dy} = 4$
9. $u\dfrac{du}{dv} = v+2$
10. $\dfrac{y^2}{x^3}\dfrac{dy}{dx} = \ln x$
11. $e^x\dfrac{dy}{dx} = \dfrac{x}{y}$
12. $\sec x\dfrac{dy}{dx} = e^y$
13. $r\dfrac{dr}{d\theta} = \sin^2\theta$
14. $\dfrac{dv}{du} = \dfrac{v+1}{u+2}$
15. $xy\dfrac{dy}{dx} = \ln x$
16. $y(x+1) = (x^2+2x)\dfrac{dy}{dx}$
17. $v^2\dfrac{dv}{dt} = (2+t)^3$
18. $x\dfrac{dy}{dx} = \dfrac{1}{y} + y$
19. $r\dfrac{d\theta}{dr} = \cos^2\theta$
20. $y\sin^3 x\dfrac{dy}{dx} = \cos x$
21. $\dfrac{uv}{u-1} = \dfrac{du}{dv}$
22. $e^x\dfrac{dy}{dx} = e^{y-1}$
23. $\tan x\dfrac{dy}{dx} = 2y^2\sec^2 x$
24. $\dfrac{dy}{dx} = \dfrac{x(y^2-1)}{(x^2+1)}$

CALCULATION OF THE ARBITRARY CONSTANT

We saw on pp. 539–40 that

$$3y\frac{dy}{dx} = 5x^2 \quad \Leftrightarrow \quad \tfrac{3}{2}y^2 = \tfrac{5}{3}x^3 + A$$

The equation $\frac{3}{2}y^2 = \frac{5}{3}x^3 + A$ represents a family of curves with similar characteristics. Each value of A gives one particular member of the family, i.e. a *particular solution.*
The value of A cannot be found from the differential equation alone; further information is needed.

Suppose that we require the equation of a curve which satisfies the differential equation $2\frac{dy}{dx} = \frac{\cos x}{y}$ and which passes through the point (0, 2).

We want one member of the family of curves represented by the differential equation, i.e. the particular value of the arbitrary constant must be found.

The general solution has to be found first so, separating the variables, we have

$$\int 2y\,dy = \int \cos x\,dx \quad \Rightarrow \quad y^2 = A + \sin x$$

In order to find the required curve we need the value of A such that the general solution is satisfied by

$$x = 0 \text{ and } y = 2$$

i.e. $$4 = A + 0 \quad \Rightarrow \quad A = 4$$

Hence the equation of the specified curve is $y^2 = 4 + \sin x$

Examples 32d

1. Describe the family of curves represented by the differential equation $y = x\frac{dy}{dx}$ and sketch any three members of this family.
   Find the particular solution for which $y = 2$ when $x = 1$ and sketch this member of the family on the same axes as before.

By separating the variables,

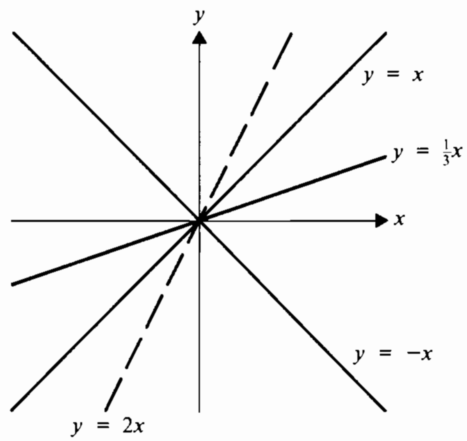
$$y = x\frac{dy}{dx} \quad \Rightarrow \quad \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \ln|y| = \ln|x| + \ln A$$

i.e. the general solution is $y = Ax$

This equation represents a family of straight lines through the origin, each line having a gradient A, as shown in the following diagram.

If $y = 2$ when $x = 1$ then $A = 2$ and the corresponding member of the family is the line $y = 2x$



2. A curve is such that the gradient is proportional to the product of the x and y coordinates. If the curve passes through the points (2, 1) and $(4, e^2)$, find its equation.

First find the general solution

$$\frac{dy}{dx} = kxy \quad \text{where } k \text{ is a constant of proportion.}$$

$$\therefore \quad \int \frac{1}{y}\,dy = \int kx\,dx \quad \Rightarrow \quad \ln y = \tfrac{1}{2}kx^2 + A$$

There are *two* unknown constants this time so we need two extra pieces of information; these are

(i) $y = 1$ when $x = 2 \quad \Rightarrow \quad \ln 1 = 2k + A$

$\ln 1 = 0$ so $A + 2k = 0$

(ii) $y = e^2$ when $x = 4 \quad \Rightarrow \quad \ln e^2 = 8k + A$

$\ln e^2 = 2$ so $A + 8k = 2$

Solving these equations for A and k we get $k = \frac{1}{3}$ and $A = -\frac{2}{3}$

$\therefore$ the equation of the specified curve is $\ln y = \frac{1}{6}x^2 - \frac{2}{3}$

$$= \tfrac{1}{6}(x^2 - 4)$$

or $$y^6 = e^{(x^2-4)}$$

EXERCISE 32d

Find the particular solution of each of the following differential equations.

1. $y^2\dfrac{dy}{dx} = x^2 + 1$ and $y = 1$ when $x = 2$

2. $e^t\dfrac{ds}{dt} = \sqrt{s}$ and $s = 4$ when $t = 0$

3. $\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}$ and $y = 3$ when $x = 2$

4. A curve passes through the origin and its gradient function is $2x - 1$. Find the equation of the curve and sketch it.

5. A curve for which $e^{-x}\dfrac{dy}{dx} = 1$, passes through the point $(0, -1)$. Find the equation of the curve.

6. A curve passes through the points $(1, 2)$ and $(\frac{1}{5}, -10)$ and its gradient is inversely proportional to x^2. Find the equation of the curve.

7. If $y = 2$ when $x = 1$, find the coordinates of the point where the curve represented by $\dfrac{2y}{3}\dfrac{dy}{dx} = e^{-3x}$ crosses the y-axis.

8. Find the equation of the curve whose gradient function is $\dfrac{y+1}{x^2 - 1}$ and which passes through the point $(-3, 1)$.

9. The gradient function of a curve is proportional to $x + 3$. If the curve passes through the origin and the point $(2, 8)$, find its equation.

10. Solve the differential equation $(1 + x^2)\dfrac{dy}{dx} - y(y + 1)x = 0$, given that $y = 1$ when $x = 0$

MIXED EXERCISE 32

Integrate w.r.t. x

1. $x(1 + x^2)^4$
2. xe^{-3x}
3. $\cos 2x \cos 3x$
4. $\dfrac{x+3}{x+2}$
5. $\dfrac{x^2}{(x^3+1)^2}$
6. $\dfrac{3}{(x-4)(x-1)}$
7. $\dfrac{(x+1)}{x(2x+1)}$
8. $\dfrac{x-1}{(x^2+1)}$
9. $\dfrac{\sin x}{\sqrt{(\cos x)}}$

Evaluate

10. $\displaystyle\int_{\pi/2}^{\pi} (\sin \tfrac{1}{2}x + \cos 2x)\,dx$
11. $\displaystyle\int_2^5 x\sqrt{(x-1)}\,dx$
12. $\displaystyle\int_0^{\pi/4} \tan^3 x\,dx$
13. $\displaystyle\int_1^2 x\sqrt{(5 - x^2)}\,dx$
14. $\displaystyle\int_4^6 \frac{5}{x^2 - x - 6}\,dx$
15. $\displaystyle\int_2^3 \frac{1}{x \ln x}\,dx$
16. $\displaystyle\int_{-2}^{-1} \frac{2-x}{x(1-x)}\,dx$
17. $\displaystyle\int_0^1 \frac{x(1-x)}{2+x}\,dx$

18. Solve the differential equation $\dfrac{dy}{dx} = 3x^2y^2$ given that $y = 1$ when $x = 0$

19. If $\dfrac{dy}{dx} = x(y^2 + 1)$ and $y = 0$ when $x = 2$ find the particular solution of the differential equation.

20. Find the equation of the curve which passes through the point $(\frac{1}{2}, 1)$ and is defined by the differential equation $ye^{y^2}\dfrac{dy}{dx} = e^{2x}$. Show that the curve also passes through the point $(2, 2)$ and sketch the curve.

CHAPTER 33

APPLICATIONS OF CALCULUS

GENERAL RATES OF INCREASE

We have already seen that

$\dfrac{dy}{dx}$ represents the rate at which y increases compared with x

Whenever the variation in one quantity, p say, depends upon the changing value of another quantity, q, then the rate of increase of p compared with q can be expressed as $\dfrac{dp}{dq}$

There are many every-day situations where such relationships exist, e.g.

1) liquid expands when it is heated so, if V is the volume of a quantity of liquid and T is the temperature, then the rate at which the volume increases with temperature can be written $\dfrac{dV}{dT}$

2) if the profit, P, made by a company selling radios depends upon the number, n, of radios sold, then $\dfrac{dP}{dn}$ represents the rate of increase of profit compared with the increase in sales.

MOTION IN A STRAIGHT LINE

Consider a particle which moves in a straight line so that, at any time t, its displacement from a fixed point on that line is s, its velocity is v and its acceleration is a

Velocity is the rate at which the displacement is increasing with time,

i.e. $$v = \frac{ds}{dt} \quad \text{and hence} \quad \int v\, dt = s$$

Acceleration is the rate at which the velocity is increasing with time,

i.e. $$a = \frac{dv}{dt} \quad \text{and hence} \quad \int a\, dt = v$$

Examples 33a

1. A particle, P, is moving along a straight line and Q is a fixed point on that line. The acceleration of P after t seconds is given by $3t^2 - 2$. When $t = 0$, P is at O, and has a velocity of 2 m/s. Find an expression for the velocity of P at any time t and find the velocity after 4 seconds.

$$a = 3t^2 - 2$$

$$v = \int a\, dt = \int (3t^2 - 2)\, dt$$

$\therefore$ $$v = t^3 - 2t + K$$

When $t = 0$, $v = 2$ $\Rightarrow$ $2 = K$

$\therefore$ $$v = t^3 - 2t + 2$$

Hence, after 4 seconds, i.e. when $t = 4$,

$$v = 4^3 - 2(4) + 2 = 58$$

The velocity after 4 seconds is 58 m/s.

2. The velocity, at any time t, of a particle moving in a straight line is given by $v = 1 + e^{2t}$. Initially (i.e. when $t = 0$) the particle is at O, a fixed point on the line. Find
   (a) an expression for the acceleration of the particle at time t,
   (b) the displacement of the particle from O after 3 seconds.

$$v = 1 + e^{2t}$$

(a) $$a = \frac{dv}{dt} = 2e^{2t}$$

i.e. at any time t, the acceleration is $2e^{2t}$

(b) $$s = \int v\,dt = \int (1 + e^{2t})\,dt$$

$$= t + \tfrac{1}{2}e^{2t} + K$$

When $t = 0$ the particle is at O, i.e. $s = 0$

$\therefore$ $$0 = 0 + \tfrac{1}{2}e^0 + K \quad \Rightarrow \quad K = -\tfrac{1}{2}$$

$\Rightarrow$ $$s = t + \tfrac{1}{2}e^{2t} - \tfrac{1}{2}$$

Hence, when $t = 3$, $s = 3 + \frac{1}{2}e^6 - \frac{1}{2}$

i.e. after 3 seconds the displacement from O is $\frac{1}{2}(5 + e^6)$

---

3. A particle starts from rest and travels in a straight line so that its acceleration is $(t + 3)$ m/s² at any time t. Find the distance travelled in the interval of time from $t = 1$ to $t = 3$

$$a = t + 3$$

$$v = \int (t + 3)\,dt = \tfrac{1}{2}t^2 + 3t + K$$

The particle starts from rest so $v = 0$ when $t = 0$ $\quad\Rightarrow\quad K = 0$

$\therefore$ $$v = \tfrac{1}{2}t^2 + 3t$$

$$s = \int (\tfrac{1}{2}t^2 + 3t)\,dt = \tfrac{1}{6}t^3 + \tfrac{3}{2}t^2 + K'$$

where s is the displacement at time t from a fixed point.

For positive values of t, s is always positive so the *distance* travelled in any time interval is given by the difference of the displacements at the beginning and end of that interval.

When $t = 1$, $\quad s_1 = \frac{1}{6} + \frac{3}{2} + K' = \frac{5}{3} + K'$

When $t = 3$, $\quad s_3 = \frac{1}{6}(27) + \frac{3}{2}(9) + K' = 18 + K'$

$\therefore\ s_3 - s_1 = (18 + K') - (\frac{5}{3} + K') = 16\frac{1}{3}$

i.e. the distance travelled between $t = 1$ and $t = 3$ is $16\frac{1}{3}$ m.

EXERCISE 33a

1. A particle moving in a straight line starts from rest at a point O on the line, and, t seconds later, has an acceleration $(t - 6)$ m/s². Find expressions for the velocity and the displacement of the particle from O after t seconds. Find also the velocity and the displacement from O after 6 seconds.

2. A particle, P, moves in a straight line with acceleration $(3t - 1)$ m/s² where t is the time in seconds. If P has a velocity of 3 m/s when $t = 2$, find its velocity at time t and when $t = 5$

3. The velocity of a particle moving in a straight line is v m/s after t seconds where $v = 6t^2 + 1$. Find how far the particle travels in the interval from $t = 1$ to $t = 4$

4. A particle moves in a straight line with velocity v m/s where, after t seconds, $v = 2 + 1/(t + 1)^2$. Initially the particle is at a fixed point O on the line.
   (a) Find an expression for the displacement of the particle from after t seconds
   (b) Show that the velocity has a limiting value (i.e. v approaches a fixed value as t approaches infinity), and state this value.

5. The velocity of a particle moving in a straight line is $3t^2$ m/s at any time, t seconds. What is the acceleration of the particle at time t? If the particle passes through a fixed point A on the line, with velocity 12 m/s, find the displacement from A 6 seconds later.

NATURAL OCCURRENCE OF DIFFERENTIAL EQUATIONS

Differential equations often arise when a physical situation is interpreted mathematically (i.e. when a mathematical model is made of the physical situation).

Consider the following examples.

1) Suppose that a body falls from rest in a medium which causes the velocity to decrease at a rate proportional to the velocity.

Using v for velocity and t for time, the rate of *decrease* of velocity can be written as $-\dfrac{dv}{dt}$.

Thus the motion of the body satisfies the differential equation

$$-\frac{dv}{dt} = kv$$

2) During the initial stages of the growth of yeast cells in a culture, the number of cells present increases in proportion to the number already formed.

Thus n, the number of cells at a particular time t, can be found from the differential equation

$$\frac{dn}{dt} = kn$$

3) Suppose that a chemical mixture contains two substances A and B whose weights are W_A and W_B and whose combined weight remains constant. B is converted into A at a rate which is inversely proportional to the weight of B and proportional to the square of the weight of A in the mixture at any time t. The weight of B present at time t can be found using

$$\frac{d}{dt}(W_B) = \frac{k}{W_B} \times (W_A)^2$$

But $W_A + W_B$ is constant, W say

Hence $$\frac{d}{dt}(W_B) = \frac{k(W - W_B)^2}{W_B}$$

This differential equation now relates W_B and t.

Note. In forming (and subsequently solving) differential equations from naturally occuring data, it is not actually necessary to understand the background of the situation or experiment.

EXERCISE 33b

In Questions 1 to 4 form, but *do not solve*, the differential equation representing the given data.

1. A body moves with a velocity v which is inversely proportional to its displacement s from a fixed point.

2. The rate at which the height h of a certain plant increases is proportional to the natural logarithm of the difference between its present height and its final height H.

3. The manufacturers of a certain brand of soap powder are concerned that the number, n, of people buying their product at any time t has remained constant for some months. They launch a major advertising programme which results in the number of customers increasing at a rate proportional to the square root of n. Express as differential equations the progress of sales.
   (a) before advertising
   (b) after advertising.

4. In an isolated community, the number, n, of people suffering from an infectious disease is N_1 at a particular time. The disease then becomes epidemic and spreads so that the number of sick people increases at a rate proportional to n, until the total number of sufferers is N_2. The rate of increase then becomes inversely proportional to n until N_3 people have the disease. After this, the total number of sick people decreases at a constant rate. Write down the differential equation governing the incidence of the disease.
   (a) for $N_1 \leqslant N_2$
   (b) for $N_2 \leqslant N_3$
   (c) for $n \geqslant N_3$.

5. Two chemicals, P and Q, are involved in a reaction. The masses of P and Q present at any time t, are p and q respectively. The rate at which p is increasing at time t is k times the product of the two masses. If the masses of P and Q have a constant sum s, find a differential equation expressing $\frac{dp}{dt}$ in terms of p, s and k.

FINDING COMPOUND AREAS

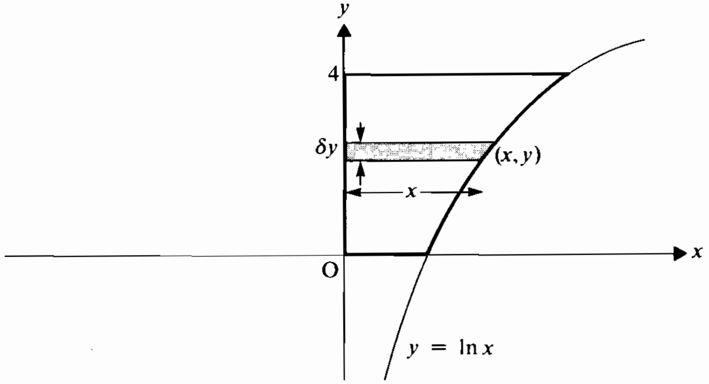
In Chapter 28 we saw how integration can be used to calculate an area bounded by the x-axis, part of a known curve and two vertical lines. The method introduced there is based on the summation of the areas of vertical strips, or elements.

We also saw that an area bounded by two horizontal lines, the y-axis and part of a curve can be found in a similar way but in this case we use horizontal strips.

Only curves with algebraic equations were used in Chapter 28 but the methods apply to the graphs of other functions, as the following example shows.

Examples 33c

1. A plane region is defined by the line $y = 4$, the x and y axes and part of the curve $y = \ln x$. Find the area of the region.



A vertical element is unsuitable in this case as the top and bottom are not always on the same boundaries, but a horizontal element is satisfactory.

The area, δA, of a typical horizontal element is given by $\delta A \approx x\,\delta y$. Because the width of our element is δy we will have to integrate w.r.t y, so we need the equation of the curve in the form $x = f(y)$

$$y = \ln x \quad \Rightarrow \quad x = e^y$$

$\therefore \quad \delta A \approx e^y\,\delta y$

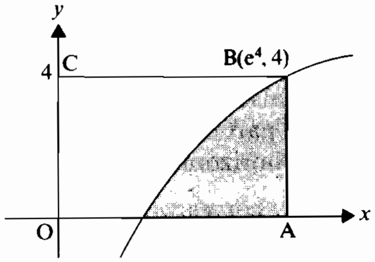
$$\Rightarrow \quad A = \lim_{\delta y \to 0} \sum_{y=0}^{y=4} e^y\,\delta y = \int_0^4 e^y\,dy$$

$$= \Big[e^y\Big]_0^4$$

$$= e^4 - e^0$$

The defined area is $(e^4 - 1)$ square units

Note that this area can also be found by subtracting the shaded region from the area of the rectangle OABC but this alternative is not always suitable.



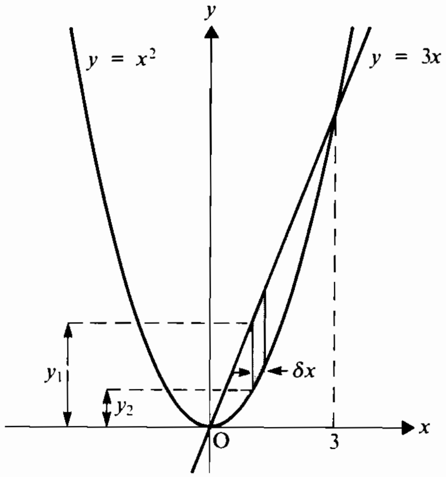
A similar approach can be made in a variety of circumstances provided that an element can be found

1) which has the same format throughout, i.e. the ends of all the elements are on the same boundaries;

2) whose length and width are measured parallel to the x and y axes.

2. Find the area between the curve $y = x^2$ and the line $y = 3x$

The line and curve meet where $x^2 = 3x$,
i.e. where $x = 0$ and $x = 3$



A vertical strip always has its top on the line and its foot on the curve so it is a suitable element. It is approximately a rectangle whose width is δx and whose height is the vertical distance between the line and the curve. The area of the element, δA, is given by

$$\delta A \approx (y_1 - y_2)\delta x = (3x - x^2)\delta x$$

$$\therefore \qquad A = \lim_{\delta x \to 0} \sum_{x=0}^{x=3} (3x - x^2)\delta x = \int_0^3 (3x - x^2)\,dx$$

$$= \Big[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\Big]_0^3$$

$$= 4\tfrac{1}{2}$$

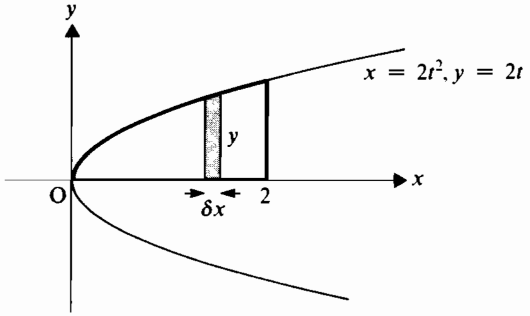
The required area is $4\frac{1}{2}$ square units.

The area bounded partly by a curve whose equation is given parametrically can also be found by summing the areas of suitable elements.

---

3. Find the area bounded in the first quadrant by the x-axis, the line $x = 2$ and part of the curve with parametric equations $x = 2t^2$, $y = 2t$

The sketch of this curve need not be an accurate shape but it is important to realise that it goes through the origin because when $t = 0$, both x and y are zero.



A suitable element is a vertical strip which is approximately a rectangle of height, y, width δx and area δA where $\delta A \approx y\,\delta x$

Considering $x = 2t^2$ gives

$$\delta x \approx \left(\frac{dx}{dt}\right)(\delta t) = 4t\delta t \quad \text{and} \quad \delta t \to 0 \text{ when } \delta x \to 0$$

it also shows that, when $x = 2$, $t = 1$

$t \neq -1$, as it gives $y = -2$ which is not in the first quadrant.

$$\therefore \qquad \delta A \approx (2t)(4t\delta t)$$

Then $$A = \lim_{\delta x \to 0} \sum_{x=0}^{x=2} 8t^2\delta t = \lim_{\delta t \to 0} \sum_{t=0}^{t=1} 8t^2\delta t$$

$$\Rightarrow \qquad A = \int_0^1 8t^2\,dt = \left[\tfrac{8}{3}t^3\right]_0^1 = \tfrac{8}{3}$$

The required area is $\frac{8}{3}$ square units.

EXERCISE 33c

1. Calculate the area bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 3$

2. Find, by integration, the area bounded by
   (a) the x-axis, the line $x = 2$ and the curve $y = x^2$
   (b) the y-axis, the line $y = 4$ and the curve $y = x^2$
   Sketch these two areas on the same diagram and hence check the sum of the answers to (a) and (b).

3. Calculate the area in the first quadrant between the curve $y^2 = x$ and the line $x = 9$

4. Find the area between the y-axis and the curve $y^2 = 1 - x$

5. Find the area of the region whose boundaries are the y-axis, the line $y = \frac{1}{2}\pi$ and the curve $y = \arcsin x$

6. A region in the xy plane is bounded by the lines $y = 1$ and $x = 1$, and the curve $y = e^x$. Find its area.

7. Find the area bounded by the inequalities $y \leqslant 1 - x^2$ and $y \geqslant 1 - x$

8. Calculate the area bounded by the curve $y = \sin x$ and the lines $y = \frac{1}{2}$ and $x = \frac{1}{2}\pi$

9. Find the area of the region of the xy plane defined by
   (a) $y \geqslant e^x$, $x \geqslant 0$, $y \leqslant e$ (b) $1 \geqslant y \geqslant 1/(x+1)$, $x \leqslant 2$

10. The equations of a curve are $x = 2t$, $y = 2/t$. Find the area bounded by this curve, the x-axis and the ordinates at $x = 1$, $x = 4$

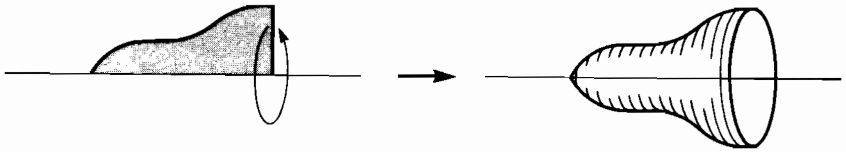
The next two questions are a little harder.

11. Evaluate the area between the line $y = x - 1$ and the curve
    (a) $y = x(1 - x)$ (b) $y = (2x + 1)(x - 1)$

12. Calculate the area of the region of the xy plane defined by the inequalities $y \geqslant (x + 1)(x - 2)$ and $y \leqslant x$

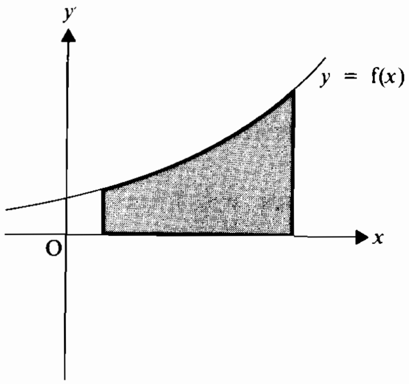
VOLUME OF REVOLUTION

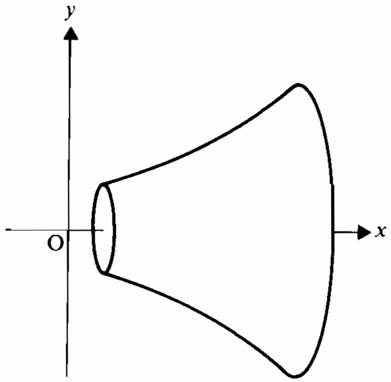
If an area is rotated about a straight line, the three-dimensional object so formed is called a *solid of revolution*, and its volume is a *volume of revolution*.



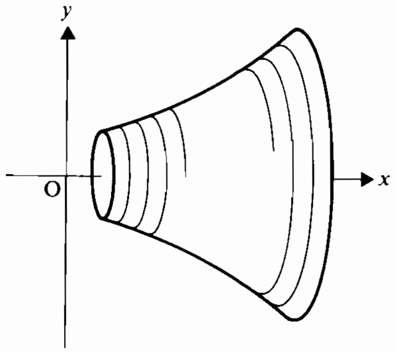
The line about which rotation takes place is always an axis of symmetry for the solid of revolution. Also, any cross-section of the solid which is perpendicular to the axis of rotation, is circular.

Consider the solid of revolution formed when the area shown in the diagram is rotated about the x-axis.

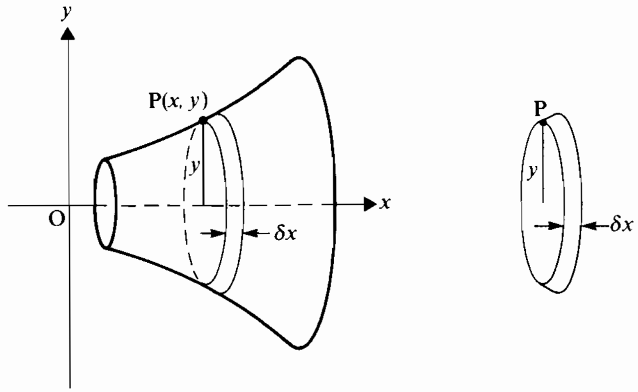




To calculate the volume of this solid we can divide it into 'slices' by making cuts perpendicular to the axis of rotation.



If the cuts are reasonably close together, each slice is approximately cylindrical and the approximate volume of the solid can be found by summing the volumes of these cylinders.



Consider an element formed by one cut through the point $P(x, y)$ and the other cut distant δx from the first.
The volume, δV, of this element is approximately that of a cylinder of radius y and 'height' δx

i.e. $$\delta V \approx \pi y^2 \,\delta x$$

Then the total volume of the solid is V, where

$$V \approx \sum \pi y^2 \,\delta x$$

The smaller δx is, the closer is this approximation to V,

i.e. $$V = \lim_{\delta x \to 0} \sum \pi y^2 \,\delta x = \int \pi y^2 \,dx$$

Now if the equation of the rotated curve is given, this integral ca evaluated and the volume of the solid of revolution found,
e.g. to find the volume generated when the area between part of curve $y = e^x$ and the x-axis is rotated about the x-axis, we use

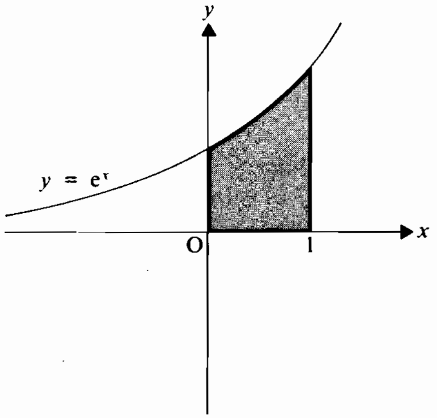
$$\int \pi (e^x)^2 \,dx = \pi \int e^{2x} \,dx$$

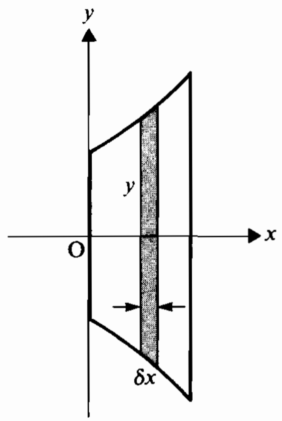
When an area rotates about the y-axis we can use a similar method based on slices perpendicular to the y-axis, giving

$$V = \int \pi x^2 \,dy$$

Examples 33d

1. Find the volume generated when the area bounded by the x and y axes, the line $x = 1$ and the curve $y = e^x$ is rotated through one revolution about the x-axis.





The volume, δV, of the element shown is approximately that of a cylinder of radius y and thickness δx, therefore

$$\delta V \approx \pi y^2 \delta x$$

$\therefore$ the total volume is V, where $V \approx \sum_{x=0}^{x=1} \pi y^2 \delta x$

$$\Rightarrow \qquad V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} \pi y^2 \delta x = \int_0^1 \pi y^2 \, dx$$

$$= \pi \int_0^1 (e^x)^2 \, dx$$

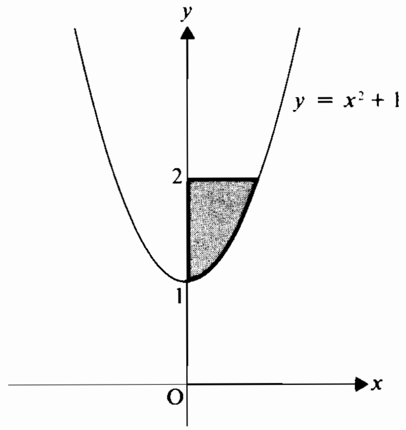
$$= \pi \int_0^1 e^{2x} \, dx$$

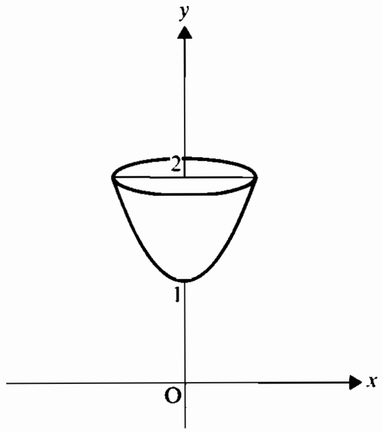
$$= \pi \left[\tfrac{1}{2} e^{2x} \right]_0^1$$

$$= \tfrac{1}{2}\pi(e^2 - e^0)$$

i.e. the specified volume of revolution is $\frac{1}{2}\pi(e^2 - 1)$ cubic units.

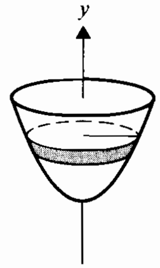
2. The area defined by the inequalities $y \geqslant x^2 + 1, x \geqslant 0, y \leqslant 2$, is rotated completely about the y-axis. Find the volume of the solid generated.

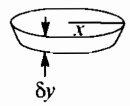




Rotating the shaded area about the y-axis gives the solid shown.

This time we use horizontal cuts to form elements which are approximately cylinders with radius x and thickness δy





$$\delta V \approx \pi x^2 \delta y \quad \Rightarrow \quad V \approx \sum_{y=1}^{y=2} \pi x^2 \delta y$$

$$\therefore \qquad V = \lim_{\delta y \to 0} \sum_{y=1}^{y=2} \pi x^2 \delta y = \int_1^2 \pi x^2 \, dy$$

Using the equation $y = x^2 + 1$ gives $x^2 = y - 1$

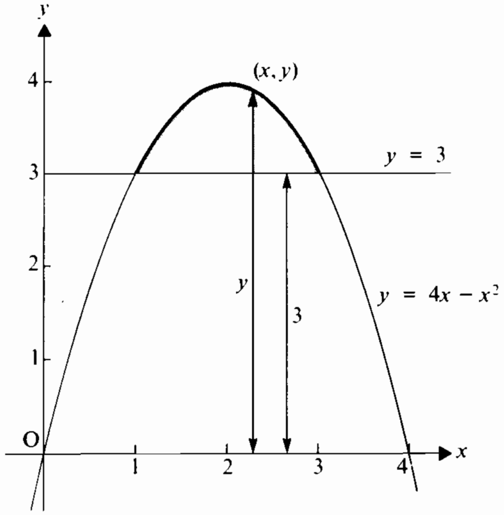
$$\therefore \qquad V = \pi \int_1^2 (y - 1) \, dy = \pi \left[\tfrac{1}{2} y^2 - y \right]_1^2$$

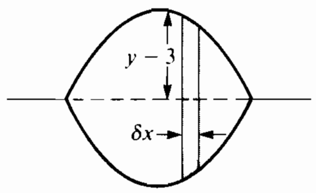
$$= \pi \{(2 - 2) - (\tfrac{1}{2} - 1)\}$$

i.e. the volume of the specified solid is $\frac{1}{2}\pi$ cubic units.

3. The area enclosed by the curve $y = 4x - x^2$ and the line $y = 3$ is rotated about the line $y = 3$. Find the volume of the solid generated.

The line $y = 3$ meets the curve $y = 4x - x^2$ at the points $(1, 3)$ and $(3, 3)$, therefore the volume generated is as shown in the diagram.





The element shown is approximately a cylinder with radius $(y - 3)$ and thickness δx, so its volume, δV, is given by $\delta V \approx \pi(y-3)^2\delta x$

i.e. $$V = \lim_{\delta x \to 0} \sum_{x=1}^{x=3} \pi(y-3)^2\delta x = \pi\int_1^3 (y-3)^2\,dx$$

$\Rightarrow$ $$V = \pi\int_1^3 (4x - x^2 - 3)^2\,dx$$

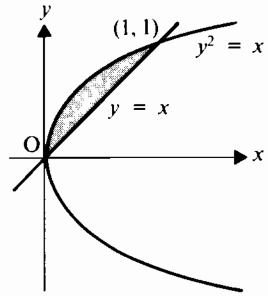
$$= \pi\int_1^3 (9 - 24x + 22x^2 - 8x^3 + x^4)\,dx$$

$$= \pi\left[9x - 12x^2 + \tfrac{22}{3}x^3 - 2x^4 + \tfrac{1}{5}x^5\right]_1^3$$

$$= \tfrac{16}{15}\pi$$

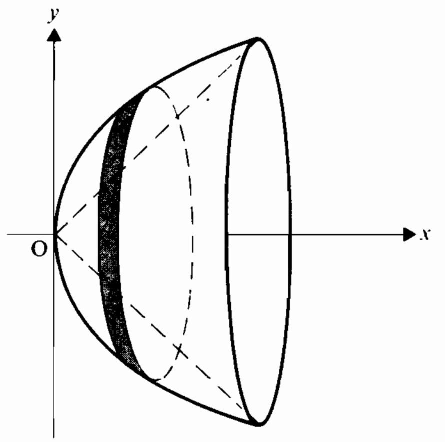
$\therefore$ the required volume is $\frac{16}{15}\pi$ cubic units.

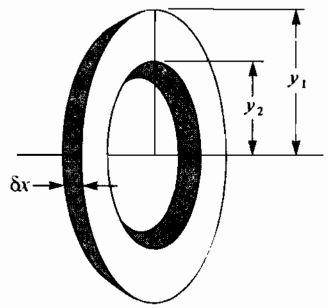
4. Find the volume generated when the area between the curve $y^2 = x$ and the line $y = x$ is rotated completely about the x-axis.



The defined area is shown in the diagram.

When this area rotates about Ox, the solid generated is bowl-shaped on the outside, with a conical hole inside.
The cross-section this time is not a simple circle but is an annulus, i.e. the area between two concentric circles.





For a typical element the area of cross-section is $\pi y_1^2 - \pi y_2^2$

Therefore the volume of an element is given by $\delta V \approx \pi\{y_1^2 - y_2^2\}\delta x$

$\therefore$ $$V \approx \sum_{x=0}^{x=1} \pi\{y_1^2 - y_2^2\}\delta x = \pi\int_0^1 (y_1^2 - y_2^2)\,dx$$

Now $y_1 = \sqrt{x}$ and $y_2 = x$

$\therefore$ $$V = \pi\int_0^1 (x - x^2)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{6}\pi$$

The volume generated is $\frac{1}{6}\pi$ cubic units.

Note that the volume specified in Example 4 could be found by calculating separately

1) the volume given when the curve $y^2 = x$ rotates about the x-axis;

2) the volume, by formula, of a cone with base radius 1 and height 1 and subtracting **(1)** from **(2)**.

The method in which an annulus element is used, however, applies whatever the shape of the hollow interior.

EXERCISE 33d

In each of the following questions, find the volume generated when the area defined by the following sets of inequalities is rotated completely about the x-axis.

1. $0 \leqslant y \leqslant x(4-x)$

2. $0 \leqslant y \leqslant e^x, \quad 0 \leqslant x \leqslant 3$

3. $0 \leqslant y \leqslant \dfrac{1}{x}, \quad 1 \leqslant x \leqslant 2$

4. $0 \leqslant y \leqslant x^2, \quad -2 \leqslant x \leqslant 2$

5. $y^2 \leqslant x, \quad x \leqslant 2$

In each of the following questions, the area bounded by the curve and line(s) given is rotated about the y-axis to form a solid. Find the volume generated.

6. $y = x^2, \; y = 4$

7. $y = 4 - x^2, \; y = 0$

8. $y = x^3, \; y = 1, \; y = 2,$ for $x \geqslant 0$

9. $y = \ln x, \; x = 0, \; y = 0, \; y = 1$

10. Find the volume generated when the area enclosed between $y^2 = x$ and $x = 1$ is rotated about the line $x = 1$

11. The area defined by the inequalities

$$y \geqslant x^2 - 2x + 4, \quad y \leqslant 4$$

is rotated about the line $y = 4$. Find the volume generated.

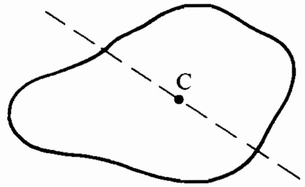
12. The area enclosed by $y = \sin x$ and the x-axis for $0 \leqslant x \leqslant \pi$ is rotated about the x-axis. Find the volume generated.

13. An area is bounded by the line $y = 1$, the x-axis and parts of the curve $y = 3 - x^2$. Find the volume generated when this area rotates completely about the y-axis.

14. The area enclosed between the curves $y = x^2$ and $y^2 = x$ is rotated about the x-axis. Find the volume generated.

CENTROID OF AREA

The centroid of an area is the *point* about which the area is evenly distributed.



It follows that

the centroid lies on any line of symmetry.

Hence, the centroid of a rectangle is the point of intersection of the diagonals, the centroid of a circle is the centre and the centroid of a triangle is the point of intersection of the medians.

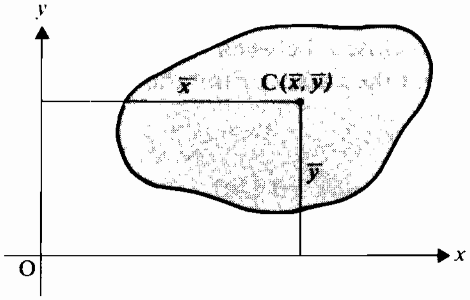
If we have a lamina (i.e. a thin flat plate) whose mass per unit area is constant, the centroid of the area of the lamina is also the *centre of mass of the lamina*, i.e. the point at which the lamina could be supported in perfect balance.

The same point *may* also be the *centre of gravity* of the lamina but only if, in addition to constant mass per unit area, gravitational attraction is constant over the whole area. This is really a problem of mechanics and outside the scope of this book.

First Moment of Area

If the point C is the centroid of an area A then the first moment of A about a particular axis is defined as

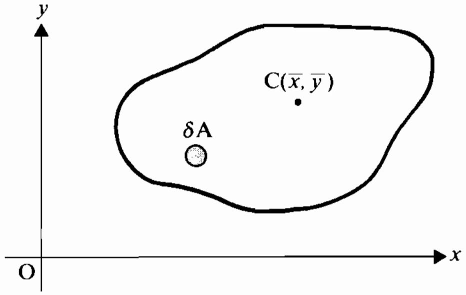
$$A \times (\text{distance of C from that axis})$$



For an area A in the xy plane, the coordinates of the centroid are usually denoted by $(\bar{x}, \bar{y})$. So for the area A shaded in the diagram

$$\begin{cases} \text{the first moment of } A \text{ about O}x \text{ is } A\bar{y} \\ \text{the first moment of } A \text{ about O}y \text{ is } A\bar{x} \end{cases}$$

TO FIND THE COORDINATES OF THE CENTROID OF AN AREA



If $C(\bar{x}, \bar{y})$ is the centroid of the area A and if δA is a small element of that area, then the first moment of A about a given axis is given by

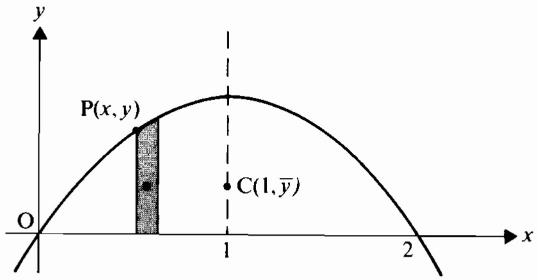
$$\Sigma(\text{first moment of } \delta A \text{ about that axis})$$

Hence $\quad A\bar{x} = \Sigma(\text{first moment of } \delta A \text{ about O}y)$

and $\quad A\bar{y} = \Sigma(\text{first moment of } \delta A \text{ about O}x)$

Examples 33e

1. For the area between the x-axis and the curve $y = x(2-x)$, find
   (a) the first moment about Ox
   (b) the coordinates of the centroid.



If we take as our element of area the vertical strip through $P(x,y)$ shown in the diagram, then

the area of the element, δA, is given by $\delta A \approx y\,\delta x$
and the distance of the centroid of the element from Ox $\approx \frac{1}{2}y$

$\therefore$ the first moment of the element about Ox is approximately $(y\delta x)(\frac{1}{2}y)$

(a) The first moment of the whole area A about Ox is $A\bar{y}$, where

$$A\bar{y} \approx \sum \tfrac{1}{2}y^2\,\delta x$$

$$\Rightarrow \quad A\bar{y} = \lim_{\delta x \to 0} \sum_{x=0}^{x=2} \tfrac{1}{2}y^2\,\delta x = \tfrac{1}{2}\int_0^2 y^2\,dx = \tfrac{1}{2}\int_0^2 x^2(2-x)^2\,dx$$

$$= \tfrac{1}{2}\left[\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\right]_0^2 = \tfrac{8}{15}$$

(b) From symmetry, $\bar{x} = 1$ and $\bar{y}$ is given by $A\bar{y} = \frac{8}{15}$

Now $\qquad A = \int_0^2 y\,dx$

$\therefore \qquad \bar{y}\int_0^2 y\,dx = \frac{8}{15}$

i.e. $\qquad \bar{y}\int_0^2 x(2-x)\,dx = \bar{y}\left[x^2 - \frac{1}{3}x^3\right]_0^2 = \frac{8}{15}$

$\Rightarrow \qquad \frac{4}{3}\bar{y} = \frac{8}{15} \qquad \Rightarrow \qquad \bar{y} = \frac{2}{5}$

i.e. the centroid of the area is the point $(1, \frac{2}{5})$ and the first moment of the area about Ox is $\frac{8}{15}$

CENTROID OF VOLUME OF REVOLUTION

The centroid of a volume is the *point* about which the volume is evenly distributed.
Hence *the centroid of a volume of revolution lies on the axis of rotation.*

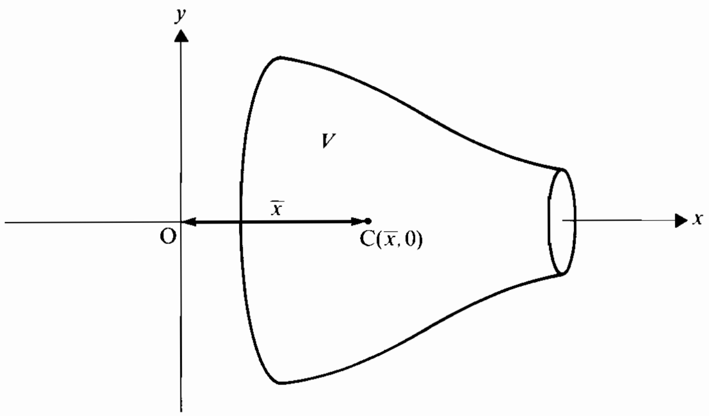
For a solid whose mass per unit volume is constant, the centroid is also the centre of mass.

First Moment of a Volume of Revolution

If the point C is the centroid of a volume V the first moment of V about a particular axis is defined as

$$V \times (\text{distance of C from that axis})$$

If a volume V is formed by rotating an area about the x-axis, then the centroid of V must lie on the x-axis, i.e. C is the point $(\bar{x}, 0)$



So, for the volume V, shown in the diagram,

$$\begin{cases} \text{the first moment about } Ox \text{ is zero} \\ \text{the first moment about } Oy \text{ is } V\bar{x} \end{cases}$$

TO FIND THE POSITION OF THE CENTROID OF A VOLUME OF REVOLUTION

If $C(\bar{x}, 0)$ is the centroid of the volume V shown on p. 566 and if δV is a small element of that volume, then the first moment of V about a specified axis can be found from

$$\Sigma(\text{first moment of } \delta V \text{ about that axis})$$

But, by definition, $V\bar{x} = \Sigma(\text{first moment of } \delta V \text{ about } Oy)$

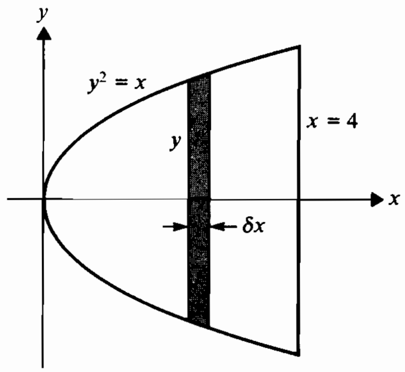
Hence, by calculating V, $\bar{x}$ can be found.

Examples 33e (continued)

2. For the volume generated when the area bounded by the x-axis, the line $x = 4$ and the curve $y^2 = x$ is rotated completely about the x-axis find

   (a) the first moment about the y-axis

   (b) the coordinates of the centroid.



If we take as our element the slice shown, which is approximately a cylinder of radius y and thickness δx then,

the volume of the element, δV, is given by $\delta V \approx \pi y^2 \delta x$
and the approximate distance of its centroid from Oy is x

so the first moment of the element about Oy is approximately $(\pi y^2 \delta x)x$

(a) The first moment of the whole volume V about Ox is $V\bar{x}$ where

$$V\bar{x} \approx \sum_{x=0}^{x=4} \pi y^2 x\,\delta x$$

$$\Rightarrow \quad V\bar{x} = \lim \sum_{x=0}^{x=4} \pi y^2 x\,\delta x = \pi \int_0^4 y^2 x\,dx$$

$$= \pi \int_0^4 x^2\,dx$$

$$= \pi \left[\tfrac{1}{3}x^3\right]_0^4$$

$$= \tfrac{64}{3}\pi$$

(b) Now $V = \int_0^4 \pi y^2\,dx = \pi \int_0^4 x\,dx = \left[\tfrac{1}{2}\pi x^2\right]_0^4 = 8\pi$

$\therefore \quad 8\pi\bar{x} = \tfrac{64}{3}\pi$

$\Rightarrow \quad \bar{x} = \tfrac{8}{3}$

The centroid is on the axis of rotation, i.e. $\bar{y} = 0$

Therefore the centroid is the point $(\tfrac{8}{3}, 0)$
and the first moment of volume about Oy is $\tfrac{64}{3}\pi$

EXERCISE 33e

1. The boundaries of a region of the xy plane are the x-axis, the lines $x = 3$ and $x = -3$ and the curve $y = x^2 + 1$. Find
   (a) the area of the region
   (b) the first moment of the area about the x-axis
   (c) the y-coordinate of the centroid of the region.

2. A region in the xy plane is defined by the inequalities $x \leqslant 4$, $0 \leqslant y \leqslant x$. Sketch the region and write down its area.
   Find the first moment of the area about (a) the x-axis (b) the y-axis.
   Determine the coordinates of the centroid of the region.

3. A region of the xy plane is defined by the inequalities
   $0 \leqslant y \leqslant \sin x,\ 0 \leqslant x \leqslant \pi$
   Find (a) the area of the region
   (b) the first moment of this area about the x-axis
   (c) the coordinates of the centroid of this area.
   Find also
   (d) the volume obtained when this area is rotated completely about the x-axis
   (e) the first moment of this volume about the y-axis
   (f) the centroid of this volume.

4. Repeat Question 1 for the region of the xy plane defined by the inequalities $0 \geqslant y \geqslant x(x-1)$

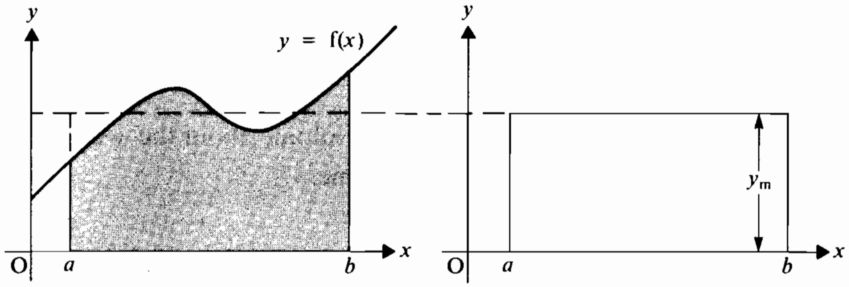
5. A region of the xy plane is bounded by the curve $y = x^2$ and the line $y = 4$
   Find: (a) the area of this region
   (b) the first moment of this area about the x-axis
   (c) the y coordinate of the centroid of the area.
   Find also:
   (d) the volume obtained when this area is rotated about the y-axis
   (e) the first moment of the volume about the x-axis
   (f) the centroid of this volume.

6. Repeat Question 5 for the area bounded by $y = \ln x$, $x = 0$, $y = 0$ and $y = 1$

7. (a) Find the x coordinate of the centroid of the area bounded by
   $$y = e^x,\ y = 0,\ x = 0,\ x = 2$$
   (b) This area is rotated about the x-axis to give a volume of revolution.
   Find (i) the volume generated
   (ii) the coordinates of the centroid of this volume.

THE MEAN VALUE OF A FUNCTION

Consider the curve $y = f(x)$ for values of x in the interval $[a, b]$. The mean value of y in this interval is the arithmetic average value of y and is denoted by y_m or $\{f(x)\}_m$



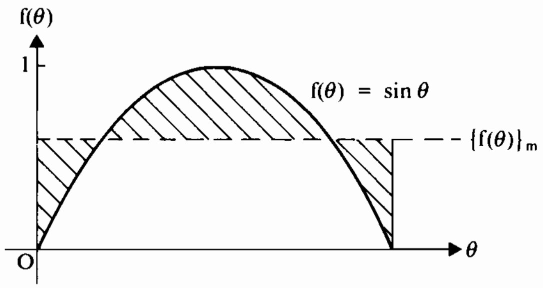
On the curve $y = f(x)$, some of the ordinates are above the line $y = y_m$ and some ordinates are below this line. As y_m is the *average* ordinate, it follows that these two cancel, i.e. the area between the line and the curve that is above $y = y_m$ is equal to the area between the line and the curve that is below the line.
Hence, the area of the rectangle in the diagram on the right is equal to the shaded area in the left-hand diagram.

i.e. $$(b - a)y_m = \int_a^b y\,dx$$

Therefore $$y_m = \frac{1}{b-a}\int f(x)\,dx$$

Examples 33f

1. Find the mean value of $\sin\theta$ within the interval $0 \leqslant \theta \leqslant \pi$



In general the mean value of $f(x)$ is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

So, for $f(\theta) = \sin\theta$ in the interval $0 \leqslant \theta \leqslant \pi$,

$$\{f(\theta)\}_m = \frac{1}{\pi - 0}\int_0^\pi \sin\theta\,d\theta$$

$$= \frac{1}{\pi}\Big[-\cos\theta\Big]_0^\pi$$

$$= \frac{2}{\pi}$$

Note that, to 2 s.f., $\dfrac{2}{\pi} = 0.64$ and that, on the graph, the line $f(\theta) = 0.64$ looks a reasonable average ordinate.

---

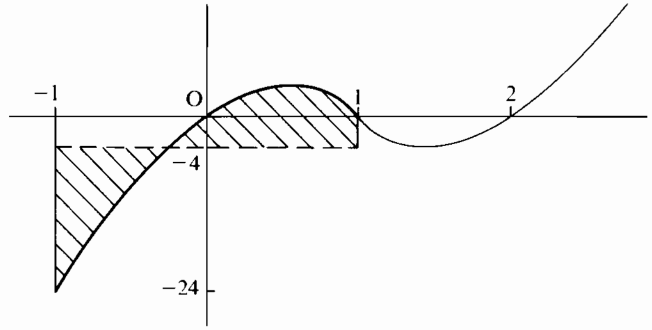
2. If $-1 \leqslant x \leqslant 1$, find the mean value of y given that $y = 4x(x-1)(x-2)$

Using $y_m = \dfrac{1}{b-a}\displaystyle\int_a^b y\,dx$ for $y = 4x(x-1)(x-2)$ in the interval $[-1, 1]$ gives

$$y_m = \frac{1}{1-(-1)}\int_{-1}^{1} 4x(x-1)(x-2)\,dx$$

$$= \tfrac{1}{2}\int_{-1}^{1}(4x^3 - 12x^2 + 8x)\,dx$$

i.e. $$y_m = \tfrac{1}{2}\Big[x^4 - 4x^3 + 4x^2\Big]_{-1}^{1} = \tfrac{1}{2}(1 - 9) = -4$$



MIXED EXERCISE 33

1. The acceleration at any time, t, of a particle moving in a straight line, is $(4t - 3)$ m/s^2. Initially the particle is at a fixed point O on the line and has a velocity of 1 m/s. Find the times when the velocity of the particle is zero. Find also the displacement of the particle from O when $t = 3$

2. A particle, P, moves in a straight line with an acceleration of $1/t^3$ m/s^2 after t seconds. When $t = 1$, P is at rest at a point O on the line of motion. Find expressions for its velocity and its displacement from O at time t and when $t = 2$

3. The velocity at time, t, of a particle P travelling in a straight line, is v m/s where $v = 2 + 1/t^2$. What is the acceleration of P when $t = 1$? Find the distance travelled by P during the time interval between $t = 2$ and $t = 6$

4. Find the area between the curves $y = x^2$ and $y^2 = x$

5. A region of the xy plane is defined by the inequalities $0 \leqslant x \leqslant 4$ and $0 \leqslant y \leqslant e^x$. Find
   (a) the area of the region
   (b) the first moment of the region about Ox
   (c) the y-coordinate of the centroid of the region.

6. Find the mean value of $2x\exp(x^2)$ for the interval $[0, 2]$.

7. Find the area of the region in the first quadrant bounded by the y-axis, the line $y = 6$ and the curve $y = x^2 + 2$
   If this area is rotated completely about Oy to form a solid, find
   (a) the volume of the solid
   (b) the coordinates of the centre of mass of the solid.

8. A plane region is bounded by the curve $y = 6 - x^2$ and the line $y = 2$. Find
   (a) the area of the region
   (b) the first moment of the region about the x-axis
   (c) the coordinates of the centroid of the region.

9. If the part of the area given in Question 8 which is in the first quadrant is rotated through one revolution about the line $y = 2$ find the volume generated.
   Find also the first moment of this volume about the y-axis.
   State the y-coordinate of the centre of mass of the solid and find the corresponding x-coordinate.

10. Find the mean value of the function $x\cos x$ for $0 \leqslant x \leqslant \frac{1}{3}\pi$

CONSOLIDATION E

SUMMARY

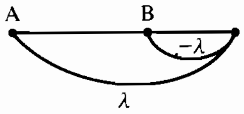
COORDINATE GEOMETRY

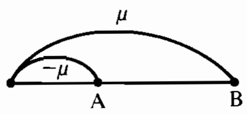
Straight Lines

For two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, the point which divides AB in the ratio $\lambda : \mu$ has coordinates

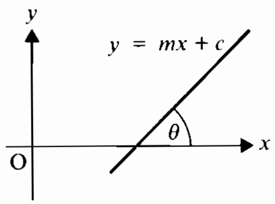
$$\left(\frac{\lambda x_2 + \mu x_1}{\lambda + \mu}, \frac{\lambda y_2 + \mu y_1}{\lambda + \mu}\right)$$

For internal division, λ and μ are both positive while, for external division, the smaller of λ and μ is negative, i.e. the ratio is negative.

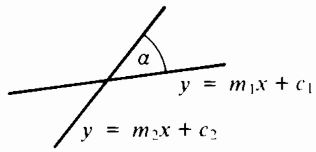




When a line with gradient m makes an angle θ with the positive x-axis, then $m = \tan\theta$



The acute angle α between two lines with gradients m_1 and m_2 is given by $\tan\alpha = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$



The distance from the point (p, q) to the line with equation $ax + by + c = 0$ is given by $\left|\dfrac{ap + bq + c}{\sqrt{(a^2 + b^2)}}\right|$

Reduction of Relationships to Linear Form

When a non-linear law, containing two unknown constants, connects two variables, the relationship can often be reduced to linear form. The aim is to produce an equation in which one term is constant and another term does not contain a constant. The law can then be expressed in the form

$$Y = mX + C$$

Some common conversions are

$p = a\sqrt{q} + b$; use $Y = p$ and $X = \sqrt{q}$

$p = aq^b$; take logs: $\ln p = \ln a + b \ln q$; use $Y = \ln p$ and $X = \ln q$

$\frac{a}{p} + \frac{b}{q} = c$; $\Rightarrow$ $\frac{1}{p} = \frac{c}{a} - \frac{b}{a}\frac{1}{q}$; use $Y = \frac{1}{p}$ and $X = \frac{1}{q}$

Circles

The equation of a circle with centre (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$

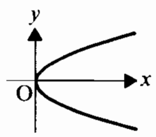
The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle
with centre $(-g, -f)$ and radius $\sqrt{(g^2 + f^2 - c)}$
provided that $g^2 + f^2 - c > 0$

Two circles with radii r_1 and r_2 touch if the distance between their centres is equal to $|r_1 \pm r_2|$

Conic Sections

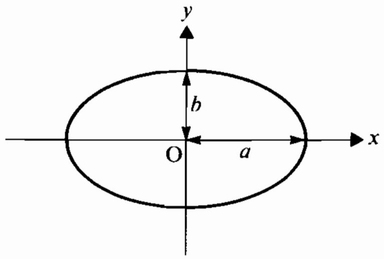
The cartesian equation of this parabola is $y^2 = 4ax$

The corresponding parametric equations are
$x = at^2$, $y = 2at$



The cartesian equation of an ellipse with its centre at O is

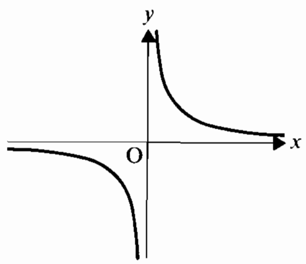
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The corresponding parametric equations are $x = a\cos\theta$, $y = b\sin\theta$

The cartesian equation of this rectangular hyperbola is $xy = c^2$

The parametric equations are
$x = ct$, $y = c/t$



ALGEBRA

Quadratic Equations

If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β

then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Any quadratic equation is of the form

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Partial Fractions

A proper fraction with a denominator which factorises can be expressed in partial fractions as follows:

The numerators which can be found by the cover-up method are screened.

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

$$\frac{f(x)}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2}$$

$$\frac{f(x)}{(x-a)(x^2+b)} = \frac{A}{(x-a)} + \frac{Bx + C}{(x^2+b)}$$

The Remainder Theorem

When a polynomial, $f(x)$, is divided by $(x - a)$ the remainder is equal to $f(a)$

The Factor Theorem

If $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$

INTEGRATION

Standard Integrals

Function	Integral
x^n	$\frac{1}{n+1}x^{n+1} \quad (n \neq -1)$
e^x	e^x
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\frac{1}{\sqrt{(1-x^2)}}$	$\arcsin x$
$\frac{1}{1+x^2}$	$\arctan x$

The following methods are general guide lines; alternative approaches can be better in individual cases.

Integrating products can be done by

a) recognition:

in particular $\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + K$

$$\int \sin^p x \cos x\,dx = \frac{1}{p+1}\sin^{p+1}x + K \qquad (p \neq -1)$$

$$\int \cos^p x \sin x\,dx = -\frac{1}{p+1}\cos^{p+1}x + K \qquad (p \neq -1)$$

$$\int \tan^p x \sec^2 x\,dx = \frac{1}{p+1}\tan^{p+1}x + K \qquad (p \neq -1)$$

b) change of variable: suitable for the type $f'(x)g\{f(x)\}$

c) by parts: $\int v\frac{du}{dx}dx = uv - \int u\frac{dv}{dx}dx$

Integration by parts can be used also to integrate $\ln x$ and inverse trig functions.

Integrating fractions can be done by

a) recognition: in particular $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + K$

b) change of variable: suitable for the type $\frac{f'(x)}{g\{f(x)\}}$

c) using partial fractions.

Integration as a Process of Summation

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x = \int_a^b f(x)\,dx$$

DIFFERENTIAL EQUATIONS

A first order linear differential equation is a relationship between x, y and dy/dx. It can be solved by collecting all the x terms, along with dx, on one side, with all y terms and dy on the other side. Then each side is integrated with respect to its own variable. A constant of integration called an arbitrary constant is introduced on one side only to give a general solution which is a family of lines or curves.
If extra information provides the value of this constant we have a particular solution, i.e. one member of the family.

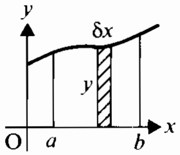
PRACTICAL APPLICATIONS OF INTEGRATION

Area

The area bounded by the x-axis, the two lines $x = a$ and $x = b$, and part of the curve $y = f(x)$ can be found by summing the areas of vertical strips of width δx and using

$$\text{Area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$$

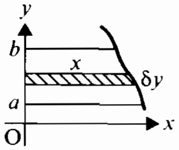
(i)



Similarly, for horizontal strips.

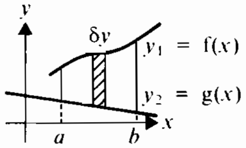
$$\text{Area} = \lim_{\delta y \to 0} \sum_{y=a}^{y=b} x\,\delta y = \int_a^b x\,dy$$

(ii)



For compound areas the length of a strip is usually a difference of two quantities.

(iii)



The area shown is given by

$$\text{Area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} (y_1 - y_2)\,\delta x = \int_a^b (y_1 - y_2)\,dx$$

An approximate value for the area shown in diagram (i) above can be found by taking strips of equal width, d, and using:

a) the Trapezium Rule

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}d[y_0 + 2y_1 + \ldots + 2y_{n-2} + y_{n-1}]$$

b) Simpson's Rule (in this case there must be an even number of strips)

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}d[(\text{1st} + \text{last}) + 4(\text{2nd} + \text{4th} + \ldots) + 2(\text{3rd} + \text{5th} + \ldots)]$$

Volume

When the area bounded by the x-axis, the ordinates at a and b, and part of the curve $y = f(x)$ rotates completely about the x-axis, the volume generated is given by

$$\text{Volume} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\,\delta x = \int_a^b \pi y^2\,dx$$

For rotation about the y-axis, $\text{volume} = \lim_{\delta y \to 0} \sum_{y=a}^{y=b} \pi x^2\,\delta y = \int_a^b \pi x^2\,dy$

Mean Value

The mean value, y_m, of a function $f(x)$ from $x = a$ to $x = b$ is given by

$$y_m = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

MULTIPLE CHOICE EXERCISE E

TYPE I

1. The equation of the line through the origin and perpendicular to $3x - 2y + 4 = 0$ is

A $3x + 2y = 0$ **C** $2x + 3y = 0$ **E** $3x - 2y = 0$
B $2x + 3y + 1 = 0$ **D** $2x - 3y - 1 = 0$

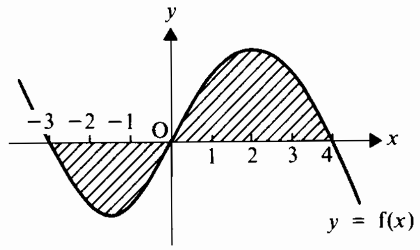
2. A point P moves so that is equidistant from A and B. The locus of the set of points P is

A a circle on AB as diameter
B a line parallel to AB
C the perpendicular bisector of AB
D a parabola

3. The point dividing A(1, 2) and B(7, −4) in the ratio 1:2 has coordinates

A $(3, -2)$ **C** $(\frac{8}{3}, -\frac{4}{3})$ **E** $(13, -10)$
B $(5, -2)$ **D** $(3, 0)$

4.



The shaded area in the diagram is given by

A $\int_{-3}^{4} f(x)\,dx$ **C** $\int_{-3}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$
B $\int_{-3}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$ **D** none of these

5. e^{x^2} could be the integral w.r.t. x of

A e^{2x} **C** $\dfrac{e^{x^2}}{2x}$ **E** none of these
B $2xe^{x^2}$ **D** $x^2e^{x^2} - 1$

6. If $\int_1^5 \dfrac{dx}{2x-1} = \ln K$, the value of K is

A 9 **B** 3 **C** undefined **D** 81 **E** 8

7. $I = \int_1^2 x\sqrt{(x^2-1)}\,dx$ is found as follows. Where does an error first occur?

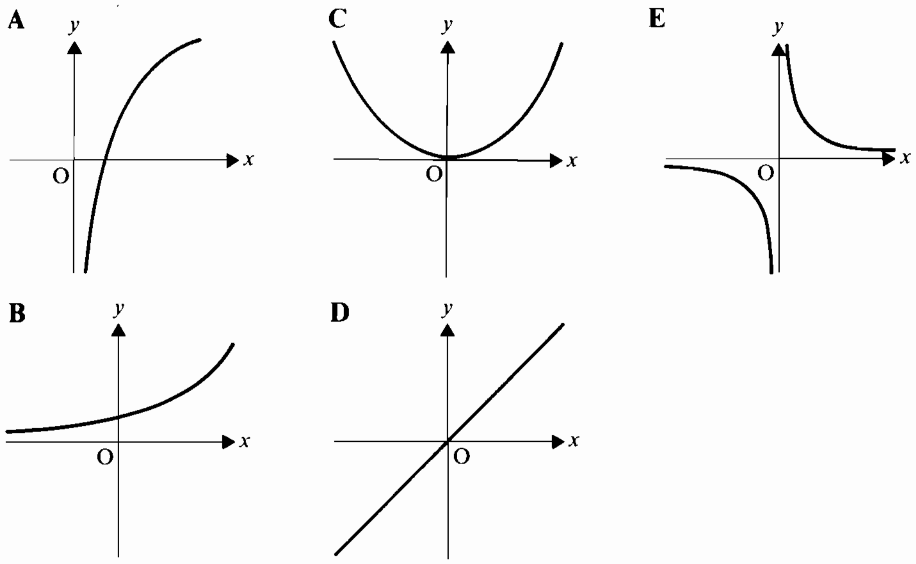
A Let $u \equiv x^2 - 1$ C $I = \frac{1}{2}\int_1^2 u^{1/2}\,du$

B $\ldots du \equiv \ldots 2x\,dx$ D $I = \frac{3}{4}\left[u^{3/2}\right]_1^2$

8. $\int_0^{\pi/6} \sin^n x \cos x\,dx = \frac{1}{64}$; n is

A 6 B 5 C 4 D 3 E none of these

9. The differential equation of a curve is $\frac{x}{y}\frac{dy}{dx} = 1$. The sketch of the curve could be



10. The value of $\int_0^2 2e^{2x}\,dx$ is

A e^4 B $e^4 - 1$ C ∞ D $4e^4$ E $\frac{1}{2}e^4$

11. $x^3 - 3x^2 + 2x - 6$ has a factor

A $x-3$ C $x-4$ E $x+2$
B $x-2$ D $x+3$

12. $x^3 - 3x^2 + 6x - 2$ has remainder 2 when divided by

A $x-1$ C x E $2x-1$
B $x+1$ D $x+2$

13.

x	1	2	3
y	0	3	6

Given the information in the table, using the trapezium rule with 3 ordinates to find $\int_1^3 y\,dx$ gives

A 12 B 9/2 C 9 D 6 E 8

14. The value of k for which $x-1$ a factor of $4x^3 - 3x^2 - kx + 2$ is

A -1 B 0 C 1 D 2 E 3

TYPE II

15. Which of the following relationships gives a straight line when $\ln y$ is plotted against $\ln x$, assuming that a and b are constants.

A $ay^3 = bx^2$ C $y = ax^b$
B $y = a + b^x$ D $y^a = b^x$

16. If the equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ represents a circle through the origin,

A $g = 0$ and $f = 0$ C $a = b$
B $c = 0$ D $a = -b$

17. $f(x) \equiv (3 - 5x)^4$

A $f(x)$ has a remainder 16 when divided by $x - 1$
B the expansion of $f(x)$ contains four terms
C the equation $f(x) = 0$ is satisfied by only one value of x

18. $f(x) \equiv 2x^2 + 3x - 2$

A $f(x)$ can be expressed as the sum of two partial fractions,
B the equation $f(x) = 0$ has two real distinct roots
C $x + 2$ is a factor of $f(x)$.

19. Using $x = \sin\theta$ transforms $\int \frac{x^2}{\sqrt{(1-x^2)}}\,dx$ into

A $\int \frac{\sin^2\theta}{\cos\theta}\,d\theta$ C $-\int \sin^2\theta\,d\theta$

B $\frac{1}{2}\int (1-\cos 2\theta)\,d\theta$ D $\frac{1}{2}\int (1+\cos 2\theta)\,d\theta$

20. Integration by parts can be used to find

A $\int x^2 e^x\,dx$ C $\int \ln x\,dx$

B $\int e^x \ln x\,dx$ D $\int (\ln x)(\sin x)\,dx$

21. Which of the following definite integrals can be evaluated?

A $\int_0^1 \frac{1}{x-1}\,dx$ C $\int_1^2 \sqrt{(1-x^2)}\,dx$

B $\int_0^{\pi/2} \sin x\,dx$ D $\int_{-2}^1 \ln x\,dx$

22. Which of the following differential equations can be solved by separating the variables?

A $x\frac{dy}{dx} = y + x$ C $e^{x+y} = y\frac{dy}{dx}$

B $xy\frac{dy}{dx} = x + 1$ D $x + \frac{dy}{dx} = \ln y$

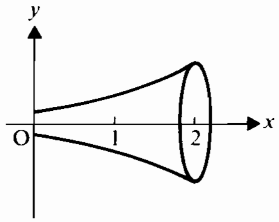
23. $\int_1^2 xe^x\,dx$

A is a definite integral C is equal to $\left[\frac{1}{2}e^{x^2}\right]_1^2$

B is equal to $xe^x - e^x$ D can be integrated by parts.

24. The centroid of this volume of revolution is

A on the x-axis
B on the y-axis
C at the point $(1, 0)$
D right of the point $(1, 0)$



TYPE III

25. $\sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$

26. If $x - a$ is a factor of $x^2 + px + q$, the equation $x^2 + px + q = 0$ has a root equal to a

27. $\int \tan x\,dx = \sec^2 x + K$

28. $\int_0^a f(y)\,dy = \lim_{\delta y \to 0} \sum_{y=0}^{y=a} f(y)\,\delta y$

29. $\left[f(x)\right]_0^a = f(a) - 0$

30. A differential equation must contain $\frac{dy}{dx}$

31. $x^2 + y^2 - 2x - 4y + 6 = 0$ is the equation of a circle.

32. The area between the curve $y = 1 - x^2$ and the x-axis is given by $\int_{-1}^1 y\,dx$

MISCELLANEOUS EXERCISE E

1. Given that $(x + 2)$ is a factor of $x^4 + kx^2 + 4x + 1$, find the value of k.

2. Express $\frac{13x + 16}{(x-3)(3x+2)}$ in partial fractions.

   Hence find the value of $\frac{d}{dx}\left[\frac{13x+16}{(x-3)(3x+2)}\right]$ when $x = 2$

   (AEB 86)

3. Given that f(x), where $f(x) \equiv x^2 + ax + 3$ and a is a constant, is such that the remainder on dividing f(x) by $x - 1$ is three times the remainder on dividing f(x) by $x + 1$, find the value of a (AEB 86)$_p$

4. Given that $f(x) \equiv x^2 + px + q$ determine the values of the constants p and q so that both

(a) $f(x)$ has a turning point when $x = -3$ and

(b) the remainder when $f(x)$ is divided by $x + 2$ is 2

Show that, with these values of p and q, $f(x) \geqslant 1$ (U of L 88)

5. (a) Write down the coordinates of the mid-point M of the line joining $A(0, 1)$ and $B(6, 5)$.

(b) Show that the line $3x + 2y - 15 = 0$ passes through M and is perpendicular to AB.

(c) Calculate the coordinates of the centre of the circle which passes through A, B and the origin O. (U of L 87)

6. Given that $$y = \frac{3x - 14}{(x - 2)(x + 6)}$$

express y as a sum of partial fractions. Hence find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Show that $\frac{dy}{dx} = 0$ for $x = 10$ and for one other value of x

Find the maximum and minimum values of y, distinguishing between them. (JMB 86)

7. The function

$$f: x \to 2x^3 + ax^2 + bx + 36 \quad x \in \mathbb{R}$$

is such that $f(3) = 0$ and the remainder when $f(x)$ is divided by $(x + 2)$ is -30. Find the values of a and b and express $f(x)$ as the product of three linear factors. (AEB 86)

8. A circle S is given by the equation

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

Find the radius of S and the coordinates of the centre of S. Calculate the length of the perpendicular from the centre of S to the line L whose equation is

$$3x + 4y = k$$

where k is a constant. Deduce the values of k for which L is a tangent to S. (JMB 87)

9. The circle with equation $(x - 5)^2 + (y - 7)^2 = 25$ has centre C. The point P(2, 3) lies on the circle. Determine the gradient of PC and hence, or otherwise, obtain the equation of the tangent to the circle at P.

Find also the equation of the straight line which passes through the point C and the point Q(−1, 4). The tangent and the line CQ intersect at R.

Determine the size of angle PRC, to the nearest 0.1° (AEB 87)

10. The straight lines $3x + 4y = 14$ and $x - 7y = 13$ intersect at the point A, and the line $4x - 3y + 23 = 0$ cuts the other two lines at the points B and C respectively.

(a) Show that the length of BC is 10 units.

(b) Calculate the value of the acute angle BAC.

(c) Find the coordinates of A and calculate the perpendicular distance from A to BC. Hence find the equation of the circle with centre A for which BC is a tangent. (AEB 86)

11. The following measurements of the volume, $V\,\text{cm}^3$, and the pressure, p cm of mercury, of a given mass of gas were taken.

V	10	50	110	170	230
p	1412.5	151.4	50.3	27.4	18.6

By plotting values of $\log_{10} p$ against $\log_{10} V$, verify graphically the relationship $p = kV^n$ where k and n are constants.

Use your graph to find approximate values for k and n, giving your answers to two significant figures. (AEB 1987)$_p$

12.

x	2.1	2.8	4.7	6.2	7.3
y	13	32	316	2000	7080

The above table shows corresponding values of variables x and y obtained experimentally. By drawing a suitable graph, show that these values support the hypothesis that x and y are connected by a relationship of the form $y = a^x$, where a is a constant. Use your graph to estimate the value of a to 2 significant figures. (U of L 85)

13. The variables x and y are known to satisfy an equation of the form $y = ab^x$, where a and b are constants. For five different values of x, corresponding approximate values of y were obtained experimentally. The results are given in the following table.

x	2.0	2.5	3.0	3.5	4.0
y	11.3	18.0	27.1	44.5	70.4

By drawing a suitable linear graph, estimate the values of a and b, giving both answers to one decimal place. (JMB 87)

14. (a) Find $\int (3x + 4)\,e^{2x}\,dx$

(b) By using the substitution $x = 2\tan\theta$, evaluate

$$\int_0^2 \frac{1}{(4 + x^2)^2}$$ (U of L 85)

15. Given that α and β are the costs of the equation $x^2 - px + 2 = 0$, express $\alpha^2 + \beta^2$ in terms of p

Without solving the given equation, find a quadratic equation whose roots are

$$\alpha^2 + \frac{\alpha}{\beta} \quad \text{and} \quad \beta^2 + \frac{\beta}{\alpha}$$

giving the coefficients in a simplified form not involving α and β. (JMB 84)

16. Find the following integrals

(a) $\int \cos^2(3x)\,dx$

(b) $\int \frac{1}{e^x + 4e^{-x}}\,dx$, by means of the substitution $u = e^x$, or otherwise

(c) $\int x e^{2x}\,dx$ (C 86)

17. Given that $\frac{x}{(1 + x)^2} = \frac{A}{(1 + x)^2} + \frac{B}{1 + x}$, find the values of the constants A and B.

Hence, or otherwise, evaluate

$$\int_0^1 \frac{x}{(1 + x)^2}\,dx$$ (U of L 85)

18. By means of the substitution $u = e^x$, or otherwise, evaluate

$$\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$$

giving the answer correct to two decimal places. (JMB 87)

19. By using a substitution, or otherwise, find the exact value of

$$\int_0^{\pi/2} \frac{\cos x}{(4 + \sin x)^2}$$ (JMB 84)

20. Given that $\frac{7x - x^2}{(2 - x)(x^2 + 1)} \equiv \frac{A}{(2 - x)} + \frac{Bx + C}{(x^2 + 1)}$, determine the values of A, B and C.

A curve has equation $y = \frac{7x - x^2}{(2 - x)(x^2 + 1)}$.

Determine the equation of the normal to the curve at the point $(1, 3)$.

Prove that the area of the region bounded by the curve, the x-axis and the line $x = 1$ is $\frac{7}{2}\ln 2 - \frac{\pi}{4}$. (AEB 87)

21. Using the same axes, sketch the curves with equations

$$y^2 = 4x \quad \text{and} \quad x^2 = 4y$$

Verify that the curves intersect at the two points $(0, 0)$ and $(4, 4)$.

Hence find the area of the finite region enclosed by the curves. (U of L 88)

22. (a) The finite region bounded by the x-axis, the curve $y = e^{-x}$ and the lines $x = \pm a$ is denoted by R Find, in terms of a, the volume of the solid generated when R is rotated through one revolution about the x-axis.

(b) (i) By using a suitable substitution, or otherwise, evaluate

$$\int_0^1 x(1 - x)^9\,dx$$

(ii) Find $\int 2x\tan^{-1}x\,dx$ (C 87)

23. Given that $x < 4$, find

$$\int \frac{8}{(4-x)(8-x)}\,dx$$

A chemical reaction takes place in a solution containing a substance S. At noon there are two grams of S in the solution and t hours later there are x grams of S. The rate of the reaction is such that x satisfies the differential equation

$$8\frac{dx}{dt} = (4-x)(8-x)$$

Solve this equation, giving t in terms of x.

Find, to the nearest minute, the time at which there are three grams of S present. (JMB 87)

24. Express $\dfrac{1}{(1+x)(3+x)}$ in partial fractions.

Hence find the solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{(1+x)(3+x)} \quad x > -1$$

given that $y = 2$ when $x = 1$.

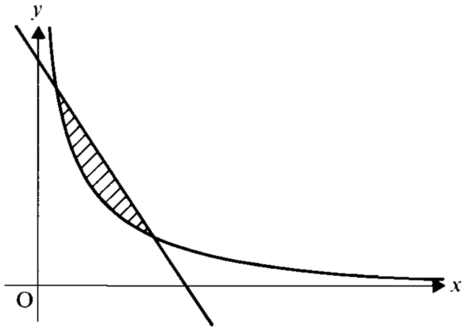
Express your answer in the form $y = f(x)$. (U of L 88)

25. Find the solution of the differential equation

$$(x^2-5)^{1/2}\frac{dy}{dx} = 2xy^{1/2}, \quad x^2 > 5$$

for which $y = 4$ when $x = 3$, expressing y in terms of x (JMB 84)

26.



The diagram shows a sketch of the curve $xy = 6$ and part of the line $y = 9 - 3x$. Use integration to find the area of the shaded region. (C 86)

27. The quadratic function $q(x)$ is given by

$$q(x) = x^2 + 2kx + k + 2, \quad x \in \mathbb{R}$$

where k is a constant. Given that the roots of the equation $q(x) = 0$ are α and β, show that

$$(\alpha - \beta)^2 = 4(k^2 - k - 2).$$

(a) Find the values of k for which the roots of the equation $q(x) = 0$ differ by 4.

(b) Given that $k \neq -2$, form a quadratic equation, with coefficients in terms of k, whose roots are

$$(1 + \alpha/\beta) \quad \text{and} \quad (1 + \beta/\alpha)$$ (U of L 87)

28. In a model to estimate the depreciation of the value of a car, it is assumed that the value, £V, at age t months, decreases at a rate which is proportional to V. Using this model, write down a differential equation relating V and t. Given that the car has an initial value of £6000, solve the differential equation and show that $V = 6000e^{-kt}$ where k is a positive constant.

The value of the car is expected to decrease to £3000 after 36 months. Calculate

(a) the value, to the nearest pound, of the car when it is 15 months old

(b) the age of the car, to the nearest month, when its value is £2000 (JMB 86)

29. The gradient of a curve at any point (x, y) on the curve is directly proportional to the product of x and y. The curve passes through the point $(1, 1)$ and at this point the gradient of the curve is 4. Form a differential equation in x and y and solve this equation to express y in terms of x. (U of L 85)

30. Use the trapezium rule, with ordinates at $x = 1$, $x = 2$ and $x = 3$, to estimate the value of

$$\int_1^3 \sqrt{(40 - x^3)}\,dx$$ (C 87)

31. By using a substitution, or otherwise, find the exact value of

$$\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{(1-x^4)}}\,dx.$$

State the mean value of $\dfrac{x}{\sqrt{(1-x^4)}}$ as x varies from 0 to $\dfrac{1}{\sqrt{2}}$. (JMB 86)

32. Shade on a sketch the finite region R in the first quadrant bounded by the x-axis, the curve $y = \ln x$ and the line $x = 5$
By means of integration, calculate the area of R.

The region R is rotated completely about the x-axis to form a solid of revolution S.

x	1	2	3	4	5
$(\ln x)^2$	0	0.480	1.207	1.922	2.590

Use the given table of values and apply the trapezium rule to find an estimate of the volume of S, giving your answer to one decimal place. (AEB 87)

33. Show that

(a) $\displaystyle\int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$

(b) $\displaystyle\int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1$

Sketch and label the region R defined by

$$x \geqslant 0 \quad y \geqslant 0 \quad y \leqslant \cos x \quad x \leqslant \frac{\pi}{2}$$

Find in terms of π,

(c) the x-coordinate of the centroid of the region R

(d) the y-coordinate of the centroid of the region R

Find also, in terms of π, the volume obtained when the region R is revolved through 2π about the x-axis.

34. (a) Find $\displaystyle\int x \ln x \, dx$.

(b) By means of the substitution $t = \tan x$, or otherwise, find

$$\int \frac{1}{1 + \cos^2 x} \, dx$$

(c) The region bounded by the curve $y = (1 + \cos x)^{-1/2}$, the x-axis, and the lines $x = 0$ and $x = \frac{1}{2}\pi$, is denoted by R. Use the trapezium rule with ordinates at $x = 0$, $x = \frac{1}{4}\pi$ and $x = \frac{1}{2}\pi$ to estimate the area of R, giving two significant figures in your answer. (C 88)

35. The roots of the equation $x^2 - 2px + q = 0$ are α and β.

(a) Find, in terms of p and q, an expression for $\alpha^2 + \beta^2$.

(b) Show that $\alpha^3 - 2p\alpha^2 + q\alpha = 0$ and $\beta^3 - 2p\beta^2 + q\beta = 0$.

(c) *Hence* show that $\alpha^3 + \beta^3 = 8p^3 - 6pq$.

CHAPTER 34

NUMBER SERIES

SEQUENCES

Consider the following sets of numbers,

$$2, 4, 6, 8, 10, \ldots$$
$$1, 2, 4, 8, 16, \ldots$$
$$4, 9, 16, 25, 36, \ldots$$

Each set of numbers, in the order given, has a pattern and there is an obvious rule for obtaining the next number and as many subsequent numbers as we wish to find.
Such sets are called *sequences* and each member of the set is a term of the sequence.

SERIES

When the terms of a sequence are added, a series is formed,
e.g., $1 + 2 + 4 + 8 + 16 + \ldots$ is a series.

If the series stops after a finite number of terms it is called a finite series,
e.g., $1 + 2 + 4 + 8 + 16 + 32 + 64$ is a finite series of seven terms.

If the series continues indefinitely it is called an infinite series,
e.g., $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots + \frac{1}{1024} + \ldots$ is an infinite series.

Consider again the series $1 + 2 + 4 + 8 + 16 + 32 + 64$

As each term is a power of 2 we can write this series in the form

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$$

All the terms of this series are of the form 2^r, so 2^r is a general term.
We can then define the series as the sum of terms of the form 2^r where r takes all integral values in order from 0 to 6 inclusive.

Using Σ as a symbol for 'the sum of terms such as' we can redefine our series more concisely as $\Sigma 2^r$, r taking all integral values from 0 to 6 inclusive, or, even more briefly,

$$\sum_{r=0}^{6} 2^r$$

Placing the lowest and highest value that r takes below and above the sigma symbol respectively, indicates that r takes all integral values between these extreme values.

Thus $\sum_{r=2}^{10} r^3$ means 'the sum of all terms of the form r^3 where r takes all integral values from 2 to 10 inclusive',

i.e. $$\sum_{r=2}^{10} r^3 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3$$

Note that a finite series, when written out, should always end with the last term even if intermediate terms are omitted, e.g. $3 + 6 + 9 + \ldots + 99$

The infinite series $$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots$$

may also be written in the sigma notation. The continuing dots after the last written term indicate that the series is infinite, i.e. there is *no* last term. Each term of this series is a power of $\frac{1}{2}$ so a general term can be written $(\frac{1}{2})^r$. The first term is 1 or $(\frac{1}{2})^0$, so the first value that r takes is zero. There is no last term of this series, so there is no upper limit for the value of r.

Therefore $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ may be written as $\sum_{r=0}^{\infty} (\frac{1}{2})^r$

Note that when a given series is rewritten in the sigma notation it is as well to check that the first few values of r give the correct first few terms of the series.

Writing a series in the sigma notation, apart from the obvious advantage of brevity, allows us to select a particular term of a series without having to write down all the earlier terms.

For example, in the series $\sum_{r=3}^{10} (2r + 5)$,

the first term is the value of $2r + 5$ when $r = 3$, i.e. $2 \times 3 + 5 = 11$
the last term is the value of $2r + 5$ when $r = 10$, i.e. 25
the fourth term is the value of $2r + 5$ when r takes its fourth value in order from $r = 3$, i.e. when $r = 6$

Thus the fourth term of $\sum_{r=3}^{10} (2r + 5)$ is $2 \times 6 + 5 = 17$

Example 34a

Write the following series in the sigma notation,

(a) $1 - x + x^2 - x^3 + \ldots$ (b) $2 - 4 + 8 - 16 + \ldots + 128$

(a) A general term of this series is $\pm x^r$, having a positive sign when r is even and a negative sign when r is odd.
Because $(-1)^r$ is positive when r is even and negative when r is odd, the general term can be written $(-1)^r x^r$

The first term of this series is 1, or x^0

Hence $1 - x + x^2 - x^3 + \ldots = \sum_{r=0}^{\infty} (-1)^r x^r$

(b) $2 - 4 + 8 - 16 + \ldots + 128 = 2 - (2)^2 + (2)^3 - (2)^4 + \ldots + (2)^7$

So a general term is of the form $\pm 2^r$, being positive when r is odd and negative when r is even,
i.e. the general term is $(-1)^{r+1} 2^r$

Hence $2 - 4 + 8 - 16 + \ldots + 128 = \sum_{r=1}^{7} (-1)^{r+1} 2^r$

EXERCISE 34a

1. Write the following series in the sigma notation:

(a) $1 + 8 + 27 + 64 + 125$

(b) $2 + 4 + 6 + 8 + \ldots + 20$

(c) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{50}$

(d) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(e) $-4 - 1 + 2 + 5 \ldots + 17$

(f) $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$

2. Write down the first three terms and, where there is one, the last term of each of the following series:

(a) $\sum_{r=1}^{\infty} \frac{1}{r}$ (b) $\sum_{r=0}^{5} r(r+1)$

(c) $\sum_{r=0}^{20} \frac{r+2}{(r+1)(2r+1)}$ (d) $\sum_{r=0}^{\infty} \frac{1}{(r^2+1)}$

(e) $\sum_{r=-1}^{8} r(r+1)(r+2)$ (f) $\sum_{r=0}^{\infty} a^r(-1)^{r+1}$

3. For the following series, write down the term indicated, and the number of terms in the series.

(a) $\sum_{r=1}^{9} 2^r$, 3rd term (b) $\sum_{r=-1}^{8} (2r+3)$, 5th term

(c) $\sum_{r=-6}^{-1} \frac{1}{(2r+1)}$, last term (d) $\sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)}$, 20th term

(e) $\sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^r$, nth term

(f) $8 + 4 + 0 - 4 - 8 - 12 \ldots - 80$ 15th term

(g) $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots + 32$, 7th term

ARITHMETIC PROGRESSION

Consider the sequence 5, 8, 11, 14, 17, ..., 29
Each term of this sequence exceeds the previous term by 3, so the sequence can be written in the form

$$5,\ (5+3),\ (5+2\times 3),\ (5+3\times 3),\ (5+4\times 3),\ \ldots,\ (5+8\times 3)$$

This sequence is an example of an arithmetic progression (AP) which is a sequence where any term differs from the preceding term by a constant, called the *common difference*.
The common difference may be positive or negative. For example, the first six terms of an AP whose first term is 8 and whose common difference is -3, are $8, 5, 2, -1, -4, -7$

In general, if an AP has a first term a, and a common difference d, the first four terms are a, $(a+d)$, $(a+2d)$, $(a+3d)$, and the nth term, u_n, is $a+(n-1)d$

Thus **an AP with n terms can be written as**

$$a,\ (a+d),\ (a+2d),\ \ldots,\ [a+(n-1)d]$$

Examples 34b

1. The 8th term of an AP is 11 and the 15th term is 21. Find the common difference, the first term of the series, and the nth term.

If the first term of the series is a and the common difference is d, then the 8th term is $a+7d$,

$\therefore$ $a + 7d = 11$ [1]

and the 15th term is $a + 14d$,

$\therefore$ $a + 14d = 21$ [2]

[2] − [1] gives $7d = 10 \Rightarrow d = \frac{10}{7}$

and $a = 1$

so the first term is 1 and the common difference is $\frac{10}{7}$

Hence the nth term is $a+(n-1)d = 1+(n-1)\frac{10}{7} = \frac{1}{7}(10n-3)$

---

2. The nth term of an AP is $12 - 4n$. Find the first term and the common difference.

If the nth term is $12-4n$, the first term $(n = 1)$ is 8
The second term $(n = 2)$ is 4

Therefore the common difference is -4

The Sum of an Arithmetic Progression

Consider the sum of the first ten even numbers, which is an AP.

Writing it first in normal, then in reverse, order we have

$$\begin{aligned} S &= 2 + 4 + 6 + 8 + \ldots + 18 + 20 \\ S &= 20 + 18 + 16 + 14 + \ldots + 4 + 2 \\ \text{Adding gives} \quad 2S &= 22 + 22 + 22 + 22 + \ldots + 22 + 22 \end{aligned}$$

As there are ten terms in this series, we have

$$2S = 10 \times 22 \quad \Rightarrow \quad S = 110$$

This process is known as finding the sum from first principles. Applying it to a general AP gives formulae for the sum, which may be quoted and used.

If S_n is the sum of the first n terms of an AP with last term l,

$$\begin{aligned} \text{then} \quad S_n &= a + (a+d) + (a+2d) + \ldots + (l-d) + l \\ \text{reversing} \quad S_n &= l + (l-d) + (l-2d) + \ldots + (a+d) + a \\ \text{adding} \quad 2S_n &= (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) \end{aligned}$$

as there are n terms we have $\quad 2S_n = n(a+l)$

$\Rightarrow \quad S_n = \frac{1}{2}n(a+l) \qquad$ i.e. $S_n =$ (number of terms) $\times$ (average term)

Also, because the nth term, l, is equal to $a + (n-1)d$, we have

$$S_n = \tfrac{1}{2}n[a + a + (n-1)d]$$

i.e.

$$S_n = \tfrac{1}{2}n[2a + (n-1)d]$$

Either of these formulae can now be used to find the sum of the first n terms of an AP.

Examples 34b (continued)

3. Find the sum of the following series,

   (a) an AP of eleven terms whose first term is 1 and whose last term is 6

   (b) $\displaystyle\sum_{r=1}^{8}\left(2 - \frac{2r}{3}\right)$

(a) We know the first and last terms, and the number of terms so we use $S_n = \frac{1}{2}n(a+l)$

$$\Rightarrow \quad S_{11} = \tfrac{11}{2}(1+6) = \tfrac{77}{2}$$

(b) $\displaystyle\sum_{r=1}^{8}\left(2 - \frac{2r}{3}\right) = \tfrac{4}{3} + \tfrac{2}{3} + 0 - \tfrac{2}{3} - \ldots - \tfrac{10}{3}$

This is an AP with 8 terms where $a = \frac{4}{3}$, $d = -\frac{2}{3}$

Using $S_n = \frac{1}{2}n[2a + (n-1)d]$ gives

$$S_8 = 4\left[\tfrac{8}{3} + 7\left(-\tfrac{2}{3}\right)\right] = -8$$

4. In an AP the sum of the first ten terms is 50 and the 5th term is three times the 2nd term. Find the first term and the sum of the first 20 terms.

If a is the first term and d is the common difference, and there are n terms, using $S_n = \frac{1}{2}n[2a + (n-1)d]$ gives

$$S_{10} = 50 = 5(2a + 9d) \qquad [1]$$

Now using $u_n = a + (n-1)d$ gives

$$u_5 = a + 4d \quad \text{and} \quad u_2 = a + d$$

Therefore $\quad a + 4d = 3(a+d) \qquad [2]$

From [1] and [2] we get $d = 1$ and $a = \frac{1}{2}$
so the first term is $\frac{1}{2}$ and the sum of the first 20 terms is S_{20} where

$$S_{20} = 10(1 + 19 \times 1) = 200$$

5. Show that the terms of $\sum_{r=1}^{n} \ln 2^r$ are in arithmetic progression.

Find the sum of the first 10 terms of this series.

By taking $r = 1, 2, 3 \ldots$ we have

$$\sum_{r=1}^{n} \ln 2^r = \ln 2 + \ln 2^2 + \ln 2^3 + \ldots + \ln 2^n$$
$$= \ln 2 + 2\ln 2 + 3\ln 2 + \ldots + n\ln 2$$

We now see that there is a common difference of $\ln 2$ between successive terms, so the terms of this series are in arithmetic progression.

Hence $$\sum_{r=1}^{10} \ln 2^r = \ln 2 + 2\ln 2 + 3\ln 2 + \ldots + 10\ln 2$$
$$= (1 + 2 + 3 + \ldots + 10)\ln 2$$
$$= \tfrac{10}{2}(1 + 10)\ln 2 = 55\ln 2$$

Note that the sum of the first n natural numbers.

i.e. $1 + 2 + 3 + \ldots + n$

is an AP in which $a = 1$ and $d = 1$ so

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

This is a result that may be quoted, unless a proof is specifically asked for.

---

6. The sum of the first n terms of a series is given by $S_n = n(n+3)$ Find the fourth term of the series and show that the terms are in arithmetic progression.

If the terms of the series are $a_1, a_2, a_3, \ldots a_n$

then $S_n = a_1 + a_2 + \ldots + a_n = n(n+3)$

So $S_4 = a_1 + a_2 + a_3 + a_4 = 28$

and $S_3 = a_1 + a_2 + a_3 = 18$

Hence the fourth term of the series, a_4, is 10

Now $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n = n(n+3)$

and $S_{n-1} = a_1 + a_2 + \ldots + a_{n-1} = (n-1)(n+2)$

Hence the nth term of the series, a_n, is given by

$$a_n = n(n+3) - (n-1)(n+2) = 2n + 2$$

Replacing n by $n-1$ gives the $(n-1)$th term

i.e. $a_{n-1} = 2(n-1) + 2 = 2n$

Then $a_n - a_{n-1} = (2n+2) - 2n = 2$

i.e. there is a common difference of 2 between successive terms, showing that the series is an AP.

EXERCISE 34b

1. Write down the fifth term and the nth term of the following APs.

(a) $\sum_{r=1}^{n} (2r-1)$ (b) $\sum_{r=1}^{n} 4(r-1)$ (c) $\sum_{r=0}^{n} (3r+3)$

(d) first term 5, common difference 3

(e) first term 6, common difference -2

(f) first term p, common difference q

(g) first term 10, last term 30, 11 terms

(h) $1, 5, \ldots$ (i) $2, 1\tfrac{1}{2}, \ldots$ (j) $-4, -1, \ldots$

2. Find the sum of the first ten terms of each of the series given in Question (1).

3. The 9th term of an AP is 8 and the 4th term is 20. Find the first term and the common difference.

4. The 6th term of an AP is twice the 3rd term and the first term is 3. Find the common difference and the 10th term.

5. The nth term of an AP is $\frac{1}{2}(3-n)$. Write down the first three terms and the 20th term.

6. Find the sum, to the number of terms indicated, of each of the following APs.

(a) $1 + 2\frac{1}{2} + \ldots$, 6 terms

(b) $3 + 5 + \ldots$, 8 terms

(c) the first twenty odd integers

(d) $a_1 + a_2 + a_3 + \ldots + a_8$ where $a_n = 2n + 1$

(e) $4 + 6 + 8 + \ldots + 20$

(f) $\sum_{r=1}^{3n} (3 - 4r)$

(g) $S_n = n^2 - 3n$, 8 terms

(h) $S_n = 2n(n+3)$, m terms

7. The sum of the first n terms of an AP is S_n where $S_n = n^2 - 3n$. Write down the fourth term and the nth term.

8. The sum of the first n terms of a series is given by S_n where $S_n = n(3n-4)$. Show that the terms of the series are in arithmetic progression.

9. In an arithmetic progression, the 8th term is twice the 4th term and the 20th term is 40. Find the common difference and the sum of the terms from the 8th to the 20th inclusive.

10. How many terms of the AP, $1 + 3 + 5 + \ldots$ are required to make a sum of 1521?

11. Find the least number of terms of the AP, $1 + 3 + 5 + \ldots$ that are required to make a sum exceeding 4000.

12. If the sum of the first n terms of a series is S_n where $S_n = 2n^2 - n$,

(a) prove that the series is an AP, stating the first term and the common difference,

(b) find the sum of the terms from the 3rd to the 12th inclusive.

13. In an AP the 6th term is half the 4th term and the 3rd term is 15.

(a) Find the first term and the common difference.

(b) How many terms are needed to give a sum that is less than 65?

GEOMETRIC PROGRESSIONS

Consider the sequence

$$12,\ 6,\ 3,\ 1.5,\ 0.75,\ 0.375,\ \ldots$$

Each term of this sequence is half the preceeding term so the sequence may be written

$$12,\ 12(\tfrac{1}{2}),\ 12(\tfrac{1}{2})^2,\ 12(\tfrac{1}{2})^3,\ 12(\tfrac{1}{2})^4,\ 12(\tfrac{1}{2})^5,\ \ldots$$

Such a sequence is called a geometric progression (GP) which is a sequence where each term is a constant multiple of the preceding term. This constant multiplying factor is called the common ratio, and it may have any real value.

Hence, if a GP has a first term of 3 and a common ratio of -2 the first four terms are

$$3,\ 3(-2),\ 3(-2)^2,\ 3(-2)^3$$

or $$3,\ -6,\ 12,\ -24$$

In general if a GP has a first term a, and a common ratio r, the first four terms are

$$a,\ ar,\ ar^2,\ ar^3$$

and the nth term, u_n, is ar^{n-1}, thus

a GP with n terms can be written $a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1}$

The Sum of a Geometric Progression

Consider the sum of the first eight terms, S_8, of the GP with first term 1 and common ratio 3

i.e. $$S_8 = 1 + 1(3) + 1(3)^2 + 1(3)^3 + \ldots + 1(3)^7$$

$\Rightarrow$ $$3S_8 = 3 + 3^2 + 3^3 + \ldots + 3^7 + 3^8$$

Hence $$S_8 - 3S_8 = 1 + 0 + 0 + 0 + \ldots + 0 - 3^8$$

So $$S_8(1-3) = 1 - 3^8$$

$\Rightarrow$ $$S_8 = \frac{1-3^8}{1-3} = \frac{3^8-1}{2}$$

This process can be applied to a general GP.

Consider the sum, S_n, of the first n terms of a GP with first term a and common ratio r,

i.e. $S_n = a + ar + \dots + ar^{n-2} + ar^{n-1}$

Multiplying by r gives

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

Hence $S_n - rS_n = a - ar^n$

$\Rightarrow$ $S_n(1-r) = a(1-r^n)$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r}$$

If $r > 1$ the formula may be written $\dfrac{a(r^n-1)}{r-1}$

Examples 34c

1. The 5th term of a GP is 8, the third term is 4, and the sum of the first ten terms is positive. Find the first term, the common ratio, and the sum of the first ten terms.

For a first term a and common ratio r, the nth term is ar^{n-1}

Thus we have $ar^4 = 8$ $(n = 5)$

and $ar^2 = 4$ $(n = 3)$

dividing gives $r^2 = 2$

$\Rightarrow$ $r = \pm\sqrt{2}$ and $a = 2$

Using the formula $S_n = \dfrac{a(r^n-1)}{r-1}$ gives,

when $r = \sqrt{2}$, $S_{10} = \dfrac{2[(\sqrt{2})^{10}-1]}{\sqrt{2}-1} = \dfrac{62}{\sqrt{2}-1}$

when $r = -\sqrt{2}$, $S_{10} = \dfrac{2[(-\sqrt{2})^{10}-1]}{-\sqrt{2}-1} = \dfrac{-62}{\sqrt{2}+1}$

But we are told that $S_{10} > 0$, so we deduce that

$r = \sqrt{2}$ and $S_{10} = \dfrac{62}{\sqrt{2}-1} = 62(\sqrt{2}+1)$

2. A prize fund is set up with a single investment of £2000 to provide an annual prize of £150. The fund accrues interest at 5% p.a. paid yearly. If the first prize is awarded one year after the investment, find the number of years for which the full prize can be awarded.

After one year the value of the fund is the initial investment of £2000, plus 5% interest, less one £150 prize, i.e. £$\{(1.05)(2000) - 150\}$

If £P_n is the value of the fund after n years then

$$P_1 = 2000(1.05) - 150$$

$$P_2 = 1.05P_1 - 150 = 2000(1.05)^2 - 150(1.05) - 150$$

$$P_3 = 1.05P_2 - 150 = 2000(1.05)^3 - 150(1.05)^2 - 150(1.05) - 150$$

$$P_n = 2000(1.05)^n - 150(1.05)^{n-1} - 150(1.05)^{n-2} - \dots - 150$$

$$= 2000(1.05)^n - 150\{1 + 1.05 + \dots + (1.05)^{n-1}\}$$

The expression in square brackets is a GP of n terms with $a = 1$ and $r = 1.05$ and hence

$$P_n = 2000(1.05)^n - 150\left[\frac{(1.05)^n - 1}{1.05 - 1}\right]$$

$$= 3000 - 1000(1.05)^n$$

The fund can award the full prize as long as there is money left in the fund at the end of a year, i.e. as long as $P_n \geqslant 0$

$\Rightarrow$ $3000 - 1000(1.05)^n \geqslant 0$

$\Rightarrow$ $1.05^n \leqslant 3$

$\Rightarrow$ $n \ln 1.05 \leqslant 3$

$\Rightarrow$ $n \leqslant \ln 3 \div \ln 1.05 = 22.5$

Dividing by $\ln 1.05$ does not alter the inequality as $\ln 1.05$ is positive.

Therefore the prize fund contains some money after 22 years but would not after 23 years, so the full prize can be awarded for 22 years.

3. The sum of the first n terms of a series is 3^{n-1}. Show that the terms of this series are in geometric progression and find the first term, the common ratio and the sum of the second n terms of this series.

If the series is $a_1 + a_2 + \ldots + a_n$

then $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n = 3^n - 1$

and $S_{n-1} = a_1 + a_2 + \ldots + a_{n-1} = 3^{n-1} - 1$

therefore $a_n = 3^n - 1 - (3^{n-1} - 1)$

i.e. the nth term is $3^n - 3^{n-1} = 3^{n-1}(3-1) = (2)3^{n-1}$

Similarly $a_{n-1} = (2)3^{n-2}$ so $a_n \div a_{n-1} = 3$

showing that successive terms in the series have a constant ratio of 3
Hence this series is a GP with first term 2 and common ratio 3

The sum of the second n terms is

(the sum of the first $2n$ terms) − (the sum of the first n terms)

$$= S_{2n} - S_n$$
$$= (3^{2n} - 1) - (3^n - 1)$$
$$= 3^n(3^n - 1)$$

EXERCISE 34c

1. Write down the fifth term and the nth term of the following GPs:
(a) 2, 4, 8, ... (b) 2, 1, $\frac{1}{2}$, ... (c) 3, −6, 12, ...
(d) first term 8, common ratio $-\frac{1}{2}$
(e) first term 3, last term $\frac{1}{81}$, 6 terms

2. Find the sum, to the number of terms given, of the following GPs.
(a) $3 + 6 + \ldots$, 6 terms (b) $3 - 6 + \ldots$, 8 terms
(c) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, 20 terms
(d) first term 5, common ratio $\frac{1}{5}$, 5 terms
(e) first term $\frac{1}{2}$, common ratio $-\frac{1}{2}$, 10 terms
(f) first term 1, common ratio −1, 2001 terms.

3. The 6th term of a GP is 16 and the 3rd term is 2. Find the first term and the common ratio.

4. Find the common ratio, given that it is negative, of a GP whose first term is 8 and whose 5th term is $\frac{1}{2}$

5. The nth term of a GP is $(-\frac{1}{2})^n$. Write down the first term and the 10th term.

6. Evaluate $\sum_{r=1}^{10} (1.05)^r$

7. Find the sum to n terms of the following series.
(a) $x + x^2 + x^3 + \ldots$ (b) $x + 1 + \dfrac{1}{x} + \ldots$ (c) $1 - y + y^2 - \ldots$
(d) $x + \dfrac{x^2}{2} + \dfrac{x^3}{4} + \dfrac{x^4}{8} + \ldots$ (e) $1 - 2x + 4x^2 - 8x^3 + \ldots$

8. Find the sum of the first n terms of the GP $2 + \frac{1}{2} + \frac{1}{8} + \ldots$ and find the least value of n for which this sum exceeds 2.65

9. The sum of the first 3 terms of a GP is 14. If the first term is 2, find the possible values of the sum of the first 5 terms.

10. Evaluate $\sum_{r=1}^{10} 3(3/4)^r$

11. A mortgage is taken out for £10 000 and is repaid by annual instalments of £2000. Interest is charged on the outstanding debt at 10%, calculated annually. If the first repayment is made one year after the mortgage is taken out find the number of years it takes for the mortgage to be repaid.

12. A bank loan of £500 is arranged to be repaid in two years by equal monthly instalments. Interest, *calculated monthly*, is charged at 11% p.a. on the remaining debt. Calculate the monthly repayment if the first repayment is to be made one month after the loan is granted.

CONVERGENCE OF SERIES

If a piece of string, of length l, is cut up by first cutting it in half and keeping one piece, then cutting the remainder in half and keeping one piece, then cutting the remainder in half and keeping one piece, and so on, the sum of the lengths retained is

$$\frac{l}{2} + \frac{l}{4} + \frac{l}{8} + \frac{l}{16} + \dots$$

As this process can (in theory) be carried on indefinitely, the series formed above is infinite.

After several cuts have been made the remaining part of the string will be very small indeed, so the sum of the cut lengths will be very nearly equal to the total length, l, of the original piece of string. The more cuts that are made the closer to l this sum becomes,
i.e. if after n cuts, the sum of the cut lengths is

$$\frac{l}{2} + \frac{l}{2^2} + \frac{l}{2^3} + \dots + \frac{l}{2^n}$$

then, as $n \to \infty$, $$\frac{l}{2} + \frac{l}{2^2} + \dots + \frac{l}{2^n} \to l$$

or $$\lim_{n\to\infty}\left[\frac{l}{2} + \frac{l}{2^2} + \dots + \frac{l}{2^n}\right] = l$$

l is called the sum to infinity of this series.

In general, if S_n is the sum of the first n terms of any series and if $\lim_{n\to\infty} [S_n]$ exists and is finite, the series is said to be *convergent*.

In this case the sum to infinity, S_∞, is given by

$$S_\infty = \lim_{n\to\infty} [S_n]$$

The series $l/2 + l/2^2 + l/2^3 + \dots$, for example, is convergent as its sum to infinity is l.

However, for the series $1 + 2 + 3 + \dots + n$, we have $S_n = \frac{1}{2}n(n+1)$
As $n \to \infty$, $S_n \to \infty$ so this series does not converge and is said to be divergent.

For any AP, $S_n = \frac{1}{2}n[2a + (n-1)d]$, which always approaches infinity as $n \to \infty$. Therefore any AP is divergent.

THE SUM TO INFINITY OF A GP

Consider the general GP $a + ar + ar^2 + \dots$

Now $$S_n = \frac{a(1-r^n)}{1-r}$$

and if $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$

So $$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a(1-r^n)}{1-r}\right] = \frac{a}{1-r}$$

If $|r| > 1$, $\lim_{n\to\infty} r^n = \infty$ and the series does not converge.

Therefore, provided that $|r| < 1$, a GP converges to a sum of $\dfrac{a}{1-r}$

i.e. for a GP $$S_\infty = \frac{a}{1-r}$$

provided that $|r| < 1$

Arithmetic Mean

If three numbers, p_1, p_2, p_3, are in arithmetic progression then p_2 is called the *arithmetic mean* of p_1 and p_3

If $p_1 = a$, we may write p_2, p_3 as $a + d$, $a + 2d$ respectively,

hence $$p_1 + p_3 = 2a + 2d = 2(a+d) = 2p_2$$

$\therefore$ $$p_2 = \tfrac{1}{2}(p_1 + p_3)$$

i.e. the arithmetic mean of two numbers m and n is $\frac{1}{2}(m+n)$

Geometric Mean

If p_1, p_2, p_3 are in geometric progression, p_2 is called the *geometric mean* of p_1 and p_3

If $p_1 = a$, then we may write $p_2 = ar$, $p_3 = ar^2$

thus $$p_1p_3 = a^2r^2 = p_2^2 \quad \Rightarrow \quad p_2 = \sqrt{(p_1p_3)}$$

i.e. the geometric mean of two numbers m and n is $\sqrt{(mn)}$

Examples 34d

1. Determine whether each series converges. If it does, give its sum to infinity.

(a) $3+5+7+\dots$ (b) $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\dots$ (c) $3+\frac{9}{2}+\frac{27}{4}+\dots$

(a) $3+5+7+\dots$ is an AP ($d=2$) and so does not converge.

(b) $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\dots = 1+(-\frac{1}{4})+(-\frac{1}{4})^2+(-\frac{1}{4})^3+\dots$
which is a GP where $r=-\frac{1}{4}$, i.e. $|r|<1$

So this series converges and $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-(-\frac{1}{4})} = \dfrac{4}{5}$

(c) $3+\frac{9}{2}+\frac{27}{4}+\dots = 3+3(\frac{3}{2})+3(\frac{9}{4})+\dots = 3+3(\frac{3}{2})+3(\frac{3}{2})^2+\dots$
This series is a GP where $r=\frac{3}{2}$ and, as $|r|>1$, the series does not converge.

---

2. Find the condition satisfied by x so that $\displaystyle\sum_{r=0}^{\infty}\frac{(x-1)^r}{2^r}$ converges.

Evaluate this expression when $x=1.5$

$$\sum_{r=0}^{\infty}\frac{(x-1)^r}{2^r} = 1+\frac{x-1}{2}+\left(\frac{x-1}{2}\right)^2+\dots$$

This series is a GP with common ratio $\left|\dfrac{x-1}{2}\right|$ and so converges

if $\quad \dfrac{x-1}{2} < 1$

i.e. if $\quad -1 < \dfrac{x-1}{2} < 1$

$\Rightarrow \quad -1 < x < 3$

When $x=1.5$, the series converges

and $\quad \displaystyle\sum_{r=0}^{\infty}\frac{(x-1)^r}{2^r} = \sum_{r=0}^{\infty}(\tfrac{1}{4})^r = 1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\dots$

using $S_\infty = \dfrac{a}{1-r}$ where $r=\frac{1}{4}$ and $a=1$ gives

$$S_\infty = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

---

3. Express the recurring decimal $0.1\dot{5}7\dot{6}$ as a fraction in its lowest terms.

$$\begin{aligned}
0.1\dot{5}7\dot{6} &= 0.157\,657\,657\,657\,6\dots\\
&= 0.1+0.0576+0.000\,0576+0.000\,000\,0576+\dots\\
&= \frac{1}{10}+\frac{576}{10^4}+\frac{576}{10^7}+\frac{576}{10^{10}}+\dots\\
&= \frac{1}{10}+\frac{576}{10^4}\left[1+\frac{1}{10^3}+\frac{1}{10^6}+\dots\right]\\
&= \frac{1}{10}+\frac{576}{10^4}\left[1+\frac{1}{10^3}+\left(\frac{1}{10^3}\right)^2+\dots\right]
\end{aligned}$$

Now the series in the square bracket is a GP whose first term is 1, and whose common ratio is $\dfrac{1}{10^3}$.

Hence it has a sum to infinity of $\dfrac{1}{1-10^{-3}} = \dfrac{10^3}{999}$

$$\Rightarrow \quad 0.1\dot{5}7\dot{6} = \frac{1}{10}+\frac{576}{10^4}\times\frac{10^3}{999} = \frac{1}{10}+\frac{576}{9990} = \frac{1575}{9990} = \frac{35}{222}$$

---

4. The 3rd term of a convergent GP is the arithmetic mean of the 1st and 2nd terms.
Find the common ratio and, if the first term is 1, find, the sum to infinity.

If the series is $a+ar+ar^2+ar^3+\dots$

then $\quad ar^2 = \frac{1}{2}(a+ar)$

$a \neq 0$, so $\quad 2r^2-r-1 = 0$

$\Rightarrow \quad (2r+1)(r-1) = 0$

i.e. $\quad r = -\frac{1}{2}$ or 1

As the series is convergent, the common ratio is $-\frac{1}{2}$

When $r=-\frac{1}{2}$ and $a=1$,

$$S_\infty = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

EXERCISE 34d

1. Determine whether each of the series given below converge.

(a) $4 + \frac{4}{3} + \frac{4}{3^2} + \dots$ (b) $9 + 7 + 5 + 3 + \dots$

(c) $20 - 10 + 5 - 2.5 + \dots$ (d) $\frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \dots$

(e) $p + 2p + 3p + \dots$ (f) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \dots$

2. Find the range of values of x for which the following series converge.

(a) $1 + x + x^2 + x^3 + \dots$

(b) $x + 1 + \frac{1}{x} + \frac{1}{x^2} + \dots$

(c) $1 + 2x + 4x^2 + 8x^3 + \dots$

(d) $1 - (1 - x) + (1 - x)^2 - (1 - x)^3 + \dots$

(e) $(a + x) + (a + x)^2 + (a + x)^3 + \dots$

(f) $(a + x) - 1 + \frac{1}{a + x} + \frac{1}{(a + x)^2} + \dots$

3. Find the sum to infinity of those series in Question 1 that are convergent.

4. Express the following recurring decimals as fractions

(a) $0.1\dot{6}\dot{2}$ (b) $0.\dot{3}\dot{4}$ (c) $0.0\dot{2}\dot{1}$

5. The sum to infinity of a GP is twice the first term. Find the common ratio.

6. If $\ln y$ is the arithmetic mean of $\ln x$ and $\ln z$ show that y is the geometric mean of x and z

7. The sum to infinity of a GP is 16 and the sum of the first 4 terms is 15. Find the first four terms.

8. If a, b and c are the first three terms of a GP, prove that $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ form another GP.

FINDING THE SUMS OF SOME OTHER NUMBER SERIES

15

It is sometimes possible to find the sum of a series which is neither an AP nor a GP.

Consider the series $\sum_{r=1}^{n} \frac{1}{r(r+1)}$

The general term can be expressed as two terms using partial fractions,

i.e. $\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$ (using the cover-up method)

Hence $$\sum_{r=1}^{n} \frac{1}{r(r+1)} \equiv \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \dots + (\tfrac{1}{n-1} - \tfrac{1}{n}) + (\tfrac{1}{n} - \tfrac{1}{n-1})$$

All the terms cancel except for the first and last terms,

therefore $$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

The summation of this series was possible because we were able to express the general term of the given series as the *difference* of two consecutive terms of another series. This is known as the *method of differences* which can be summarised as follows.

If the general term, u_r, of a series can be expressed as $f(r) - f(r+1)$

then

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} [f(r) - f(r+1)]$$

$$= [f(1) - f(2)] + [f(2) - f(3)] + \dots$$

$$+ [f(n-1) - f(n)] + [f(n) - f(n+1)]$$

$$= f(1) - f(n+1)$$

Example 34e

Show that $(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1$. Hence find $\sum_{r=1}^{n}(3r^2+3r+1)$

and deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

LHS $= (r+1)^3 - r^3 \equiv r^3 + 3r^2 + 3r + 1 - r^3 \equiv 3r^2 + 3r + 1 =$ RHS

i.e. $(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1$

$$\therefore \quad \sum_{r=1}^{n}(3r^2+3r+1) = \sum_{r=1}^{n}\{(r+1)^3 - r^3\}$$
$$= (2^3 - 1^3) + (3^3 - 2^3) + \ldots + \{(n+1)^3 - n^3\}$$
$$= (n+1)^3 - 1$$

Now $$\sum_{r=1}^{n}(3r^2+3r+1) = \sum_{r=1}^{n}3r^2 + \sum_{r=1}^{n}3r + \sum_{r=1}^{n}1$$

$$\therefore \quad \sum_{r=1}^{n}3r^2 + \sum_{r=1}^{n}3r + \sum_{r=1}^{n}1 = (n+1)^3 - 1 \qquad [1]$$

But $\sum_{r=1}^{n}3r$ is an AP $(a = 3,\ d = 3)$ so using $S_n = \frac{1}{2}n(a+l)$

gives $\sum_{r=1}^{n}3r = \frac{3}{2}n(n+1)$ and $\sum_{r=1}^{n}1 = 1 + 1 + \ldots + 1 = n$

Hence [1] becomes $\sum_{r=1}^{n}3r^2 + \frac{3}{2}n(n+1) + n = (n+1)^3 - 1$

$$\Rightarrow \quad \sum_{r=1}^{n}3r^2 = (n+1)^3 - (1+n) - \tfrac{3}{2}n(n+1)$$
$$= (n+1)\left[(n+1)^2 - 1 - \tfrac{3}{2}n\right]$$
$$= (n+1)\left[\tfrac{1}{2}n(2n+1)\right]$$

Now $\sum_{r=1}^{n}3r^2 = 3\sum_{r=1}^{n}r^2$,

$$\therefore \quad \sum_{r=1}^{n}r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

EXERCISE 34e

In Questions 1 to 4 express the general term in partial fractions and hence find the sum of the series.

1. $\sum_{r=1}^{n}\dfrac{1}{r(r+2)}$

2. $\sum_{r=3}^{n}\dfrac{1}{(r+1)(r+2)}$

3. $\sum_{r=n}^{2n}\dfrac{1}{r(r+1)}$

4. $\sum_{r=1}^{n}\dfrac{r}{(2r-1)(2r+1)(2r+3)}$

5. Verify that $4r^3 + r \equiv (r+\frac{1}{2})^4 - (r-\frac{1}{2})^4$. Hence find $\sum_{r=1}^{n}(4r^3+r)$

   Deduce that $\sum_{r=1}^{n}r^3 = \frac{1}{4}n^2(n+1)^2$

6. If $f(r) \equiv \dfrac{1}{r(r+1)}$, simplify $f(r+1) - f(r)$

   Hence find $\sum_{r=1}^{n}\dfrac{1}{r(r+1)(r+2)}$

7. If $f(r) \equiv \dfrac{1}{r^2}$, simplify $f(r) - f(r+1)$

   Hence find the sum of the first n terms of the series
   $$\frac{3}{(1^2)(2^2)} + \frac{5}{(2^2)(3^2)} + \frac{7}{(3^2)(4^2)} + \ldots$$

8. Given that $\sum_{r=1}^{n}r^2 = \frac{1}{6}n(n+1)(2n+1)$, use the identity
   $$r^4 - (r-1)^4 \equiv 4r^3 - 6r^2 + 4r - 1$$
   to find the sum of the cubes of the first n natural numbers,

   i.e. $\sum_{r=1}^{n}r^3$

9. If $f(r) \equiv \cos 2r\theta$, simplify $f(r) - f(r+1)$

   Use you result to find the sum of the first n terms of the series
   $$\sin 3\theta + \sin 5\theta + \sin 7\theta + \ldots$$

NATURAL NUMBER SERIES

The natural numbers are the positive integers, i.e. 1, 2, 3, ...

The series $1 + 2 + 3 + \ldots + n$ is the sum of the first n natural numbers and can be written $\sum_{r=1}^{n} r$

This series is an AP with $a = 1$ and $d = 1$, so using $S_n = \frac{1}{2}n(a + l)$

gives
$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n + 1)$$

Now consider the series $1^2 + 2^2 + 3^2 + \ldots + n^2$ which is the sum of the squares of the first n natural numbers.

This series is written $\sum_{r=1}^{n} r^2$ and its sum was found in Example 34e,

i.e.
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n + 1)(2n + 1)$$

The series $1^3 + 2^3 + 3^3 + \ldots + n^3$ is called the sum of the cubes of the first n natural numbers and we saw in Exercise 34e that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n + 1)^2 = \left[\tfrac{1}{2}n(n + 1)\right]^2 = \left[\sum_{r=1}^{n} r\right]^2$$

The results are quotable and can be used to sum other series.

Examples 34f

1. Find $\sum_{r=1}^{n} r(r + 1)(r + 2)$

$$r(r + 1)(r + 2) \equiv r^3 + 3r^2 + 2r$$

$$\begin{aligned}\therefore\quad \sum_{r=1}^{n} r(r + 1)(r + 2) &= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r\\ &= \tfrac{1}{4}n^2(n + 1)^2 + 3\left[\tfrac{1}{6}n(n + 1)(2n + 1)\right]\\ &\quad + 2\left[\tfrac{1}{2}n(n + 1)\right]\\ &= \tfrac{1}{4}n(n + 1)(n + 2)(n + 3)\end{aligned}$$

---

2. Find $\sum_{r=5}^{10} r^2$

$$\begin{aligned}\sum_{r=5}^{10} r^2 = \sum_{r=1}^{10} r^2 - \sum_{r=1}^{4} r^2 &= \tfrac{10}{6}(10 + 1)(20 + 1) - \tfrac{4}{6}(4 + 1)(8 + 1)\\ &= 355\end{aligned}$$

---

3. Find the sum of the squares of the first n odd numbers.

The odd numbers can be represented by $2r - 1$ where $r = 1, 2, 3, \ldots$

So we want $\sum_{r=1}^{n} (2r - 1)^2$

Now $(2r - 1)^2 = 4r^2 - 4r + 1$

$$\begin{aligned}\therefore\quad \sum_{r=1}^{n} (2r - 1)^2 &= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1\\ &= 4\left[\tfrac{1}{6}n(n + 1)(2n + 1)\right] - 4\left[\tfrac{1}{2}n(n + 1)\right] + n\\ &= \tfrac{1}{3}n(4n^2 - 1)\end{aligned}$$

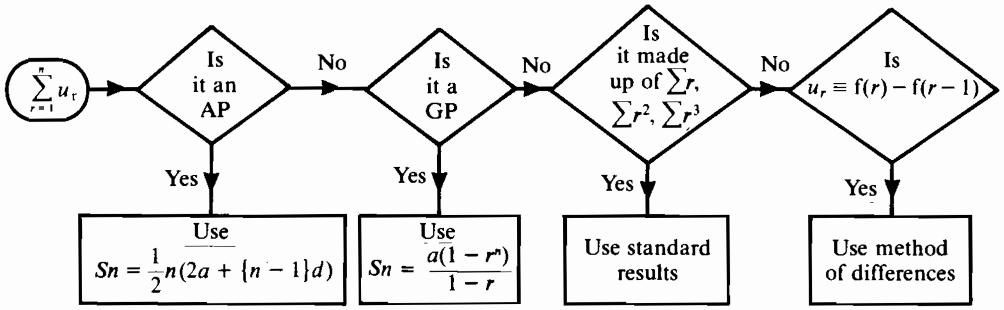
EXERCISE 34f

Find the sum of the series.

1. $\sum_{r=1}^{n} r(r + 1)$
2. $\sum_{r=1}^{n} r(r + 1)(r + 2)$
3. $\sum_{r=n}^{2n} r^2(1 + r)$
4. $\sum_{r=10}^{20} r^3$
5. $(1)(3) + (2)(4) + (3)(5) + \ldots + (n - 1)(n + 1)$
6. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots - (2n)^2$
   (*Hint.* Consider two series, the sum of the squares of even numbers and sum of the squares of odd numbers.)
7. $(1)(3) + (3)(5) + (5)(7) + \ldots + (2n - 1)(2n + 1)$

GENERAL METHODS FOR SUMMING A NUMBER SERIES

The basic method for summing a number series relies on recognition, and a systematic approach is helpful, i.e.



For example, $\sum_{r=n}^{\infty} (\frac{1}{3})^r(\frac{1}{2})^{r-2}$ can be recognised as a GP with first term $(\frac{1}{3})^n(\frac{1}{2})^{n-2}$ and common ratio $(\frac{1}{3})(\frac{1}{2})$

Similarly, $\sum_{r=1}^{2n} (1+2n)(1-n)$ can be written as $\sum_{r=1}^{2n} (1+n-2n^2)$ when it can be recognised as being made up of natural number series.

THE SUM TO INFINITY OF A NUMBER SERIES

We saw on p. 606 that if S_n is the sum of the first n terms of a series and if $\lim_{n\to\infty} S_n$ exists, then the series is convergent with a sum to infinity, S, where

$$S = \lim_{n\to\infty} S_n$$

Note that when evaluating $\lim_{n\to\infty} S_n$ certain assumptions may be made, i.e. as $n \to \infty$, then $\frac{1}{n} \to 0$, $a^n \to 0$ if $0 < a < 1$, and $a^n \to \infty$ if $a > 1$

If $S_n \to \frac{\infty}{\infty}$ or $\frac{0}{0}$, both of which are indeterminate, it may be possible to evaluate $\lim_{n\to\infty} S_n$ by expressing S_n as a proper fraction,

e.g. if $S_n = \frac{n-1}{n+1}$ then $\lim_{n\to\infty} \frac{n-1}{n+1}$ is indeterminate,

but $\frac{n-1}{n+1} = 1 - \frac{2}{n+1}$ so $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right] = 1$

Example 34g

Find the sum to infinity of the series $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \dots$

The general term of this series is $\frac{1}{(2r-1)(2r+1)}$, and using partial fractions, this becomes $\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$

Therefore the sum of the first n terms of the given series is S_n where

$$S_n = \sum_{r=1}^{\infty}\left[\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}\right]$$

$$= \frac{1}{2}\left[(\frac{1}{1} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$$

$$\Rightarrow S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

Now as $n \to \infty$, $\frac{1}{2n+1} \to 0$ so $S_n \to \frac{1}{2}$

Therefore the sum to infinity of this series is $\frac{1}{2}$

MIXED EXERCISE 34

Find the sum of each of the following series.

1. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$
2. $2 - (2)(3) + (2)(3)^2 - (2)(3)^3 + \dots + (2)(3)^{10}$
3. $\sum_{r=2}^{n} ab^{2r}$
4. $1 + 4 + 9 + 16 + \dots + 144$
5. $\sum_{r=5}^{n} 4r$
6. $\sum_{r=1}^{n} r(r^2+1)$

7. $\sum_{r=2}^{n} \frac{1}{(r-1)(r+1)}$

8. $\ln 3 + \ln 3^2 + \ln 3^3 + \ldots + \ln 3^{20}$

9. $e + e^2 + e^3 + \ldots + e^n$

10. $\sum_{r=1}^{n} r(2r+1)(r+2)$

11. $\sum_{r=n}^{2n} \frac{1}{(r+1)(r+2)}$

12. $\sum_{r=1}^{\infty} \frac{1}{2^r}$

13. The sum of the squares of the first n even numbers.

14. $\sum_{r=2}^{n} \ln\left(1 - \frac{1}{r}\right)$

15. The sum of the first n terms of a series is n^3. Write down the first four terms and the nth term of the series.

16. The fourth term of an AP is 8 and the sum of the first ten terms is 40. Find the first term and the tenth term.

17. The second, fourth and eighth terms of an AP are the first three terms of a GP. Find the common ratio of the GP.

18. Find the value of x for which the numbers $x + 1$, $x + 3$, $x + 7$, are in geometric progression.

19. The second term of a GP is $\frac{1}{2}$ and the sum to infinity of the series is 4. Find the first term and the common ratio of the series.

CHAPTER 35

PERMUTATIONS AND COMBINATIONS

ARRANGEMENTS

Suppose that three different pictures, A, B and C are to be hung, in line, on a wall. The pictures can be hung in various orders,
e.g. A B C, A C B, B A C and others.

Each of these orders is a particular arrangement of the pictures and is called a *permutation*.

A permutation is an ordered arrangement of the items in a set.

CHOICES

Now we look at a different situation. Suppose that, from a set of seven pictures, A, B, C, D, E, F and G, three pictures are to be chosen for display.

Choosing the group of three pictures does *not* involve the order in which they will be hung.
If, say, B, D and F are chosen then, although they can be hung in six different orders, the subset B, D and F comprises *only one choice*, which is called a *combination*.

i.e.
B D F
B F D
D B F
D F B
F B D
F D B
} are six different permutations but only one combination

A combination is an unordered group chosen from the items in a set.

Once a combination has been selected, its members can be arranged in different orders forming permutations should this be required.

For example, a news vendor stocks ten weekly magazines but the display stand at his kiosk has room for only five. As he cannot display all ten of his magazines he must *choose* a group of five. The order in which he picks up the five is irrelevant; the set of five is *only one combination*. However, once he has made his choice he can display them in different orders on the stand. He is now *arranging* them and each arrangement is a permutation.

EXERCISE 35a

Each question asks either for a number of combinations or for a number of permutations. Without attempting to find that number, decide whether you are looking for combinations or permutations.

1. How many arrangements of the letters X, Y and Z are there?
2. A team of four is to be chosen from nine players. How many different teams can be selected?
3. If eight records can be taken to a desert island, from a collection of one hundred records, how many different sets can be chosen?
4. Five hundred raffle tickets are sold. When the first, second and third prizes are draw, in how many different ways can the prizes be won?
5. A museum lists its exhibits by code numbers of seven digits. How many code numbers are available?
6. A door is to be painted in two shades of green paint. If six suitable shades are available, in how many ways can the two shades be selected?
7. In how many ways can fifteen books be placed on a shelf?

PERMUTATIONS

Suppose that we wish to calculate the number of permutations of the three letters, A, B and C.

Arranging them from left to right, the first letter can be

A or B or C

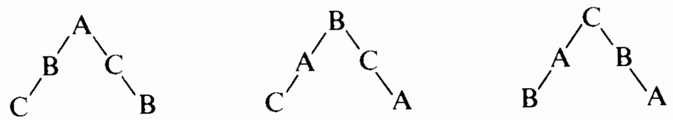
i.e. there are 3 ways of choosing the first letter.

When we come to place the second letter there are only two left to choose from,
e.g. if the first letter were B then only A or C can be in second place.

A	B	C
B or C	A or C	A or B

For *each* of the 3 ways of selecting the first letter, there are 2 ways of selecting the second letter, i.e. there are 3×2 ways of selecting the first two letters.

Once the first two are in place there is only one letter left for third place



So there are $3 \times 2 \times 1$ ways of arranging three letters in order.

This argument can be applied equally well to any number of objects (although the 'choice tree' is cumbersome for a large number).

Considering, say, the number of permutations of eight different books, we see that

> first place can be filled by any one of the 8 books (leaving seven books)
> second place can be filled by any one of the remaining 7 books
> third place can be filled by any one of 6 books
> ... and so on

Hence there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ permutations of eight items.

Because this product, and others like it, is clumsy to write out in full, a special notation is used to express it more neatly.

Factorial Notation

To represent $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ we write 8! and say 'eight factorial'. Similarly 5! means $5 \times 4 \times 3 \times 2 \times 1$

In general, the product of all the whole numbers from any number, n, down to 1 is called *n factorial* and written as $n!$

PERMUTATION PROBLEMS

The simplest problems are those in which the n items in a set are (a) all different, (b) all to be used, (c) each used only once.

In such cases, as we have already seen, there are $n!$ permutations.

There are many other situations, however, and all but the simplest problems should be worked from first principles.

Suppose, for instance, that there are eight different objects available, but that there is room for only five of those objects, then

first place can be filled in 8 ways
second place can be filled in 7 ways

- - - - - - - - - - - - - -

fifth place can be filled in 4 ways

and, as there are no more places left to be filled, we see that there are

$$8 \times 7 \times 6 \times 5 \times 4 \text{ permutations}$$

Now this product cannot be written as 8! because it stops short at 4 and $3 \times 2 \times 1$ are missing. It *can* be abbreviated however because

$$8 \times 7 \times 6 \times 5 \times 4 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{8!}{3!} = \frac{8!}{(8-5)!}$$

The number of arrangements of 5 objects when 8 different objects are available is denoted by ${}^8P_5$

Therefore $${}^8P_5 = \frac{8!}{(8-5)!}$$

In general, if n different objects are available but only r of them are arranged, then the number of arrangements is ${}^nP_r$ where

$${}^nP_r = \frac{n!}{(n-r)!}$$

So far we have considered only those arrangements of different objects in which each object is available for use only once. Certain items can be placed in an arrangement more than once, however; using the digits 0 to 9 to form telephone numbers is a good example of this situation.

Using the digits 3, 4 and 5 to form as many three-digit numbers as possible we see that the first digit can be any one of the three available.

Any one of the three digits can also be used in second place and again in third place so there are $3 \times 3 \times 3$, i.e. 27, possible numbers.

There are many other variations that arise in problems about arrangements. Some of these are illustrated in the following examples but no attempt is made to cover all possibilities. The essence of these problems is to treat each one as an individual puzzle to be thought out.

Examples 35b

1. How many arrangements of the letters in the word GROUP start with a vowel?

If the first letter is to be a vowel there are only 2 possibilities, O or U. When the first letter has been chosen, any of the remaining 4 letters (including the vowel not in first place) can be used in second place, so there are 2×4 ways of arranging the first two letters. Then there are three letters available for third place, and so on.

	1st	2nd	3rd	4th	5th
No. of ways of selecting the letter	2	4	3	2	1

The number of arrangements starting with a vowel is

$$2 \times 4 \times 3 \times 2 \times 1 = 2(4!) = 48$$

2. Four girls and two boys are to sit in a row. The two boys, Alan and Tim, insist on sitting together. In how many different ways can the six children be arranged?

As Alan and Tim must be together we first treat them as one unit, i.e. we have only five items to arrange, four girls and 'a pair of boys'.

Five different objects can be arranged in 5! ways.

In any one of these arrangements, the boys can be seated in one of two ways, i.e. Tim Alan or Alan Tim.

So the total number of arrangements is $2 \times 5!$

Note that if the two boys *refused* to sit together we would first find the number of arrangements in which they *are* next to each other (as above) and subtract this value from the total number of arrangements without any restriction, i.e. $6! - 2 \times 5!$

3. How many odd numbers between 2000 and 3000 can be formed from the digits 1, 2, 3, 4, 5 and 6?

Because the number is between 2000 and 3000, it has only four digits and the first digit can only be 2. The number is odd so the last digit must be taken from 1, 3 or 5. Once the first and last digit are placed there are four digits available for second place and three for third place

	1st	4th	2nd	3rd
Number of ways of selecting digit	1	3	4	3

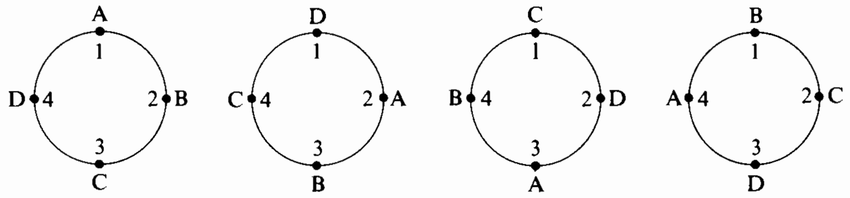
∴ there are $1 \times 3 \times 4 \times 3$, i.e. 36 numbers.

4. In how many different ways can four people stand in a circle?

No one place is special and there is no first or last place in a circle. What matters here is the order in which each person stands relative to the others.

Starting at a particular place with any one of the four people, the person standing next on the left (say) can be any one of the remaining three, and so on,
i.e. there are 4! arrangements starting from that particular place.

However, if the places are numbered 1 to 4 and the people denoted by A to D, we see that some of the arrangements accounted for above are not, in fact, different *relative* orders.
e.g. if all four people move one place clockwise they are still in the *same relative order* but so far this has been counted as a different permutation. This can be repeated giving four positions which are not different permutations

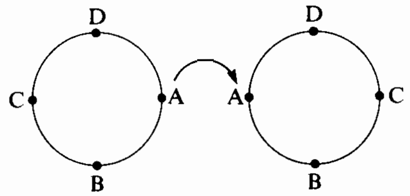


i.e. the total of 4! arrangements is 4 times too big.

So the number of different arrangements is 4!/4, i.e. 3!

In general, n objects can be arranged in a circle in $(n-1)!$ ways.

Considering four beads on a circular wire we see that the number of different orders in which they can be threaded is reduced even further because the wire can be turned over. In this way the two arrangements shown below are actually identical. So we see that the number of arrangements on a reversible ring is half the number of arrangements obtained for a circle that cannot be turned over.



So the number of different ways of threading four beads on a circular wire is $\dfrac{4!}{4 \times 2}$

Again we have a similar general result for arranging n different objects on a circular wire, i.e. there are $\dfrac{n!}{n \times 2}$ arrangements.

EXERCISE 35b

1. In how many ways can seven different trophies be arranged on a shelf?

2. How many three-digit numbers can be formed from the set $\{2, 3, 4, 5, 6\}$ if the three digits are
(a) all different (b) all the same?

3. In how many ways can six different books be placed in a rack?

4. How many numbers greater than 4000 can be formed from the integers 1, 3, 5, 7, using each of them only once in each number?

5. Stevenage telephone numbers have six digits the first of which may not be zero. What is the greatest possible number of separate lines in Stevenage?

6. How many arrangements can be made from the letters of the word PRINCE if
(a) the first letter must be a consonant
(b) the last letter must be a vowel
(c) the P and R must not be separated?

7. Given three red cubes, three blue cubes and three green cubes, in how many ways can three cubes be placed in a row if they are
   (a) of different colours (b) all the same colour?

8. In how many different ways can ten people be seated at a round table?

9. How many five-digit numbers can be formed from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if
   (a) the digits are all different
   (b) the digits are all the same
   (c) the digits are all different and the number is greater than 60 000?

10. In how many orders can twelve cows be placed in a circular milking parlour?

11. Two different maths books and three other books are to be placed on a shelf. How many different arrangements are possible if
    (a) the two maths books are together
    (b) the two maths books are separated?

12. In how many ways can five beads of different colours be arranged on a wire ring?

13. Using the digits 2, 4, 6, 8, find how many whole numbers between 3000 and 7000 can be found in which the digits are all different.

14. Twelve children stand in a row. In how many different ways can they be arranged if
    (a) one particular child has to stand at one end
    (b) two particular children must be separated?

15. How many whole numbers between 100 and 1000 can be formed from the digits 3, 5, 7, 8 if
    (a) the digits in each number are all different
    (b) each digit can be repeated as often as required?

16. How many of the numbers in 15(a) are even?

COMBINATIONS

Consider again the situation where three pictures have to be *chosen*, but not ordered, from seven available different pictures.
We are now going to investigate the number of different sets, or combinations, of three pictures.

First we will not only select three pictures but also arrange them in order as we do so.

This can be done in $7 \times 6 \times 5$ ways, i.e. in $\frac{7!}{4!}$ ways.

This number includes, as separate arrangements, the number of permutations of the pictures represented by B, D and F (see p. 619), i.e. 3! permutations.
These permutations comprise only one set, or choice. The same is true for each group of three pictures, so the *number of different sets* of three pictures that can be taken from the seven available is given by

$$\frac{\text{total number of permutations}}{\text{number of permutations of each set}}$$

The number of combinations of three objects chosen from seven different objects is denoted by ${}^7C_3$,

i.e. $${}^7C_3 = \frac{7!}{4!\,3!} = \frac{7!}{3!(7-3)!}$$

In general, if a set of r objects is chosen from n different objects then the number of combinations is denoted by ${}^nC_r$ where

$$ {}^nC_r = \frac{n!}{r!(n-r)!}$$

i.e. $${}^nC_r = \frac{(\text{total number available})!}{(\text{number chosen})!\,(\text{number not chosen})!}$$

Now if we have to choose $(n-r)$ from n different objects, the number of objects not chosen is r so

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{r!(n-r)!} = {}^nC_r$$

i.e. $${}^nC_{n-r} = {}^nC_r$$

The Meaning of 0!

There is clearly only one way of choosing n objects from n available objects. As this is represented by ${}^nC_n$ we have,

$$ {}^nC_n = 1 $$

Using the definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ in the case when $r = n$

gives $$ {}^nC_n = \frac{n!}{n!\,0!} $$

This is equal to unity only if we define $0!$ as having the value 1

i.e. $$ 0! = 1 $$

PROBLEMS ON COMBINATIONS

As was the case with permutations, problems involving combinations are very varied. Each one is a puzzle and should be carefully thought through. Some of the many variations are illustrated in the following examples.

Examples 35c

1. In how many different ways can eight cards can be dealt from a pack of fifty-two playing cards?

The fifty-two cards in the pack are all different so this question is straightforward, with no complications, and we can use ${}^nC_r$

A set of eight cards can be chosen from fifty-two in ${}^{52}C_8$ ways

i.e. the number of different hands is $\dfrac{52!}{8!(52-8)!} = \dfrac{52!}{8!\,44!}$

Note. Unless a numerical answer is specifically asked for, answers to this type of problem are usually given in factorial notation.

---

2. In how many ways can five boys be chosen from a class of twenty boys if the class prefect must be included?

One particular boy has to be chosen and the group is then completed by choosing four boys from the remaining nineteen.

Four boys can be chosen from nineteen boys in ${}^{19}C_4$ ways, i.e. in

$$ \frac{19!}{4!(19-4)!} \text{ ways} $$

$\therefore$ the number of groups of five boys including the prefect is $\dfrac{19!}{4!\,15!}$

---

3. In how many ways can a party of ten children be divided into two groups of
   (a) six children and four children (b) five children each?

(a) Each time a group of six children is chosen, those not chosen automatically form a group of four. So we need consider only forming groups of six.

Six children can be chosen from ten in ${}^{10}C_6$, i.e. $\dfrac{10!}{6!(10-6)!}$ ways.

$\therefore$ the number of ways in which the children can be divided into groups of four and six is $\dfrac{10!}{6!\,4!}$

(b) Again we begin by considering the choice of one group of five children.

The number of ways of choosing a set of five children is

$$ \frac{10!}{5!(10-5)!} = \frac{10!}{5!\,5!} $$

Each time a group of five is selected, a group is formed of the other five, i.e. each *pair* of groups is chosen twice. For example, if the children are denoted by the letters A to J then one chosen could be ABCDE leaving FGHIJ for the corresponding second group. Now the first group chosen could be FGHIJ for which the corresponding second group is ABCDE.

But dividing the children into ABCDE/FGHIJ and FGHIJ/ABCDE gives the same *pair* of groups in both cases, i.e. the number of ways of choosing a group of five children, found above, is twice the number of different pairs of groups.

$\therefore$ the number of ways of dividing the children into two groups each of five is $\left(\dfrac{1}{2}\right)\left(\dfrac{10!}{5!\,5!}\right)$

EXERCISE 35c

1. Write in factorial notation
   (a) the number of permutations of six objects if eleven different objects are available
   (b) the number of combinations of six objects chosen from eleven different objects
   (c) the number of combinations of three objects chosen from twenty different objects.

2. Describe in words a situation which could be represented by
   (a) $\dfrac{8!}{4!}$ (b) $\dfrac{14!}{5!\,9!}$ (c) $\dfrac{7!}{5!\,2!}$

3. How many different combinations of six letters can be chosen from the letters A, B, C, D, E, F, G, H if each letter may be chosen only once?

4. In how many ways can the eight letters given in Question 3 be divided into two groups containing six and two letters respectively?

5. To compete in a quiz contest a team of four is to be selected from a class of twenty children. How many different teams can be chosen if
   (a) the oldest member of the class must be included as team leader
   (b) there is a completely free choice?

6. A shop stocks nine different varieties of tinned meat. In how many ways can a shopper buy three tins if
   (a) three different varieties are chosen
   (b) there are two tins of the same variety?

7. The nine members of a committee comprise: one married couple, three more men and four more women. In how many ways can a working party of five people be selected?
   How many of these working parties are such that
   (a) at least one man and at least one woman must be chosen
   (b) the husband *or* the wife but not both, may be included
   (c) it is formed entirely of women?

8. How many different hands of five cards can be dealt from a suit of thirteen cards?

9. How many of the hands dealt in Question 8 contain the ace?

10. From a large bowl of apples, pears, oranges and bananas, three pieces of fruit are chosen. How many different selections can be made if
    (a) the three chosen are all different but must include a banana
    (b) two are the same kind and the third is different but no pears are chosen?

11. A large box packet contains nine different kinds of biscuit. In how many ways can four biscuits be chosen if
    (a) four different varieties are taken
    (b) two each of two varieties are selected
    (c) three are the same and the fourth is different
    (d) all four are the same?

USING THE FACTORIAL NOTATION

The factorial notation has proved to be very useful in permutation and combination problems, whenever we need to write the product of all the whole numbers from a given number n down to 1. There are other topics which require the use and manipulation of factorial expressions so we will now examine the ways in which factorials can be simplified or evaluated.

One of the forms in which factorials occur very frequently is, for example, $\dfrac{9!}{6!}$

Written in full, $\dfrac{9!}{6!} = \dfrac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1} = 9\times 8\times 7$

i.e. dividing the top and bottom by 6! leaves the product of numbers from 9 to 7

i.e. $\dfrac{9!}{6!}$ = the product of integers from 9 to $(6+1)$

In general $\dfrac{n!}{r!}$ is the product of the integers from n down to $(r+1)$

Examples 35d

1. Without using a calculator, evaluate (a) $\dfrac{10!}{8!}$ (b) $\dfrac{19!}{2!\,17!}$

(a) $\dfrac{10!}{8!} = 10 \times 9 = 90$

(b) $\dfrac{19!}{2!\,17!} = \dfrac{19 \times 18}{2 \times 1} = 19 \times 9 = 171$

---

2. Express $\dfrac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$ in factorial notation.

$$9 \times 8 \times 7 \times 6 = \frac{9!}{5!}$$

$$\therefore \quad \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = \frac{9!}{5!\,4!}$$

---

3. Factorise $8! - 5(7!)$

$$8! = 8 \times 7!$$

$$\therefore \quad 8! - 5(7!) = 8(7!) - 5(7!)$$

$$= 7!(8 - 5) = 3(7!)$$

---

4. Factorise $(n+1)! + n^2(n-1)!$

$$(n+1)! = (n+1) \times n!$$

and $$n^2(n-1)! = n\{n \times (n-1)!\} = n\{n!\}$$

$$\therefore \quad (n+1)! + n^2(n-1)! = (n+1)\{n!\} + n\{n!\}$$

$$= \{n!\}(n + 1 + n)$$

$$= (2n+1)n!$$

---

5. Show that the number of groups of r objects that can be chosen from n different objects is $\dfrac{n(n-1)(n-2)\dots(n-r+1)}{r!}$

The number of possible groups is given by

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!} \div r!$$

Now when $(n-r)!$ is cancelled from $n!$ the last number left in the numerator is one greater than $(n-r)$, i.e. $(n-r+1)$

$\therefore$ the number of groups is $\dfrac{n(n-1)(n-2)\dots(n-r+1)}{r!}$

EXERCISE 35d

Do not use a calculator in this exercise.

Evaluate

1. $3!$
2. $4!$
3. $5!$
4. $6!$
5. $\dfrac{6!}{4!}$
6. $\dfrac{12!}{10!}$
7. $\dfrac{7!}{3!}$
8. $\dfrac{5!}{2!}$
9. $\dfrac{8!}{6!\,2!}$
10. $\dfrac{9!}{2!\,3!\,4!}$
11. $\dfrac{8!}{(4!)^2}$
12. $\dfrac{(3!)^2}{2!\,4!}$

Write in factorial form.

13. $5 \times 4 \times 3$
14. 11×10
15. $(n+1)n(n-1)$
16. $(n+2)(n+1)$
17. $\dfrac{20 \times 19 \times 18}{3 \times 2 \times 1}$
18. $\dfrac{8 \times 7 \times 6}{6 \times 5 \times 4}$
19. $\dfrac{5 \times 4 \times 3}{3 \times 2 \times 1}$
20. $\dfrac{40 \times 39}{2 \times 1}$
21. $\dfrac{n(n-1)(n-2)}{3 \times 2 \times 1}$
22. $\dfrac{(n-1)(n-2)(n-3)(n-4)}{4 \times 3 \times 2 \times 1}$

Factorise

23. $8! + 9!$ **24.** $7! - 2(5!)$ **25.** $n! + (n-1)!$ **26.** $(n+1)! - n!$

27. Show that the number of ways in which $(r-1)$ objects can be chosen from $(n+1)$ different objects is

$$\frac{(n+1)(n)(n-1)\dots(n-r+3)}{(r-1)!}$$

28. Find an expression similar to that given in Question 27 for the number of combinations of

(a) r objects chosen from $2n$ different objects

(b) n objects chosen from $2n$ different objects

FURTHER PROBLEMS ON PERMUTATIONS AND COMBINATIONS

Permutations Involving some Identical Objects

Consider the number of possible arrangements of the letters of the word DIGIT.

This word contains two Is which are identical, but which can be distinguished by adding suffixes, i.e. D I_1 G I_2 T.

Then the number of permutations of the letters D I_1 G I_2 T is 5!

But this number includes separately the two permutations

$$D\ I_1\ G\ I_2\ T \quad \text{and} \quad D\ I_2\ G\ I_1\ T$$

so the arrangement DIGIT is counted twice. Because I_1 and I_2 can be arranged in 2! ways, every distinct arrangement of the letters DIGIT is included 2! times in the permutations of D I_1 G I_2 T.

Hence there are $\frac{5!}{2!}$ arrangements of the letters D I G I T

Now consider the number of permutations of the letters

D E F E A T E D

There are three Es and two Ds.

The number of permutations of $D_1\ E_1\ F\ E_2\ A\ T\ E_3\ D_2$ is 8!

But E_1, E_2, E_3 can be arranged in 3! ways, and D_1, D_2 can be arranged in 2! ways,

so the number of arrangements of $D_1\ E_1\ F\ E_2\ A\ T\ E_3\ D_2$ is $3! \times 2!$ times the number of arrangements of D E F E A T E D.

Therefore there are $\frac{8!}{3!\,2!}$ permutations of D E F E A T E D

Using a similar argument in a more general case we have,

there are $\frac{n!}{r!}$ permutations of n objects, r of which are identical.

and, for n objects, p of which are alike and q of which are alike (but different from the set of p objects), the number of permutations is $\frac{n!}{p!\,q!}$

Examples 35e

1. In how many of the possible permutations of the letters of the word ADDING are the two Ds **(a)** together **(b)** separated?

(a) The number of permutations in which the Ds are together can be found easily by bracketing the Ds and treating them as one item in the arrangements of A, (DD), I, N, G. There are now five different items which can be arranged in 5! ways.

(b) As the Ds are either together or separated,

(number of permutations without restriction)

− (number of arrangements with Ds together)

= (number of arrangements with Ds separated)

Now the number of arrangements without restriction is $\frac{6!}{2!}$

Hence the number of arrangements in which the Ds are separated is

$$\frac{6!}{2!} - 5! = 240$$

Note. The number of permutations in which G and A are next to each other is found in a similar way to (a) above, but these two letters can be written (GA) or (AG). Therefore there are twice as many arrangements of A, D, D, I, N, G in which A and G are adjacent than when the two Ds are adjacent.

Independent Permutations and Combinations

A security firm wishes to use a code for each of its clients, all codes are to be made up of three different letters followed by two different digits excluding zero.
The number of different codes which the firm can use is given by considering the number of permutations of three letters from the alphabet and the number of permutations of two digits from the nine digits available.

These two sets of permutations are independent of each other as the order of the letters has no effect upon the order of the digits.
There are ${}^{26}P_3$ ways of arranging the letters and ${}^{9}P_2$ ways of arranging the digits in the code.
Each of the ${}^{26}P_3$ permutations of letters can be followed by *any* of the ${}^{9}P_2$ permutations of digits.
Therefore the number of possible codes is ${}^{26}P_3 \times {}^{9}P_2$

Using a similar argument for a general case we see that

when the number of permutations, P_1, of objects from one set, is followed by the number of permutations, P_2, of objects from an *independent* set, then the total number of permutations is $P_1 \times P_2$

This result applies also when combinations of objects from independent sets are combined and can be extended to cover more than two sets.

Examples 35e (continued)

2. A cellar contains thirty different bottles of wine, fifteen different cans of beer and ten different fruit juices. Six bottles of wine, ten cans of beer and three fruit juices are chosen for a party. How many different selections of drinks are possible?

The six bottles of wine can be chosen in ${}^{30}C_6$ ways.
The ten cans of beer can be chosen in ${}^{15}C_{10}$ ways.
The three fruit juices can be chosen in ${}^{5}C_3$ ways.

The choice of each type of drink is independent of the others,

$\therefore$ there are ${}^{30}C_6 \times {}^{15}C_{10} \times {}^{5}C_3$ different selections of drinks.

Mutually Exclusive Permutations or Combinations

Consider in how many ways a number greater than 20 can be made from the integers 2, 3 and 4, no integer being repeated.

The number may contain *either* two digits *or* three digits. These permutations are mutually exclusive since the number obviously cannot contain *both* two digits *and* three digits.

We can make a two-digit number in ${}^{3}P_2$ ways and a three-digit number in ${}^{3}P_3$ ways and these two cases cover all possible permutations. So there are ${}^{3}P_2 + {}^{3}P_3$ numbers greater than 20

Extending this argument to a general case we see that

when the number of permutations, P_1, of one set of objects is combined with the number of permutations, P_2, of a mutually exclusive set, then the total number of permutations is $P_1 + P_2$

This result applies equally well to combinations and also to more than two mutually exclusive sets.

When appraising a situation, care is needed to distinguish between the types of problem which involve

1) *both* one set of objects *and* another set of objects

2) *either* one set of objects *or* another set of objects.

Examples 35e (continued)

3. How many groups of six children can be chosen from a class of twenty, if the class contains one set of twins who will not be separated.

In choosing the six children we have to consider two cases, i.e.
either six children including the twins
or six children excluding the twins

These two cases are mutually exclusive as clearly the group cannot both include *and* exclude the twins.

If the twins are included, four more children must be chosen from the remaining eighteen. This can be done in ${}^{18}C_4$ ways.
If the twins are excluded, all six children must be chosen from the other eighteen. This can be done in ${}^{18}C_6$ ways.

There are no more possible combinations so the number of different groups of six children is ${}^{18}C_4 + {}^{18}C_6$

These examples continue with assorted problems which illustrate some of the varied situations in which permutations or combinations arise.

Examples 35e (continued)

4. If the three Es must be separated, how many permutations of the letters in the word DEFEATED are there?

The most direct approach is first to remove all the Es and find the number of permutations of the letters D F A T D, i.e. of five letters, two of which are the same, so there are $\frac{5!}{2!}$ permutations.

In any one of these 60 arrangements there are six spaces into which the three Es can be inserted,

i.e. $\uparrow D \uparrow F \uparrow A \uparrow T \uparrow D \uparrow$

The three positions for the Es can be chosen in ${}^6C_3$ ways, i.e. 20 ways, so there are 20 different positions of the Es in *each* of the 60 arrangements of the other letters.
Therefore there are 60×20 permutations of the given letters in which the Es are separated

Note that the arrangements of DFATD and the choices of position for the Es are independent.

---

5. A box of counters for a board game contains three red counters, two black ones, a white one and a green one. In how many ways can three counters be chosen?

There are three mutually exclusive cases here,

(a) three counters of the same colour

(b) two of the same colour and one different

(c) all three of different colours.

For case (a), i.e. all the red counters, there is only 1 choice

For case (b) we can choose *either* two red counters *or* two black counters and for each of these the third counter must be one of the other three colours. So for case (b) there are 2×3, i.e. 6 choices.

In case (c) there are four different colours to choose from so the number of combinations is ${}^4C_3$, i.e. 4

Hence the total number of ways of choosing three counters is

$$1 + 6 + 4 = 11$$

---

6. A bookshelf holds six paperbacks and twelve hardbacks. If four books are taken from the shelf, how many combinations contain at least one paperback?

The books chosen could include *either* one paperback *or* two paperbacks *or* three paperbacks *or* four paperbacks. The number of combinations in each of these mutually exclusive cases could be found and then added to give the total required. However, realising that the only combination *not* wanted is one which *does not contain a paperback*, we see that there is a much more direct approach to this problem (and to many others of the 'containing at least ...' type).

The number of ways of choosing *any* four books is ${}^{18}C_4$ [1]

The number of ways of choosing four hardbacks is ${}^{12}C_4$ [2]

The only combinations that do not contain a paperback are those in [2] so the number of combinations containing at least one paperback is given by subtracting the number in [2] from the number in [1],

i.e. $${}^{18}C_4 - {}^{12}C_4 = \frac{18!}{4!\,14!} - \frac{12!}{4!\,8!} = 2565$$

EXERCISE 35e

1. Find how many numbers between 10 and 300 can be made from the digits 1, 2, 3, if

(a) each digit may be used only once

(b) each digit may be used more than once?

2. How many combinations of three letters taken from the letters A, A, B, B, C, C, D are there?

3. A mixed team of ten players is chosen from a class of thirty, eighteen of whom are boys and twelve of whom are girls. In how many ways can this be done if the team has five boys and five girls?

4. Find the number of permutations of the letters of the word

MATHEMATICS

5. Find the number of permutations of four letters from the word

MATHEMATICS

6. How many of the permutations in Question 5 contain two pairs of letters that are the same?

7. In how many of the permutations in Question 4 do all the consonants come together?

8. A team of two pairs, each consisting of a man and a woman, is chosen to represent a club at a tennis match. If these pairs are chosen from five men and four women, in how many ways can the team be selected?

9. Two sets of books contain five novels and three reference books respectively. In how many ways can the books be arranged on a shelf

(a) if the novels and reference books are not mixed

(b) the three reference books are all separated?

10. In a multiple choice test there is one correct answer and four wrong answers to each question. For two such questions, in how many ways is it possible to select the wrong answer to both questions?

11. How many even numbers less than 500 can be made from the integers 1, 2, 3, 4, 5, each integer being used only once?

12. Four boxes each contain a large number of identical balls, those in one box are red, those in a second box are blue, those in a third box are yellow and those in the remaining box are green. In how many ways can five balls be chosen if

(a) there is no restriction (b) at least one ball is red?

13. In how many ways can three letters from the word GREEN be arranged in a row if at least one of the letters is E?

MIXED EXERCISE 35

1. Find the number of ways in which a committee of four can be chosen from six men and six women if

(a) it must contain two men and two women

(b) it must contain at least one man

(c) either the youngest man or the youngest woman, but not both, must be included?

2. The home team's results in four hockey matches are to be forecast. Each result can be a win, a draw or a loss. Find the number of different possible forecasts and show how this number is divided into forecasts containing 0, 1, 2, 3 and 4 errors respectively.

3. Find the number of ways in which twelve trees can be planted in a row so that no two of three particular trees are planted next to each other. Give the answer in factorial notation.

4. How many numbers, greater than 6000 and divisible by 3, can be made from the digits 1, 3, 5, 6 and 7 if each digit can be used only once in each number?

5. A badminton club has to select a team of three mixed pairs for a match. If eight men and seven women are eligible for selection, in how many ways can the three pairs be chosen?

6. In how many ways can twelve children be divided into two groups of seven and five respectively? In how many ways can this be done if the two oldest children must be

(a) in the same group

(b) not in the same group?

7. Find how many numbers between 20 000 and 50 000 can be made from the digits 1, 2, 3, 4 and 5 if each digit can be used

(a) only once (b) more than once in each number.

8. In how many ways can the letters of the word COMMITTEE be arranged?

In how many of these arrangements are

(a) the two Es adjacent

(b) the C and the O adjacent

(c) all the consonants together?

CHAPTER 36

THE BINOMIAL THEOREM AND OTHER POWER SERIES

POWER SERIES

A series such as $x + x^2 + x^3 + \dots$ is called a power series because the terms involve powers of a variable quantity. Series, such as those considered in Chapter 34, each of whose terms has a fixed numerical value, are called number series.

THE BINOMIAL THEOREM

We saw in Chapter 1 that when an expression such as $(1+x)^4$ is expanded, the coefficients of the terms in the expansion can be obtained from Pascal's Triangle. Now $(1+x)^{20}$ could be expanded in the same way but, as the construction of the triangular array would be tedious we need a more general method to expand powers of $(1+x)$.

Consider $(1+x)^6 \equiv (1+x)(1+x)(1+x)(1+x)(1+x)(1+x)$

When the six brackets are expanded, each term is obtained by multiplying together either x or 1 from each of the six brackets and then collecting like terms.

Taking 1 from each bracket we get 1 as a term in the expansion.

Taking x from only one bracket and 1 from the other five, we get $1 \times x$. But this can be done six times because we can choose to take x from each of the six brackets in turn, (i.e. in ${}^6C_1$ ways). So the x term in the expansion is $6x$

Taking x from any two brackets and 1 from all the remaining brackets we get $1 \times x^2$. But this can be done ${}^6C_2$ times, as the number of ways in which two brackets can be selected from six brackets is ${}^6C_2$. So the x^2 term in the expansion is ${}^6C_2x^2$

Similarly we see that the coefficients of x^3, x^4 ... are ${}^6C_3$, ${}^6C_4$ ...

Thus, arranging the expansion of $(1+x)^6$ as a series of ascending powers of x,

$$(1+x)^6 \equiv 1 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6x^6$$

$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

The RHS of this identity is called the *series expansion* of $(1+x)^6$

Notice that the expansion has 7 terms, i.e. $(6+1)$ terms.

This argument can be generalised as follows.

If n is any positive integer, $(1+x)^n$ can be expanded to give a series of terms in ascending powers of x where the term containing x^r is obtained by multiplying together xs from r brackets and 1s from the remaining brackets. There are ${}^nC_r$ ways in which r brackets can be chosen from n brackets, so the coefficient of x^r is ${}^nC_r$

$$\therefore \quad (1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n$$

This result is known as the Binomial Theorem.

The coefficients of the powers of x are called binomial coefficients.

Now the binomial coefficient of x^r is ${}^nC_r$

and $${}^nC_r = \frac{n(n-1)\dots(n-r+1)}{r!}$$

An alternative notation for the binomial coefficient of x^r is $\binom{n}{r}$

i.e. $$\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!}$$

Hence the binomial theorem states that, if n is a positive integer,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n \qquad [1]$$

$$= 1 + nx + \frac{n(n-6)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots + x^n \qquad [2]$$

Note that [1] can be written in the form $\sum_{r=0}^{n}\binom{n}{r}x^r$

Notice that

1) the expansion of $(1+x)^n$ is a finite series with $n+1$ terms,

2) the coefficient of x^r, i.e. $\dfrac{n(n-1)(n-2)\dots(n-r+1)}{r!}$, has r factors in the numerator,

3) the term containing x^2 is the third term, the term in x^3 is the fourth term, and *the term in x^r is the $(r+1)$th term*, so the rth term is $\dbinom{n}{r-1}x^{r-1}$,

4) the form of the expansion given in [1] is useful when the coefficient of a large power of x is required, or when the general term is required. The form of the expansion given in [2] is useful when the first few terms of an expansion are required.

Now consider $(a+x)^n$, where n is a positive integer.

$$(a+x)^n \equiv a^n\left(1+\frac{x}{a}\right)^n$$

Replacing x by $\dfrac{x}{a}$ in the binomial series gives

$$\begin{aligned}(a+x)^n &= a^n\left[1+\binom{n}{1}\left(\frac{x}{a}\right)+\binom{n}{2}\left(\frac{x}{a}\right)^2+\dots+\binom{n}{r}\left(\frac{x}{a}\right)^r+\dots+\binom{n}{n}\left(\frac{x}{a}\right)^n\right]\\ &= a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\dots+\binom{n}{r}a^{n-r}x^r+\dots+\binom{n}{n}x^n\\ &= a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\dots+x^n\end{aligned}$$

i.e. **if n is a positive integer**

$$(a+x)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$$

Note that this last form need not be memorised.
The expansion of any expression in the form $(a+x)^n$ can be obtained directly from $a^n\left(1+\dfrac{x}{a}\right)^n$

Examples 36a

1. Write down the first three terms in the expansion in ascending powers of x of

(a) $\left(1-\dfrac{x}{2}\right)^{10}$ (b) $(3-2x)^8$

(a) Using the result [2] above and replacing x by $-\dfrac{x}{2}$ and n by 10 we have

$$\begin{aligned}\left(1-\frac{x}{2}\right)^{10} &= 1+(10)\left(-\frac{x}{2}\right)+\frac{10\times 9}{2!}\left(-\frac{x}{2}\right)^2+\dots\\ &= 1-5x+\tfrac{45}{4}x^2+\dots\end{aligned}$$

(b) Using the result for $(a+x)^n$ and replacing a by 3, x by $-2x$ and n by 8 gives

$$(3-2x)^8 = \sum_{r=0}^{8}\binom{8}{r}(3)^{8-r}(-2x)^r$$

Therefore the first three terms of this series $(r = 0, 1, 2)$ are

$$3^8 + 8\times(3)^7(-2x) + \frac{8\times 7}{2}(3)^6(-2x)^2$$

i.e. $$3^8 - 16\times 3^7x + 112\times 3^6x^2$$

---

2. Find the fourth term in the expansion of $(a-2b)^{20}$ as a series in ascending powers of b

$$(a-2b)^{20} = \sum_{r=0}^{20}\binom{20}{r}(a)^{20-r}(-2b)^r$$

As the first term in the series is the one for which $r = 0$, the fourth term is that for which $r = 3$,

i.e. the fourth term is $$\binom{20}{3}(a)^{17}(-2b)^3 = \frac{(20)(19)(18)}{3!}(a^{17})(8b^3) = 9120a^{17}b^3$$

---

3. Write down the first three terms in the binomial expansion of

$$(1-2x)(1+\tfrac{1}{2}x)^{10}$$

The third term in the binomial expansion is the term containing x^2, so start by expanding $(1+\frac{1}{2}x)^{10}$ as far as the term in x^2

$$(1+\tfrac{1}{2}x)^{10} = 1 + (10)(\tfrac{1}{2}x) + \frac{(10)(9)}{2!}(\tfrac{1}{2}x)^2 + \dots$$

$$= 1 + 5x + \tfrac{45}{4}x^2 + \dots$$

$$\therefore \quad (1-2x)(1+\tfrac{1}{2}x)^{10} = (1-2x)(1+5x+\tfrac{45}{4}x^2+\dots)$$

$$= 1 + 5x + \tfrac{45}{4}x^2 + \dots - 2x - 10x^2 + \dots$$

$$= 1 + 3x + \tfrac{5}{4}x^2 + \dots$$

Notice that we do not write down the product of $-2x$ and $\frac{45}{4}x^2$, as terms in x^3 are not required.

EXERCISE 36a

1. Write down the first four terms in the binomial expansion of:

(a) $(1+3x)^{12}$ (b) $(1-2x)^9$ (c) $(2+x)^{10}$

(d) $\left(1-\frac{x}{3}\right)^{20}$ (e) $\left(2-\frac{3}{2}x\right)^7$ (f) $\left(\frac{3}{2}+2x\right)^9$

2. Write down the term indicated in the binomial expansion of each of the following functions.

(a) $(1-4x)^7$, 3rd term (b) $\left(1-\frac{x}{2}\right)^{20}$, 2nd term

(c) $(2-x)^{15}$, 12th term (d) $(p-2q)^{10}$, 5th term

(e) $(3a+2b)^8$, 2nd term (f) $(1-2x)^{12}$, the term in x^4

(g) $\left(2+\frac{x}{2}\right)^9$, the term in x^5 (h) $(a+b)^8$, the term in a^3

3. Write down the binomial expansion of each function as a series of ascending powers of x as far as, and including, the term in x^2.

(a) $(1+x)(1-x)^9$ (b) $(1-x)(1+2x)^{10}$

(c) $(2+x)\left(1-\frac{x}{2}\right)^{20}$ (d) $(1+x)^2(1-5x)^{14}$

USING SERIES TO FIND APPROXIMATIONS

17

Consider $(1+x)^{20}$ and its binomial expansion,

$$(1+x)^{20} = 1 + 20x + \frac{(20)(19)}{2!}x^2 + \frac{(20)(19)(18)}{3!}x^3 + \dots + x^{20}$$

This is valid for all values of x so if, for example, $x = 0.01$ we have

$$(1.01)^{20} = 1 + 20(0.01) + \frac{(20)(19)}{2!}(0.01)^2 + \frac{(20)(19)(18)}{3!}(0.01)^3 + \dots + (0.01)^{20}$$

i.e. $(1.01)^{20} = 1 + 0.2 + 0.019 + 0.001\,14 + 0.000\,048\,45 + \dots + 10^{-40}$

Because the value of x (i.e. 0.01) is small, we see that adding successive terms of the series makes progressively smaller contributions to the accuracy of $(1.01)^{20}$.
In fact, taking only the first four terms gives $(1.01)^{20} \approx 1.22014$

This approximation is correct to three decimal places as the fifth and succeeding terms do not add anything to the first four decimal places.

In general, if x is small so that successive powers of x quickly become negligible in value, then the sum of the first few terms in the expansion of $(1+x)^n$ gives an approximate value for $(1+x)^n$

The number of terms required to obtain a good approximation depends on two considerations

1) the value of x (the smaller x is, the fewer are the terms needed to obtain a good approximation).

2) the accuracy required (an answer correct to 3 s.f. needs fewer terms than an answer correct to 6 s.f.)

When finding an approximation, the binomial expansion of $(1+x)^n$ and *not* $(a+x)^n$ should be used, e.g. to find the approximate value of $(3.006)^5$ we use $3^5(1+0.002)^5$

Examples 36b

1. By substituting 0.001 for x in the expansion of $(1-x)^7$ find the value of $(1.998)^7$ correct to five significant figures.

Now $(1.998)^7 = (2-0.002)^7 = 2^7(1-0.001)^7$

$= 2^7(1-x)^7$ when $x = 0.001$

Hence

$$(1.998)^7 = 2^7\left[1 - 7(0.001) + \frac{(7)(6)}{2!}(0.001)^2 - \frac{(7)(6)(5)}{3!}(0.001)^3 + \dots\right]$$

To give an answer correct to 5 s.f. we will work to 7 s.f. so only the first three terms need be considered.

$$\therefore \quad (1.998)^7 = 128(1 - 0.007 + 0.000\,021\,0) \quad \text{to 7 s.f.}$$

$$= 127.11 \quad \text{correct to 5 s.f.}$$

In the example above, a calculator will give the value of $(1.998)^7$ to about 8 s.f. (depending on the particular calculator). If, however, the value is required to, say, 15 s.f., the method used in the worked example will give the extra accuracy.

The next worked example illustrates how a series expansion enables us to find a simple function which can be used as an approximation to a given function when x has values that are close to zero.

---

2. If x is so small that x^2 and higher powers can be neglected show that

$$(1-x)^5\left(2+\frac{x}{2}\right)^{10} \approx 2^9(2-5x)$$

Using the binomial expansion of $(1-x)^5$ and neglecting terms containing x^2 and higher powers of x we have

$$(1-x)^5 \approx 1 - 5x$$

Similarly
$$\left(2+\frac{x}{2}\right)^{10} \equiv 2^{10}\left(1+\frac{x}{4}\right)^{10}$$

$$\approx 2^{10}\left[1 + 10\left(\frac{x}{4}\right)\right]$$

Therefore
$$(1-x)^5\left(2+\frac{x}{2}\right)^{10} \approx 2^{10}(1-5x)\left(1+\frac{5x}{2}\right)$$

$$= 2^9(1-5x)(2+5x)$$

$$\approx 2^9(2-5x)$$

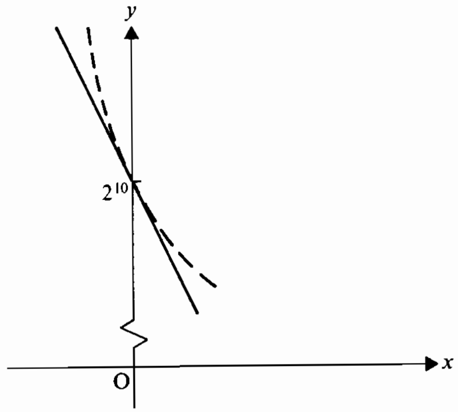
again neglecting the term in x^2

The graphical significance of the approximation in the last example is interesting.

If
$$y = (1-x)^5\left(2+\frac{x}{2}\right)^{10}$$

then, for values of x close to zero, $y \approx 2^9(2-5x)$ which is the equation of a straight line,

i.e. $y = 2^9(2-5x)$ is the tangent to $y = (1-x)^5\left(2+\frac{x}{2}\right)^{10}$ at the point where $x = 0$



Note that the function $2^9(2-5x)$ is called a *linear approximation* for the function $(1-x)^5\left(2+\frac{x}{2}\right)^{10}$ in the region where $x \approx 0$

EXERCISE 36b

1. By substituting 0.01 for x in the binomial expansion of $(1-2x)^{10}$, find the value of $(0.98)^{10}$ correct to four decimal places.

2. By substituting 0.05 for x in the binomial expansion of $\left(1+\frac{x}{5}\right)^6$, find the value of $(1.01)^6$ correct to four significant figures.

3. By using the binomial expansion of $(2+x)^7$, show that, correct to 3 d.p., $(2.08)^7 = 168.439$

4. Show that, if x is small enough for x^2 and higher powers of x to be neglected, the function $(x-2)(1+3x)^8$ has a linear approximation of $-2-47x$

5. If x is so small that x^3 and higher powers of x are negligible, show that $(2x+3)(1-2x)^{10} \approx 3-58x+500x^2$

6. By neglecting x^2 and higher powers of x find linear approximations for the following functions in the immediate neighbourhood of $x = 0$

   (a) $(1-5x)^{10}$ (b) $(2-x)^8$ (c) $(1+x)(1-x)^{20}$

EXTENDING THE BINOMIAL THEOREM

We have shown that, when n is a positive integer,

$$(1+x)^n = 1+nx+\binom{n}{2}x^2+\ldots+x^n$$

Although it cannot be proved at this stage, a very similar expansion of $(1+x)^n$ exists for *any* real value of n, i.e.

$$(1+x)^n \equiv 1+nx+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots \quad \textbf{for any real value of } n$$

provided that $-1 < x < 1$

Notice that when n is not a positive integer, the binomial expansion of $(1+x)^n$ does not terminate but carries on to infinity.

Notice also that the expansion is valid only if x is in the range $-1 < x < 1$ and this *range must always be stated.*

Finally it should be noted that the expansion is *not* valid for $(a+x)^n$.

To expand $(a+x)^n$ it must first be written in the form $a^n\left(1+\dfrac{x}{a}\right)^n$

and this expansion is valid only for $-1 < \dfrac{x}{a} < 1$

Although we cannot prove the binomial expansion of $(1+x)^n$, it can be verified when $n = -1$, as follows.

Consider the series $1-x+x^2-x^3+x^4-\ldots$

This is an infinite GP whose common ratio is $-x$

Therefore, provided that $|x| < 1$, the series converges

and the sum to infinity is $\dfrac{1}{1-(-x)} = (1+x)^{-1}$

Now consider the binomial expansion of $(1+x)^{-1}$

$$(1+x)^{-1} = 1+(-1)(x)+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3+\frac{(-1)(-2)(-3)(-4)}{4!}x^4+\ldots$$

$$= 1-x+x^2-x^3+x^4-\ldots$$

and we have shown that the sum to infinity of this series is $(1+x)^{-1}$

This series occurs frequently and is worth memorising. So also is the series obtained by replacing x by $-x$, i.e.

$$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$$

Examples 36c

1. Expand each of the following functions as a series of ascending powers of x up to and including the term in x^3 stating the set of values of x for which each expansion is valid.

   (a) $(1+x)^{1/2}$ (b) $(1-2x)^{-3}$ (c) $(2-x)^{-2}$

For $|x| < 1$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots \qquad [1]$$

(a) Replacing n by $\frac{1}{2}$ in [1] gives

$$(1+x)^{1/2} = 1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3+\ldots$$

$$= 1+\tfrac{1}{2}x+\frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\ldots$$

$$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots \quad \text{for } |x| < 1$$

(b) Replacing n by -3 and x by $-2x$ in [1] gives

$$(1-2x)^{-3} = 1+(-3)(-2x)+\frac{(-3)(-4)}{2!}(-2x)^2+\frac{(-3)(-4)(-5)}{3!}(-2x)^3+\ldots$$

$$= 1+6x+24x^2+80x^3+\ldots$$

provided that $-1 < -2x < 1$, i.e. $\frac{1}{2} > x > -\frac{1}{2}$

(c) $(2-x)^{-2} = 2^{-2}(1-\frac{1}{2}x)^{-2}$

Replacing n by -2 and x by $-\frac{1}{2}x$ in [1] gives

$$(2-x)^{-2} = \frac{1}{4}\left[1 + (-2)(-\tfrac{1}{2}x) + \frac{(-2)(-3)}{2!}(-\tfrac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3!}(-\tfrac{1}{2}x)^3 + \ldots\right]$$

$$= \tfrac{1}{4}(1 + x + \tfrac{3}{4}x^2 + \tfrac{1}{2}x^3 + \ldots)$$

$$= \tfrac{1}{4} + \tfrac{1}{4}x + \tfrac{3}{16}x^2 + \tfrac{1}{8}x^3 + \ldots$$

The expansion of $(1-\frac{1}{2}x)^{-2}$ is valid for $-1 < -\frac{1}{2}x < 1$,
i.e. for $2 > x > -2$

Therefore the expansion of $(2-x)^{-2}$ also is valid for $2 > x > -2$

---

2. Expand $\dfrac{5}{(1+3x)(1-2x)}$ as a series of ascending powers of x giving the first four terms and the range of values of x for which the expansion is valid.

Expressing $\dfrac{5}{(1+3x)(1-2x)}$ in partial fractions gives

$$\frac{5}{(1+3x)(1-2x)} = \frac{3}{(1+3x)} + \frac{2}{(1-2x)} = 3(1+3x)^{-1} + 2(1-2x)^{-1}$$

Now $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ for $-1 < x < 1$

Replacing x by $3x$ gives

$$(1+3x)^{-1} = 1 - 3x + (3x)^2 - (3x)^3 + \ldots$$

$$= 1 - 3x + 9x^2 - 27x^3 + \ldots \quad \text{for } -1 < 3x < 1$$

Also $(1-x)^{-1} = 1 + x + x^2 + \ldots$ and replacing x by $2x$ gives

$$(1-2x)^{-1} = 1 + (2x) + (2x)^2 + (2x)^3 + \ldots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \ldots \quad \text{for } -1 < -2x < 1$$

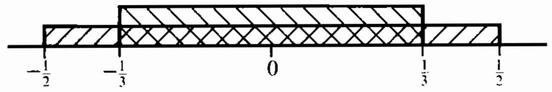
Hence $\dfrac{5}{(1+3x)(1-2x)} = 3(1+3x)^{-1} + 2(1-2x)^{-1}$

$$= (3+2) + (-9+4)x + (27+8)x^2 + (-81+16)x^3 + \ldots$$

provided that $-\frac{1}{3} < x < \frac{1}{3}$ *and* $-\frac{1}{2} < x < \frac{1}{2}$

Therefore the first four terms of the series are $5 - 5x + 35x^2 - 65x^3$

The expansion is valid for the range of values of x satisfying both $-\frac{1}{3} < x < \frac{1}{3}$ *and* $-\frac{1}{2} < x < \frac{1}{2}$



i.e. for $-\frac{1}{3} < x < \frac{1}{3}$

---

3. Expand $\sqrt{\left(\dfrac{1+x}{1-2x}\right)}$ as a series of ascending powers of x up to and including the term containing x^2

$$\sqrt{\left(\frac{1+x}{1-2x}\right)} \equiv (1+x)^{1/2}(1-2x)^{-1/2}$$

Now $(1+x)^{1/2} = \left[1 + \frac{1}{2}x + \dfrac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \ldots\right]$ for $-1 < x < 1$

and $(1-2x)^{-1/2} = \left[1 + (-\frac{1}{2})(-2x) + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-2x)^2 + \ldots\right]$
for $-1 < 2x < 1$

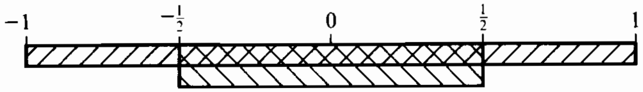
Hence $\sqrt{\left(\dfrac{1+x}{1-2x}\right)} \equiv (1+x)^{1/2}(1-2x)^{-1/2}$

$$= (1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \ldots)(1 + x + \tfrac{3}{2}x^2 + \ldots)$$

$$= 1 + (\tfrac{1}{2}x + x) + (\tfrac{1}{2}x^2 - \tfrac{1}{8}x^2 + \tfrac{3}{2}x^2) + \ldots$$

$$= 1 + \tfrac{3}{2}x + \tfrac{15}{8}x^2 + \ldots$$

provided that $-1 < x < 1$ *and* $-\frac{1}{2} < x < \frac{1}{2}$,



i.e. $-\frac{1}{2} < x < \frac{1}{2}$

It is interesting to compare the methods used in the last two examples.

In Example 2, the function is expressed as the sum of two binomials and the series is obtained by adding two binomial expansions.

In Example 3 the function is expressed as a product of two binomials and the series is obtained by multiplying two binomial expansions.

The first method has the advantage that it is very much easier to add the terms of two series than it is to multiply them.

Therefore, *whenever possible, a compound function should be expressed as a sum of simpler functions before it is expanded as a series* and, when this is not possible, a compound function should be expressed as a product of simpler functions.

Further Approximations

17 We have seen how a series can be used to find an approximate value of a rational number without having to calculate its exact value. The next example illustrates how a series can be used to find the decimal value, to any required degree of accuracy, of an irrational quantity.

---

4. Use the expansion of $(1-x)^{1/2}$ with $x = 0.02$ to find the decimal value of $\sqrt{2}$ correct to nine decimal places.

$$(1-x)^{1/2} = 1 - \tfrac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-x)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(-x)^4 + \dots$$

$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \tfrac{7}{256}x^5 - \dots$$

This is valid for $-1 < 1$ and so is valid when $x = 0.02$

Replacing x by 0.02 gives

$$(0.98)^{1/2} = 1 - 0.01 - 0.000\,05 - 0.000\,000\,5 - 0.000\,000\,006\,25 - 0.000\,000\,000\,087\,5 - \dots$$

The next term in the series is 1.3125×10^{-12} and as this does not contribute to the first ten decimal places we do not need it, or any further terms.

i.e. $\sqrt{\frac{98}{100}} = 0.989\,949\,493\,7$ to 10 d.p.

$\Rightarrow$ $\frac{7}{10}\sqrt{2} = 0.989\,949\,493\,7$ to 10 d.p.

$\therefore$ $\sqrt{2} = 1.414\,213\,562$ correct to 9 d.p.

EXERCISE 36c

Expand the following functions as series of ascending powers of x up to and including the term in x^3. In each case give the range of values of x for which the expansion is valid.

1. $(1-2x)^{1/2}$
2. $(3+x)^{-1}$
3. $\left(1+\dfrac{x}{2}\right)^{-1/2}$
4. $\dfrac{1}{(1-x)^2}$
5. $\sqrt{\left(\dfrac{1}{1+x}\right)}$
6. $(1+x)\sqrt{(1-x)}$
7. $\dfrac{x+2}{x-1}$
8. $\dfrac{2-x}{\sqrt{(1-3x)}}$
9. $\dfrac{1}{(2-x)(1+2x)}$
10. $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$
11. $\left(1+\dfrac{x^2}{9}\right)^{-1}$
12. $\left(1+\dfrac{1}{x}\right)^{-1}$ $\left[\textit{Hint.}\ \left(1+\dfrac{1}{x}\right)^{-1} \equiv \left(\dfrac{x+1}{x}\right)^{-1} \equiv \dfrac{x}{1+x}\right]$

13. Expand $\left(1+\dfrac{1}{p}\right)^{-3}$ as a series of descending powers of p, as far as and including the term containing p^{-4}. State the range of values of p for which the expansion is valid.
(*Hint.* Replace x by $\frac{1}{p}$ in $(1+x)^{-3}$)

14. By substituting 0.08 for x in $(1+x)^{1/2}$ and its expansion find $\sqrt{3}$ correct to four significant figures.

15. By substituting $\frac{1}{10}$ for x in $(1-x)^{-1/2}$ and its expansion find $\sqrt{10}$ correct to six significant figures.

16. Expand $\sqrt{\left(\dfrac{1+2x}{1-2x}\right)}$ as a series of ascending powers of x up to and including the term in x^2

17. If x is so small that x^2 and higher powers of x may be neglected show that $\dfrac{1}{(x-1)(x+2)} \approx -\frac{1}{2} - \frac{1}{4}x$

18. By neglecting x^3 and higher powers of x, find a quadratic function that approximates to the function $\dfrac{1-2x}{\sqrt{(1+2x)}}$ in the region close to $x = 0$

19. Find a quadratic function that approximates to

$$f(x) = \frac{1}{\sqrt[3]{(1-3x)^2}}$$

for values of x close to zero.

20. Use partial fractions and the binomial series to find a linear approximation for

$$\frac{3}{(1-2x)(2-x)}$$

21. If terms containing x^4 and higher powers of x can be neglected, show that

$$\frac{2}{(x+1)(x^2+1)} \approx 2(1-x)$$

22. Show that

$$\frac{12}{(3+x)(1-x)^2} \approx 4 + \tfrac{20}{3}x + \tfrac{88}{9}x^2$$

provided that x is small enough to neglect powers higher than 2

23. If x is very small, find a cubic approximation for

$$\frac{1}{(3-x)^3}$$

SERIES EXPANSIONS OF EXPONENTIAL FUNCTIONS

We have shown that functions such as $(1+x)^n$ can be expressed as infinite series of ascending powers of x and there are many other functions that can be expressed as power series.

In this section we give, without proof, the series expansion of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for } \textit{all} \text{ real values of } x$$

Note that this series is infinite and valid for any value of x

The following worked examples show how the series for e^x can be used to find the series expansion of compound exponential functions. The principles governing the expansion of compound binomial functions apply to any series expansion, i.e. whenever possible a compound function should be expressed as a *sum* of simpler functions and, when this is not possible, it should be expressed as a product of simpler functions.

Examples 36d

1. Expand each of the following functions as a series of ascending powers of x up to and including the term in x^3

(a) e^{3x} (b) $\dfrac{e^x - 1}{e^x}$ (c) e^{x-2} (d) $(1+x)e^x$

(a) Replacing x by $3x$ in the expansion of e^x gives

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$$

$$= 1 + 3x + \tfrac{9}{2}x^2 + \tfrac{9}{2}x^3 + \ldots$$

(b) $\dfrac{e^x - 1}{e^x} = \dfrac{e^x}{e^x} - \dfrac{1}{e^x} = 1 - e^{-x}$

Replacing x by $-x$ in the series for e^x gives

$$\frac{e^x - 1}{e^x} = 1 - \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\right)$$

$$= x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \ldots$$

(c) $e^{x-2} = \dfrac{e^x}{e^2} = e^{-2}e^x = e^{-2}\left(1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \ldots\right)$

$$= e^{-2} + e^{-2}x + \tfrac{1}{2}e^{-2}x^2 + \tfrac{1}{6}e^{-2}x^3 + \ldots$$

(d) $(1+x)e^x = e^x + xe^x$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$+ \qquad x + x^2 + \frac{x^3}{2!} + \ldots$$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \tfrac{2}{3}x^3 + \ldots$$

2. Show that, if x is small enough for x^2 and higher powers of x to be ignored, $e^x + e^{2x} \approx 2 + 3x$

$$e^x + e^{2x} = \left[1 + x + \frac{x^2}{2!} + \dots\right] + \left[1 + 2x + \frac{(2x)^2}{2!} + \dots\right]$$

$$= 2 + 3x + \dots$$

$$\approx 2 + 3x \quad \text{when } x \text{ is small.}$$

Note that $2 + 3x$ is a linear approximation to $e^x + e^{2x}$ and further, the line $y = 2 + 3x$ is the tangent to the curve $y = e^x + e^{2x}$ at the point where $x = 0$

EXERCISE 36d

Find the first three terms of the series expansion of each of the following functions.

1. e^{2x}
2. e^{-2x}
3. $e^{x/2}$
4. $\sqrt{(e^{3x})}$
5. $\dfrac{e^x - e^{2x}}{e^x}$
6. $\dfrac{1 - e^{2x}}{1 - e^x}$
7. $\dfrac{x - e^x}{e^x}$
8. $\dfrac{e^x}{1 + x}$
9. Using the series expansion of e^x with $x = \frac{1}{2}$, find the value of $\sqrt{e}$ correct to five decimal places.

SERIES EXPANSIONS OF LOGARITHMIC FUNCTIONS

The function given by $\ln(1 + x)$ can be expanded as a series of ascending powers of x, and again we quote this expansion without proof.

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{provided that } -1 < x \leqslant 1$$

Replacing x by $-x$ gives

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad \text{provided that } -1 \leqslant x < 1$$

Notice that both series are infinite and that they are valid for different ranges of values of x

Example 36e

Find the first four terms in the expansion of $\ln \dfrac{1 - 2x}{(1 + 2x)^2}$ as a series of ascending powers of x

$$\ln \frac{1 - 2x}{(1 + 2x)^2} = \ln(1 - 2x) - \ln(1 + 2x)^2 = \ln(1 - 2x) - 2\ln(1 + 2x)$$

Using the expansion of $\ln(1 - x)$ and replacing x by $2x$ gives

$$\ln(1 - 2x) = -(2x) - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \dots \quad \text{for } -1 \leqslant 2x < 1$$

Similarly

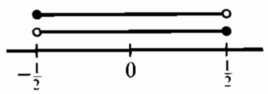
$$\ln(1 + 2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots \quad \text{for } -1 < 2x \leqslant 1$$

Hence

$$\ln \frac{1 - 2x}{(1 + 2x)^2} = \left[-(2x) - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \dots\right] - 2\left[(2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots\right]$$

$$= -6x + 2x^2 - 8x^3 + 4x^4 + \dots$$

provided that $-\frac{1}{2} < x \leqslant \frac{1}{2}$ *and* $-\frac{1}{2} \leqslant x < \frac{1}{2}$



i.e. provided that $-\frac{1}{2} < x < \frac{1}{2}$

EXERCISE 36e

For each function find the first three terms in the expansion as a series of ascending powers of x, stating the range of values of x for which it is valid.

1. $\ln(1 + 3x)$
2. $\ln(1 + x)^2$
3. $\ln(1 - \frac{1}{2}x)$
4. $\ln(1 - x^2)$
5. $\ln\sqrt{(1 - 2x)}$
6. $x\ln(1 + x)$
7. $(1 + x)\ln(1 + x)$
8. $\ln\sqrt{\left(\dfrac{1 - x}{1 + x}\right)}$
9. $\ln(1 + x)e^x$

MIXED EXERCISE 36

1. Find the first three terms and the last term in the expansion of $(1 + 2x)^9$ as a series of ascending powers of x. For what values of x is this expansion valid?

2. Expand $\dfrac{1}{1+2x}$ as a series of ascending powers of x, giving the first three terms and the range of values of x for which the expansion is valid.

3. Find the first three terms in the expansion of $\dfrac{x}{1-x}$ as a series of ascending powers of x giving the range of values of x for which the expansion is valid.

4. Express $f(x) = \dfrac{1+x}{1-2x}$ as a series of ascending powers of x, as far as the term in x^2, giving the values of x for which the series converges to $f(x)$.

5. Express $f(x) = \dfrac{1}{(1+x)(1-2x)}$ in partial fractions. Hence find a quadratic function which is approximately equal to $f(x)$ when x is small enough for powers of x greater than x^2 to be ignored.

6. Find the coefficient of x^2 when $\left(\dfrac{1+x}{1-x}\right)^2$ is expanded as a series of ascending powers of x

7. Show that $\dfrac{e^{2x}-1}{e^x} \approx 2x$ if x is small enough for x^2 and higher powers of x to be ignored.

8. Find the first two terms in the expansion of $e^x + \ln(1-x)$ as a series of ascending powers of x

9. Show that $2^x = e^{x \ln 2}$. Hence find the first three terms in the expansion of 2^x as a series of ascending powers of x. Use your series with $x = \frac{1}{3}$ to find an approximate value for $\sqrt[3]{2}$

10. (a) Expand $\dfrac{1}{\sqrt{(1-3x)}}$ as a series of ascending powers of x, giving the first three terms and the range of values of x for which the series converges.
    (b) Use the expansion in (a) to find an approximate value for $\sqrt{2}$

11. When $\dfrac{1}{(1-ax)^2} - \dfrac{1}{\sqrt{(1-4x)}}$ is expanded as a series of ascending powers of x, the first term is $-3x^2$
    (a) Find the value of a
    (b) Find the second term of the series.

CHAPTER 37

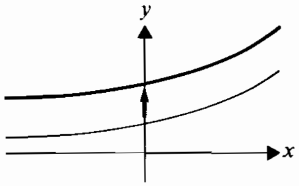
CURVE SKETCHING

In this chapter we start by looking at some methods for sketching curves without recourse to a graphics calculator.

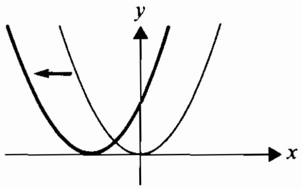
TRANSFORMATIONS

The graphs of many functions can be obtained from transformations of the curves representing basic functions. These transformations are introduced in Chapters 12 and 16. We revise them briefly here.

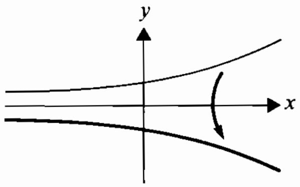
$y = f(x) + c$ is a translation of $y = f(x)$ by c units in the direction Oy



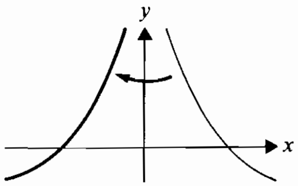
$y = f(x + c)$ is a translation of $y = f(x)$ by $-c$ units in the direction Ox



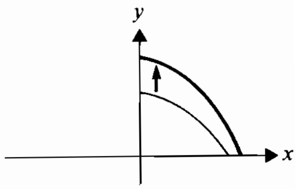
$y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis.



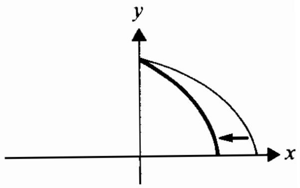
$y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.



$y = af(x)$ is a one-way stretch of $y = f(x)$ by a factor a parallel to the y-axis.



$y = f(ax)$ is a one-way stretch of $y = f(x)$ by a factor $\frac{1}{a}$ parallel to the x-axis.

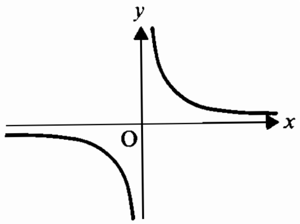


Now consider the curve $y = \dfrac{x}{x+1}$

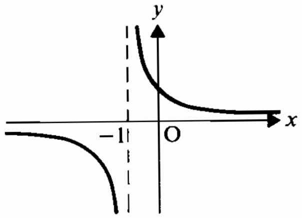
It looks as if $\dfrac{x}{x+1}$ is related to $f(x) = \dfrac{1}{x}$ and we see that we can write $y = \dfrac{x}{x+1} = 1 - \dfrac{1}{x+1}$, i.e. $y = 1 - f(x+1)$

We can now build up a picture of the curve $y = \dfrac{x}{x+1}$ in stages, with rough sketches.

1) Sketch $f(x) = \dfrac{1}{x}$

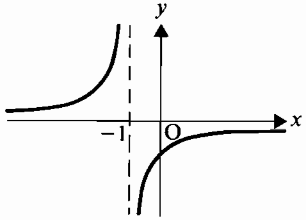


2) Sketch $g(x) = \dfrac{1}{x+1} = f(x+1)$
($f(x)$ moved one unit to the left.)



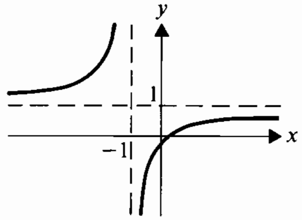
3) Sketch $h(x) = -\dfrac{1}{x+1} = -g(x)$

($g(x)$ reflected in the x-axis.)



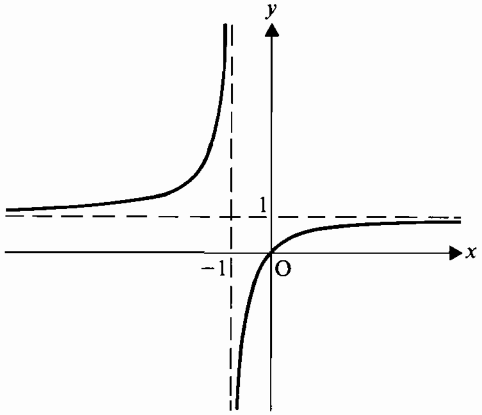
4) Sketch $y = 1 - \dfrac{1}{x+1} = 1 + h(x)$

($h(x)$ moved up one unit.)



From the last sketch, we see that the asymptotes to the curve are $y = 1$ and $x = -1$

From the equation of the curve, $y = 0$ when $x = 0$, so the curve goes through the origin and there are no other intercepts on the axes. We can now draw a more accurate sketch.



In general, a sketch graph should clearly show the following features of a curve: asymptotes, intercepts on the axes and turning points when they exist.

EXERCISE 37a

1. Find the values of a and b such that $x^2 - 4x + 1 \equiv (x-a)^2 + b$. On the same set of axes sketch the curves $y = x^2$ and $y = x^2 - 4x + 1$

2. For $-2\pi < x < 2\pi$ and on the same set of axes sketch the graphs of

(a) $y = \cos x$ and $y = 3\cos x$

(b) $y = \sin x$ and $y = \sin 2x$

(c) $y = \cos x$ and $y = \cos(x - \frac{1}{6}\pi)$

(d) $y = \sin x$ and $y = 2\sin(\frac{1}{6}\pi - x)$

3. On the same set of axes sketch the graphs of $y = 1/x$, $y = 3/x$ and $y = 1/3x$

4. Show that $\dfrac{x-2}{x-3} \equiv 1 + \dfrac{1}{x-3}$. Sketch the curve $y = \dfrac{x-2}{x-3}$ clearly showing the asymptotes and the intercepts on the axes.

Sketch the following curves clearly showing any asymptotes, turning points and intercepts on the axes.

5. $y = \dfrac{1}{x-1}$

6. $y = 1 - 2\sin x$

7. $y = 1 - (x-2)^2$

8. $y = \dfrac{1}{1-x}$

9. $y = \dfrac{1-x}{x}$

10. $y = 1 - x^3$

11. $y = \dfrac{2x+1}{x}$

12. $y = \dfrac{x+1}{x-1}$

13. $y = 3 - (x-2)^2$

14. $y = 2\sin(x - \frac{1}{3}\pi)$

15. $y = 3\cos(x + \frac{1}{6}\pi)$

16. $y = \dfrac{1+2x}{1-x}$

17. $y = 3x^3 - 4$

18. $y = 3 - 2x^4$

19. $y = 3 - (x + 2)^3$

20. $y = (x - 1)^4$

RECIPROCAL CURVES

SG

Consider the curve $y = \dfrac{1}{f(x)}$ when the graph of $f(x)$ is known.

The following simple properties of reciprocals enable the graph of $1/f(x)$ to be deduced from the graph of $f(x)$.

1) For a given value of x, $f(x)$ and $1/f(x)$ both have the same sign.

2) When the value of $f(x)$ is increasing the value of $1/f(x)$ is decreasing, i.e. when $\dfrac{d}{dx}f(x)$ is positive, $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right)$ is negative and conversely.

3) If $f(x) = 1$ then $\dfrac{1}{f(x)} = 1$ also.

Similarly when $f(x) = -1$, $\dfrac{1}{f(x)} = -1$

4) If $f(x) \to \infty$ then $\dfrac{1}{f(x)} \to 0$ from above,

and if $f(x) \to -\infty$ then $\dfrac{1}{f(x)} \to 0$ from below.

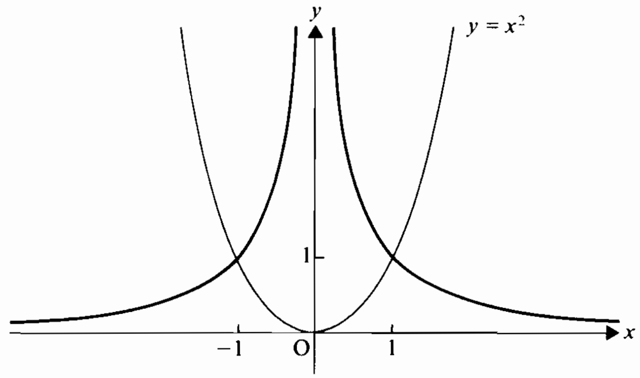
5) If $f(x) \to 0$ from above then $\dfrac{1}{f(x)} \to \infty$,

and if $f(x) \to 0$ from below then $\dfrac{1}{f(x)} \to -\infty$

6) If $f(x)$ has a maximum value then $\dfrac{1}{f(x)}$ has a minimum value, and conversely.

Examples 37b

1. Use the curve $y = x^2$ to sketch the curve whose equation is $y = 1/x^2$



From the sketch of $y = x^2$ we see that

(a) $x^2 > 0$ for all values of x, so $1/x^2 > 0$ for all values of x;

(b) when $x > 0$, x^2 is increasing so $1/x^2$ is decreasing, and when $x < 0$, x^2 is decreasing so $1/x^2$ is increasing;

(c) when $x \to \pm\infty$, $x^2 \to \infty$ so $1/x^2 \to 0$
and when $x \to 0$, $x^2 \to 0$ so $1/x^2 \to \infty$

Notice that when $x = 0$, $1/x^2$ is undefined so the fact that $f(x) = x^2$ has a minimum value when $x = 0$ has no relevance in this case.

From this information the graph of $y = 1/x^2$ can be drawn.

---

2. Sketch the curve $y = \dfrac{1}{x^2 + 2}$

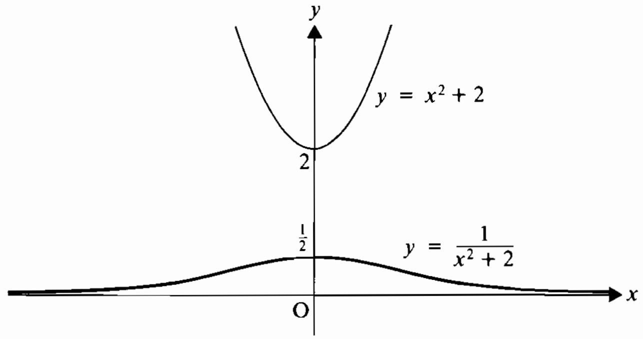
Sketching the graph of $f(x) = x^2 + 2$ and then making the following observations enables the curve $y = \dfrac{1}{f(x)}$ to be drawn.

(a) $f(x) > 0$ for all values of x, therefore so is $1/f(x)$;

(b) for $x < 0$, $f'(x)$ is negative so the gradient of $1/f(x)$ is positive;
for $x > 0$, $f'(x)$ is positive so the gradient of $1/f(x)$ is negative;

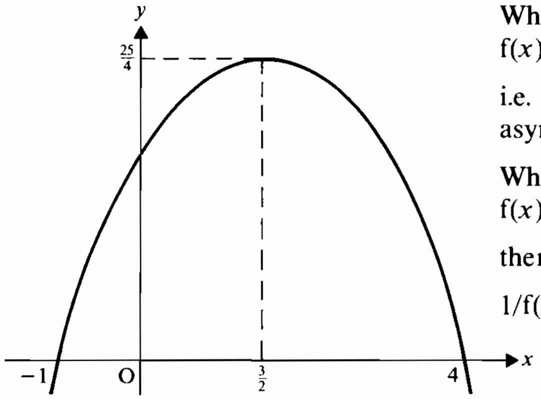
(c) when $x \to \pm\infty$, $f(x) \to \infty$ so $1/f(x) \to 0$;

(d) when $x = 0$, $f(x)$ has a minimum value of 2 so $1/f(x)$ has a maximum value of $\frac{1}{2}$



3. Sketch the curve $y = \dfrac{1}{(1+x)(4-x)}$ clearly showing asymptotes and any turning points.

First we sketch the graph of $f(x) = (1+x)(4-x)$



When $x = -1$ and 4,
$f(x) = 0 \Rightarrow 1/f(x) \to \infty$

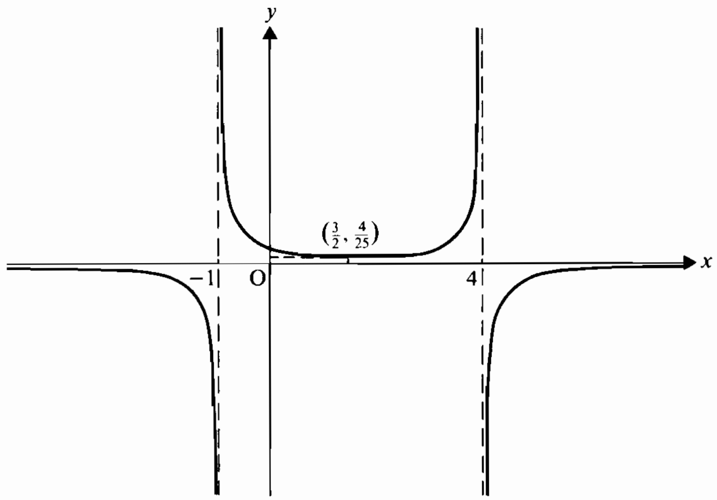
i.e. $x = -1$ and $x = 4$ are asymptotes to $y = 1/f(x)$

Where $x = \frac{3}{2}$,
$f(x)$ has a maximum value of $\frac{25}{4}$

therefore

$1/f(x)$ has a minimum value of $\frac{4}{25}$

Remembering that when $f(x)$ is +ve, $1/f(x)$ is positive also and conversely, and noting that as $x \to \pm\infty$, $f(x) \to -\infty$, so $1/f(x) \to 0$ from below, we can now sketch the curve $y = \dfrac{1}{(1+x)(4-x)}$



EXERCISE 37b

Sketch each curve, clearly showing any asymptotes, turning points and intercepts on the axes.

1. $y = \dfrac{1}{x^2 - 3x + 2}$

2. $y = \dfrac{1}{x^2 - 4}$

3. $y = \dfrac{1}{\sin x}$

4. $y = \dfrac{1}{1 - x^2}$

5. $y = \dfrac{1}{\ln x}$

6. $y = \dfrac{1}{x^3}$

7. $y = \dfrac{1}{x^3 - 1}$

8. $y = \dfrac{1}{x(x-1)(x-2)}$

9. $y = 1 - \dfrac{1}{x^2}$

10. $y = \dfrac{x^2}{x^2 - 1}$

11. $y = \dfrac{1}{e^x - 1}$

12. $y = 1 - \dfrac{1}{x^3 + 5}$

Next we look at some functions with interesting properties.

EVEN FUNCTIONS

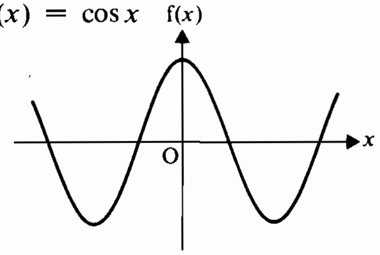
A function is even if $f(x) = f(-x)$

Since the curve $y = f(-x)$ is the reflection of the curve $y = f(x)$ in the y-axis, it follows that

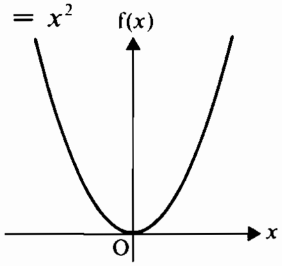
when $f(x)$ is an even function,
the curve $y = f(x)$ is symmetrical about Oy

Some familiar even functions and their graphs are shown below.

$f(x) = \cos x$



$f(x) = x^2$



ODD FUNCTIONS

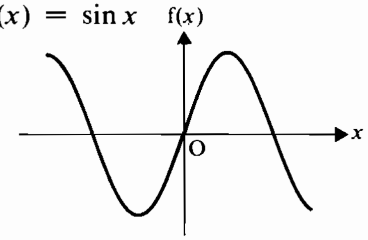
A function is odd if $f(x) = -f(-x)$

As the curve $y = -f(-x)$ is a reflection of the curve $y = f(x)$ in Oy followed by a reflection in Ox, it follows that

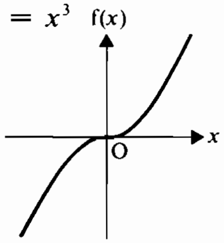
when $f(x) = -f(-x)$ the curve $y = f(x)$ has rotational symmetry of order 2 about the origin.

Some familiar odd functions and their graphs are shown below.

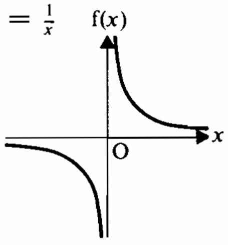
$f(x) = \sin x$



$f(x) = x^3$



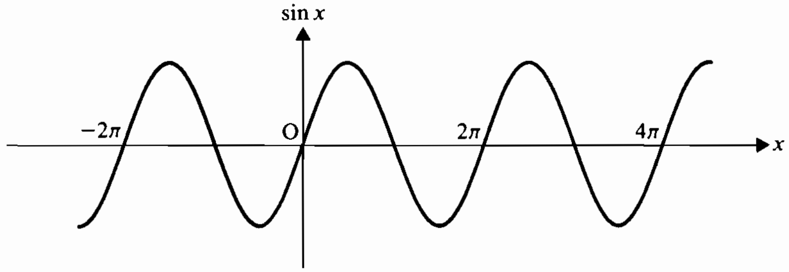
$f(x) = \frac{1}{x}$



PERIODIC FUNCTIONS

A function whose graph consists of a basic pattern which repeats at regular intervals is called a *periodic* function. The width of the basic pattern is the *period* of the function.

$f(x) = \sin x$, for example, is periodic and its period is 2π



If $f(x)$ is periodic with period a, then it follows that

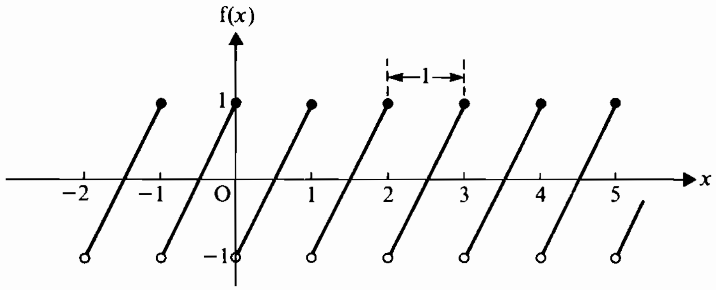
$$f(x + a) = f(x) \quad \text{for all values of } x$$

Therefore the definition of $f(x)$ within one period (e.g. for $0 < x \leqslant a$) together with the definition $f(x + a) = f(x)$ for $x \in \mathbb{R}$, defines a periodic function.

For example, if $\begin{cases} f(x) = 2x - 1 & \text{for } 0 < x \leqslant 1 \\ f(x+1) = f(x) & \text{for all values of } x \end{cases}$

then we know that the function is periodic with a period of 1

The graph of this function can be sketched by drawing $f(x) = 2x - 1$ for $0 < x \leqslant 1$ and repeating the pattern at unit intervals in either direction.



The basic pattern in the graph of a periodic function can be made up of two or more different definitions. The next worked example illustrates such a compound periodic function.

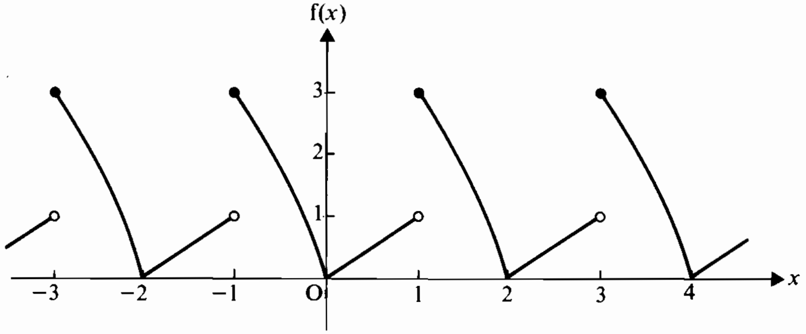
Example 37c

Sketch the graph of the function f defined by

$$\begin{aligned} f(x) &= x && \text{for } 0 \leqslant x < 1 \\ f(x) &= 4 - x^2 && \text{for } 1 \leqslant x < 2 \\ f(x+2) &= f(x) && \text{for all real values of } x \end{aligned}$$

From the last line of the definition, f is periodic with period 2

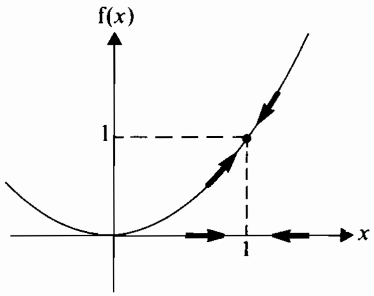
The graph of this function is built up by first drawing $f(x) = x$ in the interval $0 \leqslant x < 1$, then drawing $f(x) = 4 - x^2$ in the interval $1 \leqslant x < 2$. This pattern is then repeated every 2 units along the x-axis.



CONTINUOUS FUNCTIONS

A function f is continuous at $x = a$ if $f(a)$ is defined *and* if $f(x) \to f(a)$ as $x \to a$ from above and from below.

For example, $f(x) = x^2$ is continuous at $x = 1$ because $f(1)$ is defined and is equal to 1 and $f(x) \to f(1)$ as $x \to 1$ from above and below.

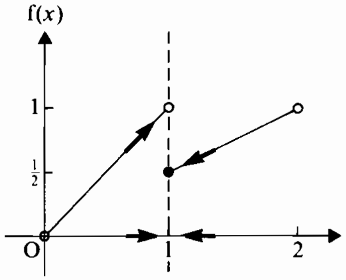


Now consider the function f defined by

$$f(x) = x \quad \text{for } 0 \leqslant x < 1$$

and $\quad f(x) = \frac{1}{2}x \quad \text{for } 1 \leqslant x < 2$

When $x = 1$, $f(1)$ is defined and is equal to $\frac{1}{2}$



but

as $x \to 1$ from below, $f(x) \to 1$
as $x \to 1$ from above, $f(x) \to \frac{1}{2}$
so $f(x)$ is not continuous at $x = 1$

A *continuous function* satisfies the conditions for continuity at all values of x in its domain.

EXERCISE 37c

1. Sketch each function and state whether it is even, odd and/or periodic.
   (a) $\cos x$ (b) $\tan x$ (c) e^{-x} (d) $\ln(1+x)$
   (e) $\cot x$ (f) $(x-1)^2$ (g) $x - 1$ (h) $1/x$

2. Sketch the graph of $f(x)$ within the interval $-4 < x \leqslant 6$ if
   $$f(x) = 4 - x^2 \quad \text{for } 0 < x \leqslant 2$$
   and $\quad f(x) = f(x-2) \quad$ for all values of x

3. If
   $$\begin{aligned} f(\theta) &= \sin\theta && \text{for } 0 < x \leqslant \tfrac{1}{2}\pi \\ f(\theta) &= \cos\theta && \text{for } \tfrac{1}{2}\pi < x \leqslant \pi \end{aligned}$$
   and $\quad f(\theta + \pi) = f(\theta) \quad$ for all values of θ
   sketch the function $f(\theta)$ for the range $-2\pi < \theta \leqslant 2\pi$

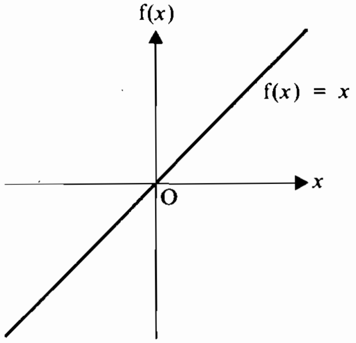
4. A function $f(x)$ is periodic with a period of 4. Sketch the graph of the function for $-6 \leqslant x \leqslant 6$, given that
   $$\begin{aligned} f(x) &= -x && \text{for } 0 < x \leqslant 3 \\ f(x) &= 3x - 12 && \text{for } 3 < x \leqslant 4 \end{aligned}$$

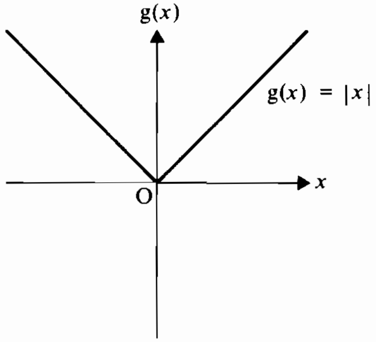
THE MODULUS OF A FUNCTION

SG

When $f(x) = x$, $f(x)$ is negative when x is negative.
But if $g(x) = |x|$, g takes the *positive* numerical value of x, e.g. when $x = -3$, $|x| = 3$, so $g(x)$ is always positive.
Therefore the graph of $g(x) = |x|$ can be obtained from the graph of $f(x) = x$ by changing, to the equivalent positive values, the part of the graph of $f(x) = x$ for which $f(x)$ is negative.

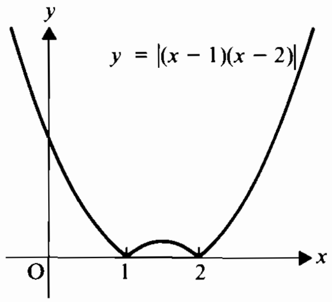
Thus for negative values of $f(x)$, the graph of $g(x) = |x|$ is the reflection of $f(x) = x$ in the y-axis.

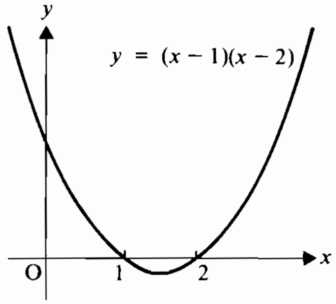




In general, the curve C_1 whose equation is $y = |f(x)|$ is obtained from the curve C_2 with equation $y = f(x)$, by reflecting in the x-axis the parts of C_2 for which $f(x)$ is negative. The remaining sections of C_1 are not changed.

For example, to sketch $y = |(x-1)(x-2)|$ we start by sketching the curve $y = (x-1)(x-2)$. We then reflect in the x-axis the part of this curve which is below the x-axis.





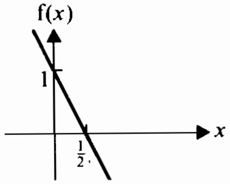
Note that for any function f, the mapping $x \to |f(x)|$ is also a function.

Example 37d

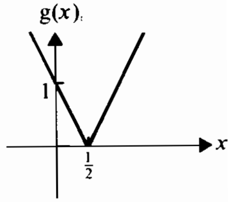
Sketch the graph of $y = 3 - |1 - 2x|$.

We can use transformations to build up the picture in stages.

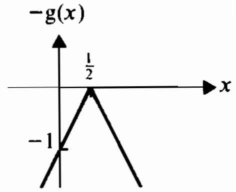
1. Draw $f(x) = 1 - 2x$



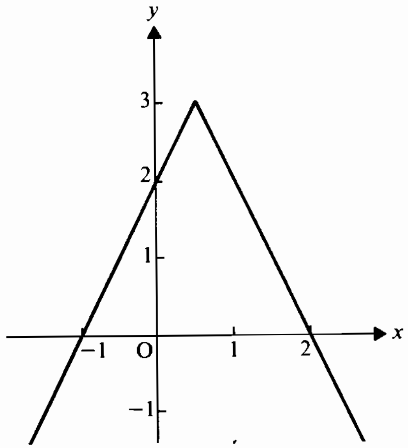
2. Draw $g(x) = |1 - 2x|$



3. Draw $-|1 - 2x|$
   ($-g(x)$ is the reflection of $g(x)$ in Ox.)



4. $y = 3 - |1 - 2x|$ is the graph of $-g(x)$ translated 3 units upwards.



EXERCISE 37d

Sketch the following graphs.

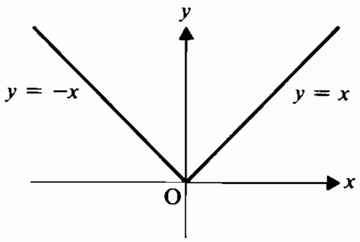
1. $y = |2x-1|$
2. $y = |x(x-1)(x-2)|$
3. $y = |x^2-1|$
4. $y = |x^2+1|$
5. $y = |\sin x|$
6. $y = |\ln x|$
7. $y = |\cos x|$
8. $y = 3+|x+1|$
9. $y = |2x+5|-4$
10. $y = |x^2-x-20|$
11. $y = 1+|2-x^2|$
12. $y = |\tan x|$

The Effect of a Modulus Sign on a Cartesian Equation

When a section of the curve $y = f(x)$ is reflected in Oy, the equation of that part of the curve becomes $y = -f(x)$,

e.g., if $y = |x|$ for $x \in \mathbb{R}$ we can write this equation as

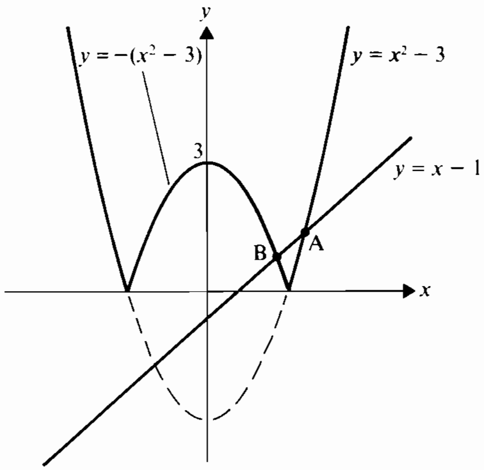
$$\begin{cases} y = x & \text{for } x \geqslant 0 \\ y = -x & \text{for } x < 0 \end{cases}$$



INTERSECTION

To find the points of intersection between two graphs whose equations involve a modulus, we first sketch the graphs to locate the points roughly. Then we identify the equations in non-modulus form for each part of the graph. If these equations are written on the sketch then the correct pair of equations for solving simultaneously can be identified.

For example, the points common to $y = x-1$ and $y = |x^2-3|$ can be seen from the sketch.



We can also see from the sketch that

the coordinates of A satisfy the equations

$$y = x-1 \quad \text{and} \quad y = x^2-3 \qquad [1]$$

and the coordinates of B satisfy the equations

$$y = x-1 \quad \text{and} \quad y = 3-x^2 \qquad [2]$$

Solving equations [1] gives $x^2-x-2 = 0 \quad \Rightarrow \quad x = -1$ or 2

It is clear from the diagram that $x \neq 1$, so A is the point $(2, 1)$

Similarly, solving equations [2] gives $x^2+x-4 = 0$

$$\Rightarrow \qquad x = -\tfrac{1}{2}(1 \pm \sqrt{17})$$

Again from the diagram, it is clear that the x-coordinate of B is positive, so at B, $x = -\frac{1}{2}(1-\sqrt{17})$
Then using $y = x-1$ gives $y = \frac{1}{2}(-3+\sqrt{17})$

This example illustrates the importance of checking solutions to see if they are relevant to the given problem.

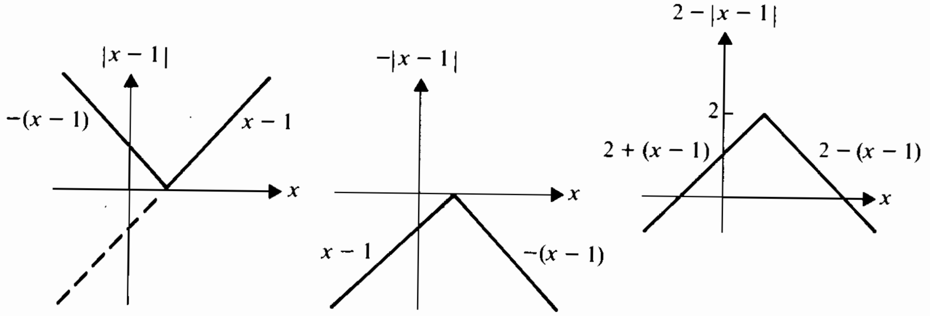
Example 37e

Find the coordinates of the points of intersection of the graphs

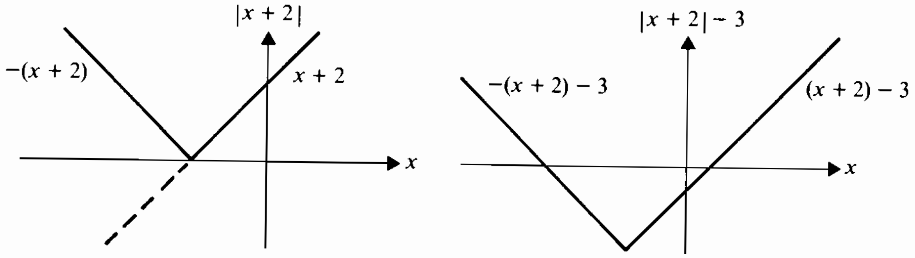
$$y = 2 - |x - 1| \quad \text{and} \quad y = |x + 2| - 3$$

The sequence of diagrams show how the graphs and the non-modulus forms of the equations are built up from transformations.

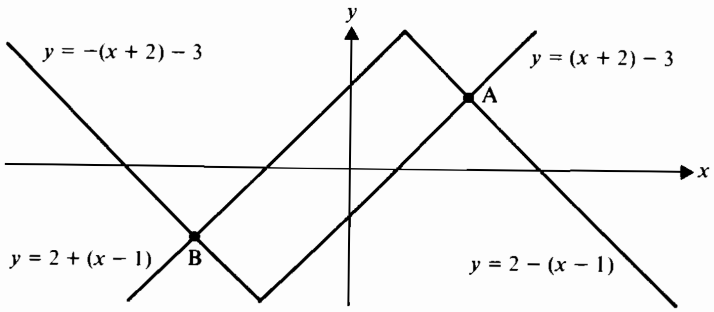
For $y = 2 - |x - 1|$



For $y = |x + 2| - 3$



Both graphs are then drawn on the same set of axes.



From this graph we can see that the coordinates of A satisfy the equations

$$y = (x + 2) - 3 \quad \text{and} \quad y = 2 - (x - 1) \quad \Rightarrow \quad x = 2 \quad \text{and} \quad y = 1$$

and the coordinates of B satisfy the equations

$$y = 2 + (x - 1) \quad \text{and} \quad y = -(x + 2) - 3 \quad \Rightarrow \quad x = -3 \quad \text{and} \quad y = -2$$

Therefore the points of intersection are $(2, 1)$ and $(-3, -2)$

EXERCISE 37e

Find the points of intersection of the graphs.

1. $y = |x|$ and $y = 1 - |x|$

2. $y = x$ and $y = |x^2 - 2x|$

3. $y = 2|x|$ and $y = 3 + 2x - x^2$

4. $y = |1/x|$ and $y = |x|$

5. $y = |x^2 - 4|$ and $y = 2x + 1$

Solve the following equations.

6. $|x^2 - 1| - 1 = 3x - 2$

7. $2 - |x + 1| = |4x - 3|$

8. $2|1 - x| = x$

9. $|2 - x^2| + 2x + 1 = 0$

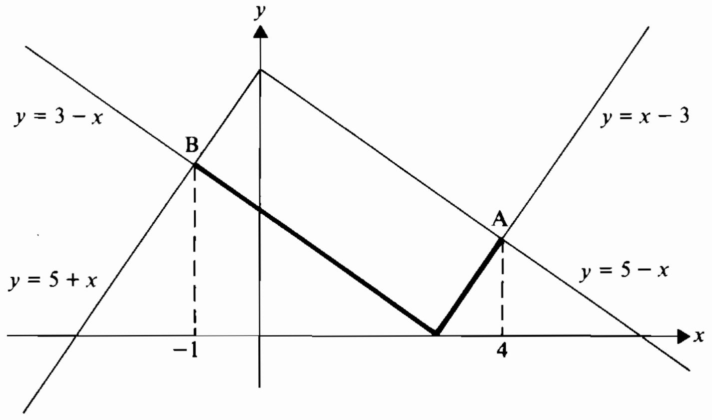
INEQUALITIES

Many inequalities can be solved easily with the aid of sketch graphs. The following worked examples illustrate how to use graphs to solve a variety of inequality problems. In some cases they provide an alternative method, though not always as direct an approach, for solving the similar problems discussed in Chapter 13.

Examples 37f

1. Find the set of values of x for which $|x-3| < 5-|x|$

We start by drawing on the same set of axes the graphs $y = |x-3|$ and $y = 5-|x|$



At A $\quad x-3 = 5-x \quad \Rightarrow \quad x = 4$

and at B $\quad 3-x = 5+x \quad \Rightarrow \quad x = -1$

From the graph we see that $|3-x| < 5-|x|$ for $-1 < x < 4$

---

2. Find the set of values of x for which $\dfrac{(x-1)^2}{x+5} < 1$

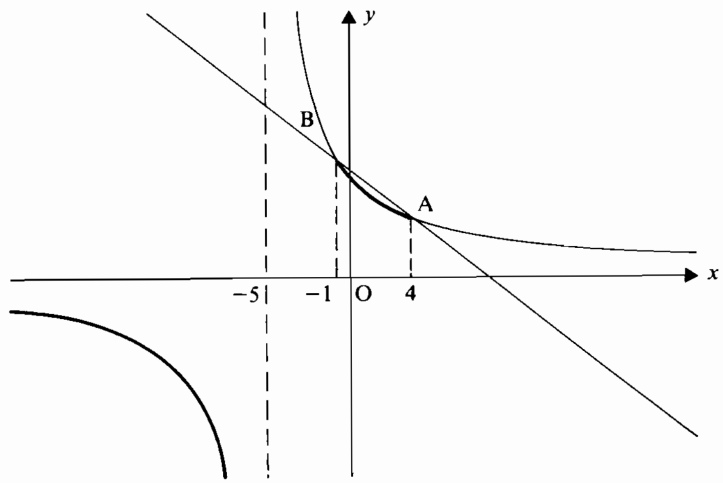
Dividing out $\dfrac{(x-1)^2}{x+5}$ so that it contains only proper fractions gives

$$\frac{(x-1)^2}{x+5} = x-7+\frac{36}{x+5}$$

The inequality then becomes $x-7+\dfrac{36}{x+5} < 1$

$$\Rightarrow \qquad \frac{36}{x+5} < 8-x$$

On the same set of axes we now draw sketches of $y = 8-x$ and $y = \dfrac{36}{x+5}$



The x coordinates of A and B are the solutions of the equation

$$\frac{36}{x+5} = 8-x \quad \Rightarrow \quad x^2-3x-4 = 0 \quad \Rightarrow \quad x = -1 \text{ and } 4$$

$\dfrac{36}{x+5} < 8-x$ where the curve is below the line.

From the graph we see that this is

when $x < -5$ and when $-1 < x < 4$

Notice that we adjusted the inequality so that each side was an expression whose graph was easily recognised.

EXERCISE 37f

Solve the following inequalities.

1. $\dfrac{1}{x-1} < x$

2. $\dfrac{1}{x} > x^2$

3. $\dfrac{x}{x-1} < 0$

4. $\dfrac{x+1}{x} < 0$

5. $\dfrac{x+1}{x+2} < 0$

6. $\dfrac{1+x^2}{x} > 0$

7. $|x| < 1 - |x|$

8. $|x-1| < |x+2|$

9. $2|x| < |1-x|$

10. $|x+1| < 2x$

11. $3x - 1 < 1 + |x|$

12. $|3x+2| > 2 - |x+1|$

13. $1 + x^2 > 2x + 1$

14. $1 + x^2 > |2x+1|$

15. $|1-x^2| < 2x + 1$

16. $1 - |x| > x^2 - 1$

17. Find the set of values of x between 0 and 2π for which

$$|\sin x| < |\cos x|$$

18. On the same set of axes, sketch the curves whose equations are

$$y = (x-2)(x-4) \quad \text{and} \quad y = \frac{1}{(x-2)(x-4)}$$

clearly showing any asymptotes and turning points.

Hence find the range of f where $f: x \to \dfrac{1}{(x-2)(x-4)}$

19. On the same set of axes sketch the curves whose equations are

$$y = x \quad \text{and} \quad y = \frac{1}{x}$$

Deduce the shape of the curve whose equation is $y = x + \dfrac{1}{x}$

and the range of $f: x \to x + \dfrac{1}{x}$

20. Use any method to find the range of the function, f, where

(a) $f(x) = \dfrac{1}{1-x}$ (b) $f(x) = \dfrac{1}{(x-2)(x-6)}$

(c) $f(x) = \dfrac{x}{(1-x)^2}$ (d) $f(x) = \dfrac{4}{(x-1)(x-3)}$

MIXED EXERCISE 37

1. Sketch the curve whose equation is $y = 1 - \dfrac{1}{x-3}$

Give the equations of the asymptotes and the coordinates of the intercepts on the axes.

2. The function f is defined by

$$f(x) = \sin x \quad \text{for} \quad 0 \leqslant x < \tfrac{1}{2}\pi$$
$$f(x) = \pi - x \quad \text{for} \quad \tfrac{1}{2}\pi \leqslant x < \pi$$
$$f(x) = f(x+\pi) \quad \text{for} \quad x \in \mathbb{R}$$

Sketch the graph of $f(x)$ for $0 \leqslant x < 4\pi$

3. Draw a sketch of the curve whose equation is $y = x^2 - 1$.
Hence superimpose a sketch of the curve whose equation is

$$y = \frac{1}{(x^2-1)}$$

4. Sketch the graph of $g(x) = \dfrac{x+2}{x+1}$.

On the same set of axes sketch the graph of the function g^{-1} and hence state the range of g^{-1}

5. Find the set of values of x for which $|x-1| > 1 + |1+x|$

6. Find the range of values of x for which $\dfrac{x^2-2}{x} > 1$

7. The function f is periodic with period 2 and

$$f(x) = x^2 \quad \text{for} \quad 0 \leqslant x < 1$$
$$f(x) = 3 - 2x \quad \text{for} \quad 1 \leqslant x < 2$$

Sketch the graph of f for $-2 \leqslant x \leqslant 4$
For which values of x in this range is $f(x)$ not continuous?

8. State whether $f(x)$ is odd, even, periodic (in which case give the period) or none of these.

(a) $f(x) = \tan x$ (b) $f(x) = (x+1)(x)(x-1)$

(c) $f(x) = x^4$ (d) $f(x) = \sin(x - \tfrac{1}{2}\pi)$

29. Using the same axes, sketch the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{x}{x+2}$$

State the equations of any asymptotes, the coordinates of any points of intersection with the axes and the coordinates of any points of intersection of the two curves.

Hence, or otherwise, find the set of values of x for which

$$\frac{1}{x} > \frac{x}{x+2}$$

(U of L 86)

30. Use the binomial expansion to express $x^4(1-x)^4$ as a polynomial in x.

Hence, or otherwise, verify that

$$x^4(1-x)^4 \equiv (1+x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4$$

Use this result to evaluate

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

and deduce that $\pi < 22/7$. (U of L 86)

31. Given that $y = \dfrac{1+3x}{(1-2x)^4}$ and $|x| < \frac{1}{3}$, find in simplified form the first three non-zero terms in ascending powers of x of the series expansions of

(a) y

(b) $\ln y$ (AEB 86)

32. Find, in ascending powers of x, as far as the term in x^2, series expansions for e^{-px} and $(1+2x)^{-q}$.

Given that the first non-zero term in the series expansion of

$$e^{-px} - (1+2x)^{-q}$$

in ascending powers of x is $-4x^2$, find the value of p and the value of q. (AEB 88)

33. Write down and simplify the first three terms in the series expansion of $\left(1 + \dfrac{x}{3}\right)^{-1/2}$ in ascending powers of x.

State the set of values of x for which the series is valid.

Given that x is so small that terms in x^3 and higher powers of x may be neglected, show that

$$e^{-x}\left(1+\frac{x}{3}\right)^{-1/2} = 1 - \frac{7}{6}x + \frac{17}{24}x^2$$

(AEB 87)

CHAPTER 38

SOLUTION OF EQUATIONS

Equations of various forms have appeared at intervals throughout this book, usually when some new mathematics has provided the means to solve a different form of equation. In these situations, the method of solution was usually obvious. However, equations often arise from mathematical models of practical situations and may not be connected obviously to the mathematical knowledge needed to solve them.

This chapter starts by bringing together some categories of equation which have exact solutions and, by looking at the form of the equation, suggesting possible ways of solving them. It must be remembered though, that it is not always possible to find exact solutions even when they exist, so this chapter ends with some methods for finding approximate solutions.

TRIGONOMETRIC EQUATIONS

Successful solutions of trig equations depend on three factors; recognising and knowing the trig identities, correctly classifying the equation so that the first attempt at solution is likely to be successful and, finally, experience.

There are two general approaches to trig equations. First see whether the equation can be factorised, either in its given form or by applying an appropriate identity. If this is not possible (and often it is not) try to reduce the equation to a form which involves only one variable, i.e. one trig ratio of one angle.

In this section we give most of the common categories of trig equation followed by an appropriate method of solution. This list is neither exhaustive nor infallible, but it covers most forms of equation met at this level.

A. Equations Containing One Angle Only

Form of Equation	Method
1. $a\cos\theta + b\sin\theta = 0$	Divide by $\cos\theta$, provided that $\cos\theta \neq 0$
2. $a\cos\theta + b\sin\theta = c$	Write LHS as $R\cos(\theta+\alpha)$ or use the 'little t' identities.
3. $a\cos^2\theta + b\sin\theta = c$ $a\sin^2\theta + b\cos\theta = c$ $a\tan^2\theta + b\sec\theta = c$	Use the Pythagorean identities to express in terms of one ratio only.
4. $a\cos\theta + b\tan\theta = 0$ $a\sin\theta + b\tan\theta = 0$	Multiply by $\cos\theta$

Note that any of the equations in Section A can be solved by using the 'little t' identities. However this is often not the simplest method and sometimes leads to a polynomial in t whose roots are far from obvious.

B. Equations Containing Multiples of One Angle

Form of Equation	Method
1. $a\cos\theta + b\cos 2\theta = c$ $a\sin\theta + b\cos 2\theta = c$	Use the double angle formulae to reduce to Section A type.
2. $\cos\theta = \sin a\theta$	Express $\sin a\theta$ as $\cos(\frac{1}{2}\pi - a\theta)$ and use $\cos A = \cos B$
3. $\cos a\theta + \cos b\theta + \cos c\theta = 0$ $\sin a\theta + \sin b\theta + \sin c\theta + \sin d\theta = 0$	Use the factor formulae on a pair(s) of terms to give a common factor.

C. Equations Containing Different Angles

Form of Equation	Method
1. $\cos\theta + \sin(\theta - \alpha) = 0$ etc.	Reduce to $\cos A = \cos B$ or to $\tan A = \tan B$ and quote solution, or use factor formulae to factorise.
2. $\cos\theta\sin(\theta-\alpha) = c$	Use compound angle formulae to reduce to a section A type equation

Note that equations of the form given in B and C can often be reduced to equations involving one angle only, by using double and compound angle formulae. However this rarely leads to an easy solution so should only be tried as a last resort.

It is important to realise that a method given in this list does not represent the only way of solving a particular equation, nor does it always lead to the quickest solution. Sometimes an equation can be simplified quickly when part of it is recognised as part of a trig identity. Sometimes it may be necessary to classify each side of an equation independently. This is illustrated in the next worked example.

Example 38a

Find the general solution of the equation

$$\cos 2\theta \sin\theta + \sin 2\theta \cos\theta = \cos 4\theta$$

$$\cos 2\theta \sin\theta + \sin 2\theta \cos\theta = \cos 4\theta$$

The LHS is recognised as the expansion of $\sin(2\theta + \theta)$, i.e. $\sin 3\theta$, giving

$$\sin 3\theta = \cos 4\theta$$

$\sin 3\theta$ may be expressed as $\cos(\frac{1}{2}\pi - 3\theta)$, giving

$$\cos(\tfrac{1}{2}\pi - 3\theta) = \cos 4\theta$$

Using the solution to $\cos A = \cos B$ gives

$$\tfrac{1}{2}\pi - 3\theta = 2n\pi \pm 4\theta$$

$$\therefore \quad \theta = \tfrac{1}{14}\pi(1-4n) \quad \text{or} \quad \tfrac{1}{2}\pi(4n-1)$$

The equation in the worked example illustrates that one approach does not always lead directly to the solution. The situation should be reappraised at each step.

EXERCISE 38a

Find the general solution in radians of each equation.

1. $\sin 2x \cos x + \cos 2x \sin x = 1$
2. $\cos 3x = \sin x$
3. $2\sin^2 x + \cos x = 1$
4. $5\cos x + 12\sin x = 13$
5. $2\sin x + \sin 2x = 0$
6. $\cos x + \cos 2x + \cos 3x = 0$
7. $\cos^2 x + 2\sin^2 x = 2$
8. $4\sin x - 5 = 3\cos x$
9. $\sin^2 x = 2\cos x + 1$
10. $\cos x = 2\tan x$

Solve each equation for $0 \leqslant x < 360°$ giving answers correct to 1 d.p. where necessary.

11. $\cos 2x + \sin x = 1$
12. $2\sin x - \cos x = 1$
13. $\sin x + \tan x = 0$
14. $\cos 2x + 2\sin x = 0$
15. $\tan x + 2\sec^2 x = 3$
16. $2\cos x + \cos 2x = 2$
17. $\cos(x + 30°) + \sin x = 0$
18. $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$
19. $\cos x \cos 2x - \sin x \sin 2x = \frac{1}{2}$

EXPONENTIAL AND LOGARITHMIC EQUATIONS

When the unknown quantity forms part of an index, taking logs will often transform the index into a factor.

For example, if $5^x = 10$ then taking logs of both sides gives

$$x \ln 5 = \ln 10 \quad \Rightarrow \quad x = \frac{\ln 10}{\ln 5}$$

Taking logs can work when the terms containing a power involving x can be expressed as a single term, but this is not always possible. Consider for example the equation

$$2^{2x} + 3(2^x) - 4 = 0$$

Now $2^{2x} + 3(2^x)$ cannot be simplified into a single term. However recognising that 2^{2x} is $(2^x)^2$, the substitution $y = 2^x$ can be made,

i.e. $$y^2 + 3y - 4 = 0$$

This is a quadratic equation which can now be solved.

When an equation contains logarithms involving the unknown, first make sure that all logs are to the same base, then check to see if a simple substitution will reduce the equation to a recognisable form. Sometimes the best policy is to remove the logarithms. Equations of each of these types are considered in the following worked examples.

Examples 38b

1. Solve the equation $\log_3 x - 4\log_x 3 + 3 = 0$

The two log terms have different bases so we begin by changing the base of the log in the second term to 3. The given equation then becomes

$$\log_3 x - \frac{4}{\log_3 x} + 3 = 0$$

$$\Rightarrow \quad (\log_3 x)^2 - 4 + 3\log_3 x = 0$$

Substituting y for $\log_3 x$ gives $y^2 - 4 + 3y = 0 \Rightarrow y^2 + 3y - 4 = 0$

$$\therefore \quad (y+4)(y-1) = 0 \Rightarrow y = 1 \text{ or } -4$$

i.e. $\log_3 x = 1$ or $\log_3 x = -4 \Rightarrow x = 3$ or $x = 3^{-4} = \frac{1}{81}$

2. Solve for x and y the equations

$$yx = 16 \text{ and } \log_2 x - 2\log_2 y = 1$$

$$yx = 16 \qquad [1]$$

$$\log_2 x - 2\log_2 y = 1 \qquad [2]$$

Using the laws of logs, equation [2] can be written as

$$\log_2 \frac{x}{y^2} = 1 \Rightarrow \frac{x}{y^2} = 2, \text{ i.e. } x = 2y^2$$

Substituting $2y^2$ for x in equation [1] gives

$$2y^3 = 16 \Rightarrow y^3 = 8$$

$\therefore$ $y = 2$ and, from [1], $x = 8$

EXERCISE 38b

Solve the equations.

1. $3^x = 6$
2. $5^x = 4$
3. $2^{2x} = 5$
4. $3^{x-1} = 7$
5. $4^{2x+1} = 3$
6. $5^x(5^{x-1}) = 10$
7. $2(2^{2x}) - 5(2^x) + 2 = 0$
8. $3^{2x+1} - 26(3^x) - 9 = 0$
9. $4^x - 6(2^x) - 16 = 0$
10. $\log_2 x + \log_x 2 = 2$
11. $\log_2 x = \log_4(x+6)$
12. $4\log_3 x = \log_x 3$

Solve the equations simultaneously.

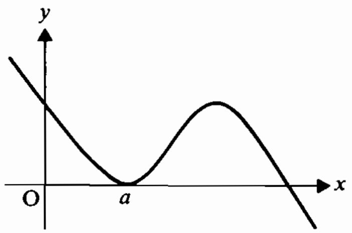
13. $2\ln y = \ln 2 + \ln x$ and $2^y = 4^x$
14. $\log_x y = 2$ and $xy = 8$
15. $\log_3 x = y = \log_9(2x-1)$
16. $\lg(x+y) = 0$ and $2\lg x = \lg(y+1)$ (lg means $\log_{10}$)

POLYNOMIAL EQUATIONS

If a quadratic equation has real roots then these roots can always be found either by factorisation or by using the formula. If the equation is of higher degree, real roots can *sometimes* be found using the factor theorem but only if these roots are integers or simple rational numbers. When the factor theorem fails to find *exact* solutions, they can be found by other methods in some special cases, one of which we look at now.

Equations with a Repeated Root

If the equation $f(x) = 0$ has a repeated root, a, then $(x-a)^2$ is a factor of $f(x) = 0$. This means that the curve $y = f(x)$ *touches* the x-axis at $x = a$



As $y = f(x)$ has a turning point at $x = a$, $f'(x) = 0$ when $x = a$, i.e. $f'(a) = 0$, therefore

$$f(a) = 0 \text{ and } f'(a) = 0 \quad \Longleftrightarrow \quad f(x) \text{ has a repeated factor.}$$

Examples 38c

1. Solve the equation $18x^3 - 111x^2 + 224x - 147 = 0$ given that it has two equal roots.

The repeated root of the given equation

$$f(x) = 18x^3 - 111x^2 + 224x - 147 = 0 \qquad [1]$$

also satisfies the equation

$$f'(x) = 54x^2 - 222x + 224 = 0$$

$$\Rightarrow \quad 27x^2 - 111x + 112 = 0$$

$$\Rightarrow \quad (9x-16)(3x-7) = 0$$

So *either* $x = \frac{16}{9}$ *or* $x = \frac{7}{3}$ is a solution of the given equation.

To check which of these values is the repeated root of the given equation, each is substituted in turn into equation [1].
We find that $f(\frac{16}{9}) \neq 0$ and that $f(\frac{7}{3}) = 0$

Hence $f(x)$ and $f'(x)$ are both zero when $x = \frac{7}{3}$

So $x = \frac{7}{3}$ is the repeated root of the given equation.

Hence $18x^3 - 111x^2 + 224x - 147 = (3x-7)^2(ax+b)$

Comparing coefficients of x^3 gives $18 = 9a \Rightarrow a = 2$

Comparing constants gives $-147 = 49b \Rightarrow b = -3$

Therefore $18x^3 - 111x^2 + 224x - 147 = (3x-7)^2(2x-3)$

So $(3x-7)^2(2x-3) = 0$

$$\Rightarrow \quad x = \tfrac{7}{3} \text{ or } \tfrac{3}{2}$$

Equations Involving Square Roots

When an equation contains square roots involving the unknown, we must eliminate those square roots. However when both sides of an equation are squared, an extra equation, and hence extra solutions, is introduced

For example if $x = 2$ then squaring gives $x^2 = 4$
But $x^2 = 4$ includes both $x = 2$ and $x = -2$, therefore

it is essential that whenever a solution involves squaring, all roots must be checked in the original equation.

Examples 38c (continued)

2. Solve the equation $\sqrt{(x+8)} - \sqrt{(x+3)} = \sqrt{(2x-1)}$

Squaring both sides of the equation gives

$$x + 8 - 2\sqrt{(x+8)}\sqrt{(x+3)} + x + 3 = 2x - 1$$

$$\Rightarrow \quad 6 = \sqrt{(x+8)}\sqrt{(x+3)}$$

Squaring again gives $\quad 36 = (x+8)(x+3)$

$$\Rightarrow \quad x^2 + 11x - 12 = 0$$

$$\Rightarrow \quad x = 1 \text{ or } -12$$

Checking these values of x in the original equation shows that
when $x = 1$, LHS $= \sqrt{9} - \sqrt{4} = 1$ and RHS $= \sqrt{1} = 1$
so $x = 1$ is a solution of the given equation,

when $x = -12$, RHS $= \sqrt{(-25)}$ which is not real
so $x = -12$ is not a solution of the given equation.

Therefore, $x = 1$ is the only solution.

Note that we have looked at a very small sample of the possible types of equation which have exact solutions. Although there are rules for solving a few particular forms of equation, in most cases such rules do not exist. The solution of equations is an art; experience will suggest a likely form of attack but there is never any guarantee that the method will produce a solution and other methods have to be tried. In the following exercise, all the equations have exact solutions and most of them are of a type that has been discussed either in this chapter or earlier in the book. Just a few may be unfamiliar.

EXERCISE 38c

1. Find the values of x that satisfy the equation $80x^3 + 88x^2 - 3x - 18 = 0$ given that it has a repeated root.

2. Show that $x = 0$ is not a root of the equation
$$5x^4 - 16x^3 - 42x^2 - 16x + 5 = 0$$
Divide the equation by x^2, then use $y = x + 1/x$ to show that
$$5(y^2 - 2) - 16y - 42 = 0$$
Hence solve the given equation.

3. The equation $20x^3 - 52x^2 + 21x + 18 = 0$ has a repeated root. Solve the equation

Solve the given equations using any suitable methods.

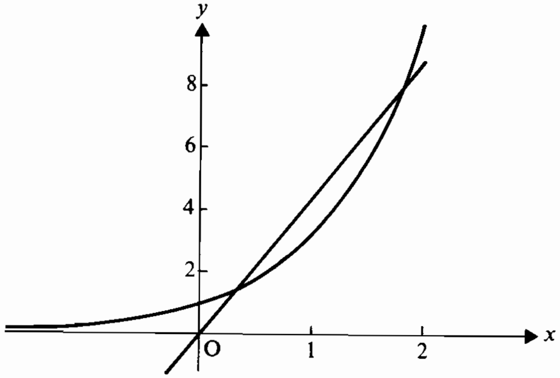
4. $x^3 - 6x^2 + 11x - 6 = 0$
5. $1 + \sqrt{x} = \sqrt{(3x-3)}$
6. $\sqrt{(2x-5)} - \sqrt{(x-2)} = 1$
7. $x^2 - 3x = 8$
8. $1 - \sqrt{x} + \sqrt{(x-3)} = 0$
9. $\sqrt{(3x+1)} + \sqrt{(x-1)} = \sqrt{(7x+1)}$
10. $x^4 - 12x^2 + 27 = 0$
11. $x^{4/3} - 5x^{2/3} + 4 = 0$
12. $|x| = 3 - |1 - x|$
13. $2x^3 - x^2 + 20 = 0$
14. $\dfrac{x^2}{4} + y^2 = 1$ and $xy = 1$
15. $x^2 + y^2 + 4x - 6y = 3$ and $y = x + 1$
16. $x^2 + y^2 + 8x - 4y + 15 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$
17. $x^4 - x^3 - 12x^2 - 4x + 16 = 0$ (*Hint*. Use $y = x + 4/x$)

APPROXIMATE SOLUTIONS

When the roots of an equation cannot be found exactly, we can find approximate solutions. The first step is to locate the roots roughly and this can be done graphically.

Consider, for example, the equation $e^x = 4x$

The roots of this equation are the values of x where the curve $y = e^x$ and the line $y = 4x$ intersect.

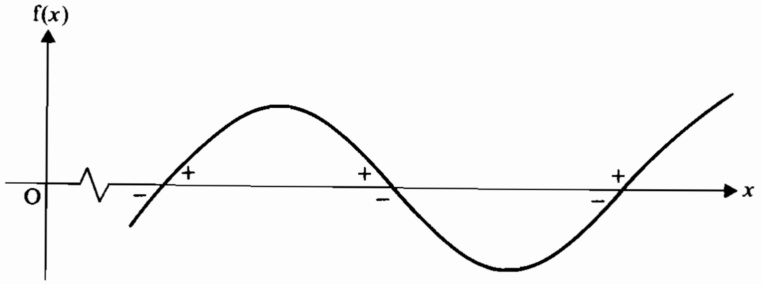


From the sketch we can see that there is one root between 0 and 0.5 and another root somewhere near 2

We now need a way to locate the roots more accurately.

Locating the Roots of an Equation

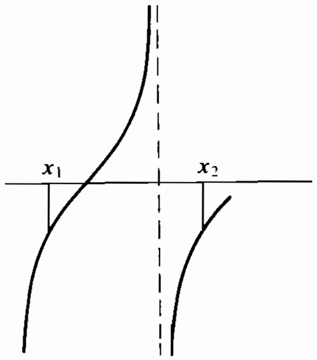
Suppose that we have very roughly located the roots of an equation $f(x) = 0$
Now consider the curve $y = f(x)$
The roots of the equation $f(x) = 0$ are the values of x where this curve crosses the x-axis, e.g.



Each time that the curve crosses the x-axis, the sign of y changes. So

if one root only of the equation $f(x) = 0$ lies between x_1 and x_2, and if the curve $y = f(x)$ is unbroken between the points where $x = x_1$ and $x = x_2$, then $f(x_1)$ and $f(x_2)$ are opposite in sign.

The condition that the curve $y = f(x)$ must be unbroken between x_1 and x_2 is essential as we can see from the curve below.



This curve crosses the x-axis between x_1 and x_2 but $f(x_1)$ and $f(x_2)$ have the same sign because the curve is broken between these values.

Returning to the equation $e^x = 4x$, we will now locate the larger root a little more precisely.

First we write the equation in the form $f(x) = 0$, i.e. $f(x) = e^x - 4x$, then we find where there is a change in the sign of $f(x)$

We know that there is a root in region of $x = 2$, so we will see if it lies between 1.8 and 2.2

Using $f(x) = e^x - 4x$, gives $f(1.8) = e^{1.8} - 4(1.8) = -1.1 \ldots$
and $f(2.2) = e^{2.2} - 4(2.2) = 0.2 \ldots$

Therefore the larger root of the equation lies between 1.8 and 2.2 (and is likely to be nearer to 2.2 as f(2.2) is nearer to zero then f(1.8) is).

Example 38d

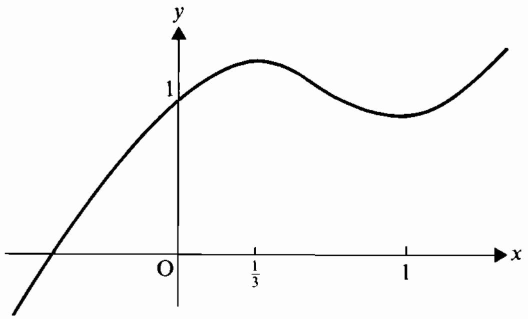
Find the turning points on the curve $y = x^3 - 2x^2 + x + 1$. Hence sketch the curve and use the sketch to show that the equation $x^3 - 2x^2 + x + 1 = 0$ has only one real root. Find two consecutive integers between which this root lies.

At turning points $\frac{dy}{dx} = 0$, i.e. $3x^2 - 4x + 1 = 0$

$$\Rightarrow \quad (3x - 1)(x - 1) = 0 \quad \text{so} \quad x = \tfrac{1}{3} \text{ and } 1$$

When $x = \frac{1}{3}$, $y = \frac{31}{27}$ and when $x = 1$, $y = 1$

As the curve is a cubic, and as $y \to \infty$ as $x \to \infty$, we deduce that the curve has a maximum point at $(\frac{1}{3}, \frac{31}{27})$ and a minimum point at $(1, 1)$



From the sketch, we see that the curve $y = x^3 - 2x^2 + x + 1$ crosses the x-axis at one point only, therefore

the equation $x^3 - 2x^2 + x + 1 = 0$ has only one real root.

Also from the sketch it appears that this root lies between $x = -1$ and $x = 0$

Using $f(x) = x^3 - 2x^2 + x + 1$ gives $f(-1) = -3$ and $f(0) = 1$

As $f(-1)$ and $f(0)$ are opposite in sign, the one real root of $x^3 - 2x^2 + x + 1 = 0$ lies between $x = -1$ and $x = 0$

EXERCISE 38d

1. Use sketch graphs to determine the number of real roots of each equation. (Some may have an infinite set of roots.)

(a) $\sin x = \dfrac{1}{x}$ (b) $\cos x = x^2 - 1$ (c) $2^x = \tan x$

(d) $2^x \sin x = 1$ (e) $(x^2 - 4) = \dfrac{1}{x}$ (f) $x2^x = 1$

(g) $x \ln x = 1$ (h) $\sin x = x^2$ (i) $\ln x + 2^x = 0$

2. For each equation in Question 1 with a finite number of roots, locate the root, or the larger root where there is more than one, within an interval of half a unit.

3. Find the turning points on the curve whose equation is $y = x^3 - 3x^2 + 1$. Hence sketch the curve and use your sketch to find the number of real roots of the equation $x^3 - 3x^2 + 1 = 0$

4. Using a method similar to that given in Question 3, or otherwise, determine the number of real roots of each equation.

(a) $x^4 - 3x^3 + 1 = 0$ (b) $x^3 - 24x + 1 = 0$ (c) $x^5 - 5x^2 + 4 = 0$

5. Show that the equation $3^{-x} = x^2 + 2$ has just one root and find this root to the nearest integer.

6. Find the successive integers between which the smallest root of the equation $2^x = \frac{1}{2}(x + 3)$ lies.

ITERATIVE APPROXIMATIONS

113

There are many ways in which successive numerical approximations can be used to find a root of an equation to any degree of accuracy required. The two given below are both common iterative methods.

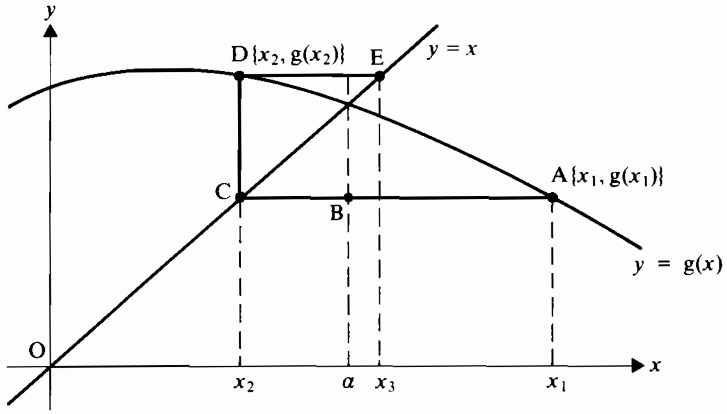
Whatever iterative procedure is used we must first find an interval in which the required root lies. This can be done using the methods described in the last section.

Method 1 $x = g(x)$

115

This method can often be used to find a root of an equation $f(x) = 0$ which can be written in the form $x = g(x)$.
The roots of the equation $x = g(x)$ are the values of x at the points of intersection of the line $y = x$ and the curve $y = g(x)$.

Taking x_1 as a first approximation to a root α then in the diagram,
A is the point on the *curve* where $x = x_1$, $y = g(x_1)$
B is the point where $x = \alpha$, $y = g(x_1)$
C is the point on the *line* where $x = x_2$, $y = g(x_1)$



If, in the region of α, the slope of $y = g(x)$ is less steep than that of the line $y = x$, i.e. provided that $|g'(x)| < 1$

then $\quad CB < BA$

so x_2 is closer to α than is x_1 and x_2 is a better approximation to α

But C is on the line $y = x$

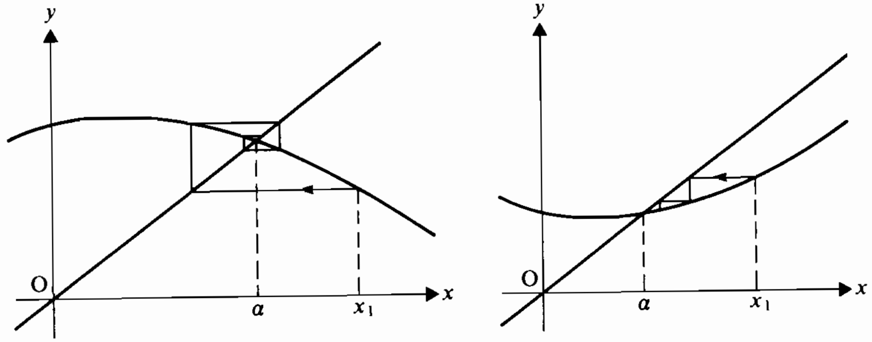
therefore $\quad x_2 = g(x_1)$

Now taking the point D on the curve where $x = x_2$, $y = g(x_2)$ and repeating the argument above we find that x_3 is a better approximation to α than is x_2

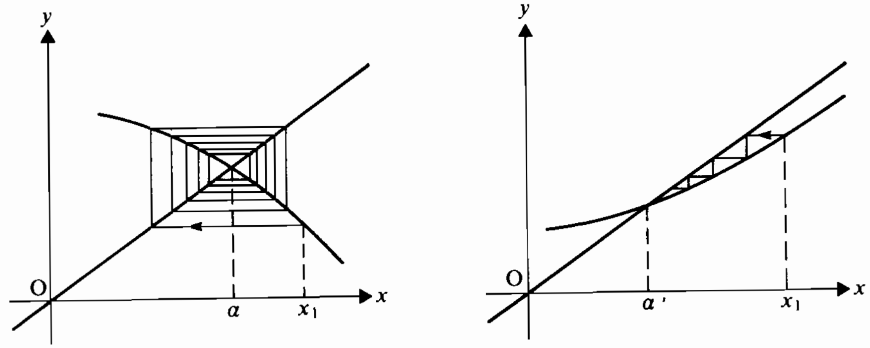
where $\quad x_3 = g(x_2)$

This process can be repeated as often as necessary to achieve the required degree of accuracy.
The rate at which these approximations converge to α depends on the value of $|g'(x)|$ near α. The smaller $|g'(x)|$ is, the more rapid is the convergence.

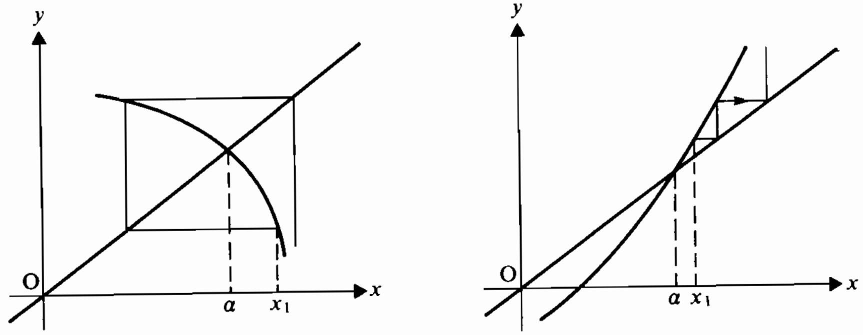
It should be noted that this method fails if $|g'(x)| > 1$ near α
The following diagrams illustrate some of the factors which determine the success, or otherwise, of this method.



Rapid rate of convergence ($|g'(x)|$ small).



Slow rate of convergence ($|g'(x)| < 1$ but close to 1)

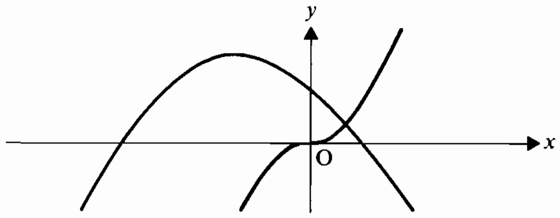


Divergence, i.e. failure, ($|g'(x)| > 1$)

As an example consider the equation

$$x^3 + 2x^2 + 5x - 1 = 0$$

The equation can be written $x^3 = 1 - 5x - 2x^2$ so that a sketch of $y = x^3$ and $y = 1 - 5x - 2x^2$ shows the number of roots.



From the sketch we see that there is only one root and it is near the origin.

The given equation can be written in the form

$$x = g(x)$$

where $\quad g(x) = -\frac{1}{5}(x^3 + 2x^2 - 1)$

We will take $x_1 = 0$ as our first approximation.

A better approximation, x_2, is found from

$$x_2 = g(x_1) = -\tfrac{1}{5}[0^3 + 2(0)^2 - 1] = 0.2$$

Further improvements are obtained by repeating this step,

i.e. $\quad x_3 = g(x_2) = -\frac{1}{5}[(0.2)^3 + 2(0.2)^2 - 1]$
$\quad = 0.1824$

$x_4 = g(x_3) = -\frac{1}{5}[(0.1824)^3 + 2(0.1824)^2 - 1]$
$\quad = 0.1855 \quad$ (to 4 d.p.)

$x_5 = g(x_4) = -\frac{1}{5}[(0.1855)^3 + 2(0.1855)^2 - 1]$
$\quad = 0.1850 \quad$ (to 4 d.p.)

and so on.

The degree of accuracy at any stage can be checked by determining the sign of $f(x)$ on either side of the value so far obtained for the root, e.g., taking $x \approx 0.1850$ we find that $f(0.1846)$ is negative and $f(0.1854)$ is positive, so $x = 0.185$ correct to 3 d.p.

Note. This does *not* show that $x = 0.1850$ to 4 d.p.

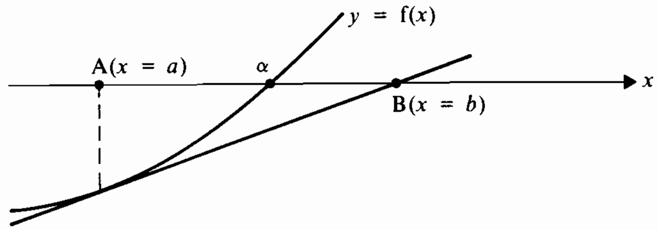
Method 2 The Newton–Raphson or Newton's Method

116

This method is based on determining a linear approximation for a function. Suppose that the equation $f(x) = 0$ has a root α and that a is an approximation for α

The curve $y = f(x)$ cuts the x-axis where $x = \alpha$. If we consider the tangent to $y = f(x)$ at the point where $x = a$ then the point B where this tangent cuts the x-axis will, in most circumstances, be nearer to the point $x = \alpha$

i.e.



So if this tangent cuts the x-axis at B where $x = b$, then b is a better approximation to α than a is.

The gradient of $y = f(x)$ at the point A is $f'(a)$.
The coordinates of A are $(a, f(a))$.

So the equation of the tangent at A is

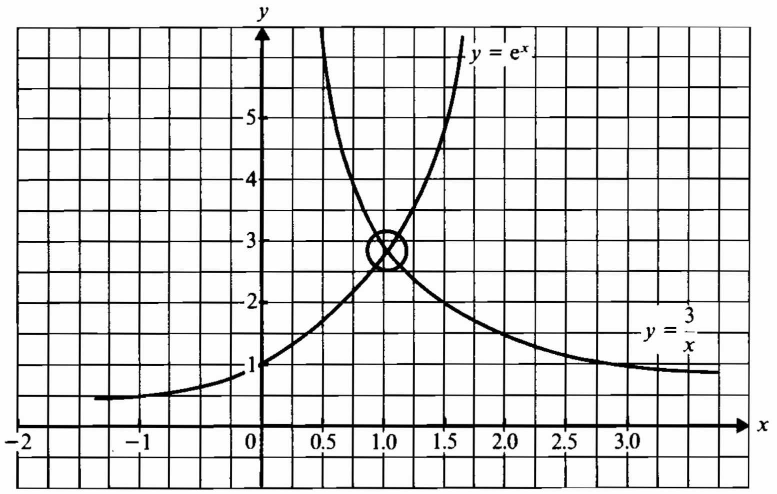
$$y - f(a) = f'(a)(x - a)$$

This line cuts the x-axis where $y = 0$, i.e. where $x = a - \dfrac{f(a)}{f'(a)}$

So if a is an approximation for a root of the equation $f(x) = 0$, then $b = a - \dfrac{f(a)}{f'(a)}$ is a better approximation.

As an example we will use Newton's Method to find the root of $xe^x = 3$ correct to three decimal places.

A first approximation to the root is found by drawing the graphs of $y = \dfrac{3}{x}$ and $y = e^x$



From these graphs we see that $xe^x = 3$ has a root α which is approximately 1

Now $\quad f(x) = xe^x - 3$

$\Rightarrow \quad f'(x) = (x+1)e^x$

Using $b = a - \dfrac{f(a)}{f'(a)}$, and taking $a = 1$ as our first approximation to α, the second approximation is

$$1 - \frac{e-3}{2e} = 1.0518 \quad \text{(to 4 d.p.)}$$

Taking $a = 1.0518$ and repeating the procedure, the third approximation is

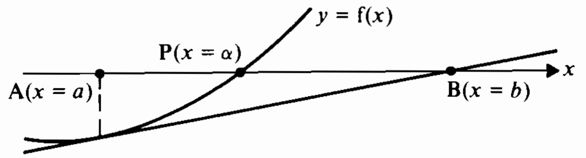
$$1.0518 - \frac{(1.0518)e^{1.0518} - 3}{(2.0518)e^{1.0518}} = 1.0499 \quad \text{(to 4 d.p.)}$$

So, to three decimal places, the root is likely to be 1.050 and we can check this by calculating f(1.0495) and f(1.0504)
Now f(1.0495) is negative and f(1.0504) is positive. Thus the root lies between these values and, correct to 3 d.p., is 1.050

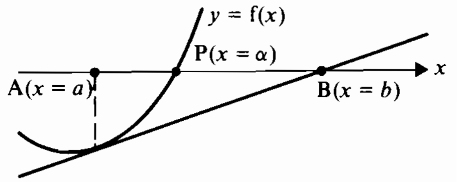
The rate of convergence using Newton's Method depends on the crudeness, or otherwise, of the first approximation and on the shape of the curve in the neighbourhood of the root. In extreme cases these factors may lead to failure. Some of these cases are illustrated by the following graphs.

In the following diagrams, α is a root of $f(x) = 0$; a is the first approximation for α and b is the second approximation for α given by Newton's Method.

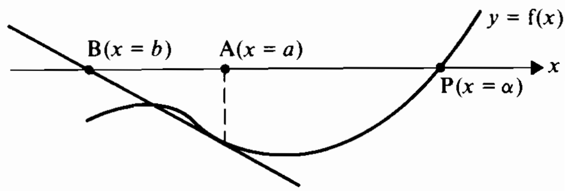
(a) $f'(a)$ is too small.



(b) $f'(x)$ increases too rapidly.



(c) A is too far from P.



EXERCISE 38e

Show that each of the following equations has a root between $x = 0$ and $x = 1$. Using $x = g(x)$ find this root correct to 2 decimal places.

1. $x^3 - x^2 + 10x - 2 = 0$

2. $3x^3 - 2x^2 - 9x + 2 = 0$

3. $2x^3 + x^2 + 6x - 1 = 0$

4. $x^2 + 8x - 8 = 0$

Use the change of sign of $f(x)$ to find, correct to 2 significant figures, the smallest root of each of the following equations.

5. $4 + 5x^2 - x^3 = 0$

6. $x^4 - 4x^3 - x^2 + 4x - 10 = 0$

Use a graphical method to find a first approximation to the root(s) of each equation. Then apply two stages of Newton's Method to give a better approximation. State the accuracy of each of your results.

7. $\tan x = 2x$ (the smallest positive root)

8. $x^3 - 6x + 3 = 0$ (the negative root)

9. $e^x = 2x + 1$

10. $\sin x = 1 - x$

11. $x^2 = \ln(x + 1)$

12. $e^x(1 + x) = 2$

13. $e^x = 3x + 1$

14. $x = 1 + \ln x$

15. $3 + x - 2x^2 = e^x$

16. $x^3 - 3x^2 - 1 = 0$

17. $e^x = 2\cos x$ (the roots between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$)

MIXED EXERCISE 38

Solve the following equations giving exact roots.

1. $3^{2x} = 10$

2. $4^x + 2^x - 6 = 0$

3. $2\log_2 x = \log_x 2$

4. $3\sin x - 4\cos x = 2$

5. $x^2 + y^2 = 4$ and $x + y = 1$

6. $\sin x - \sin 3x = \cos 2x$

7. $x^3 - 3x^2 + 3x - 1 = 0$

8. $|x| = 1 - 2|x|$

9. $\cos 2x + \cos x = 2$

10. $2x^4 - 15x^3 + 38x^2 - 39x + 18 = 0$ given that it has a repeated root.

11. Show that the equation $x^3 - 2x^2 - 1 = 0$ has a root between 2 and 3. Taking 2 as a first approximation to this root, use the Newton–Raphson method twice to obtain a better approximation.

12. Show that the equation $e^x = x^2 + 2$ has only one root and find this root correct to two significant figures.

13. Show that the equation $x^4 + x^2 - x = 0$ has two roots. By writing the equation in the form $x = \dfrac{x}{x^3 + x}$ find the larger root correct to one significant figure.

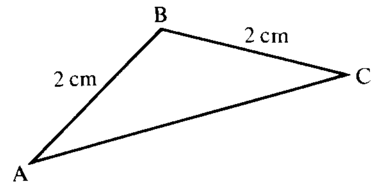
CHAPTER 39

WORKING IN THREE DIMENSIONS

VECTORS

Although we usually assume that two and two make four, this is not always the case.

If, for example, a point B is 2 cm from a point A and C is 2 cm from B then, in general, C is *not* 4 cm from A.



AB, BC and AC are displacements. Each of them has a magnitude and is related to a definite direction in space and so is called a *vector*.

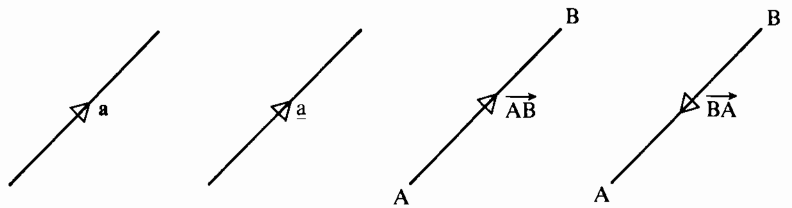
A vector is a quantity which has both magnitude and a specific direction in space.

A scalar quantity is one that is fully defined by magnitude alone. Length, for example, is a scalar quantity as the length of a piece of string does not depend on its direction when it is measured.

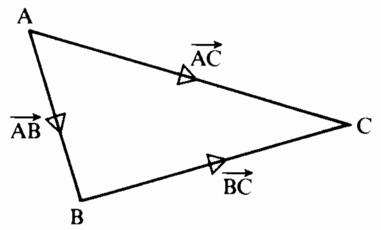
Vector Representation

A vector can be represented by a section of a straight line, whose length represents the magnitude of the vector and whose direction, indicated by an arrow, represents the direction of the vector. Such vectors can be denoted by a letter in bold type, e.g. **a** or, when hand-written, by $\underline{a}$.

Alternatively we can represent a vector by the magnitude and direction of a line joining A to B. When we denote the vector by $\overrightarrow{AB}$ or **AB**, the vector in the opposite direction, i.e. from B to A, is written $\overrightarrow{BA}$ or **BA**.



Equivalent Displacements



The displacement from A to B, followed by the displacement from B to C, is equivalent to the displacement from A to C.

This is written as the vector equation

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Note that, in vector equations like the one above,
+ means 'together with' and = means 'is equivalent to'

Many quantities other than displacements behave in the same way and all of them are vectors.

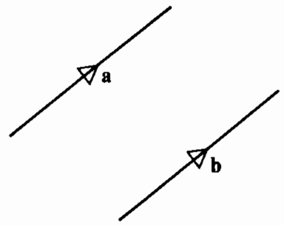
PROPERTIES OF VECTORS

The Modulus of a Vector

The *modulus* of a vector **a** is its magnitude and is written $|\mathbf{a}|$ or a
i.e. $|\mathbf{a}|$ is the length of the line representing **a**

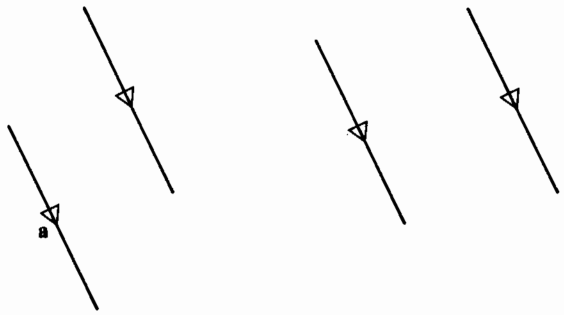
Equal Vectors

Two vectors with the same magnitude and the same direction are equal.



i.e. $\mathbf{a} = \mathbf{b} \iff \begin{cases} |\mathbf{a}| = |\mathbf{b}| & \text{and} \\ \text{the directions of } \mathbf{a} \text{ and } \mathbf{b} \text{ are the same.} \end{cases}$

It follows that a vector can be represented by *any* line of the right length and direction, regardless of position, i.e. each of the lines in the diagram below represents the vector **a**



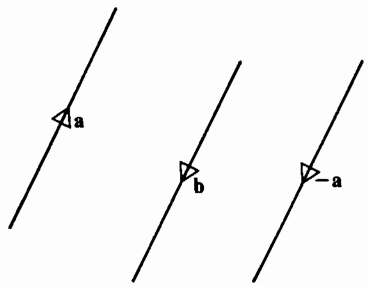
Negative Vectors

If two vectors, **a** and **b**, have the same magnitude but opposite directions we say that

$$\mathbf{b} = -\mathbf{a}$$

i.e. $-\mathbf{a}$ is a vector of magnitude $|\mathbf{a}|$ and in the direction opposite to that of **a**

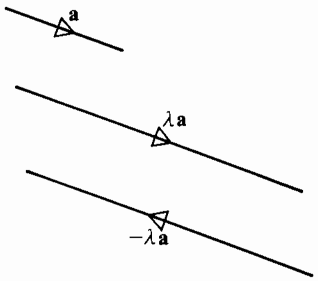
We also say that **a** and **b** are *equal and opposite* vectors.



Multiplication of a Vector by a Scalar

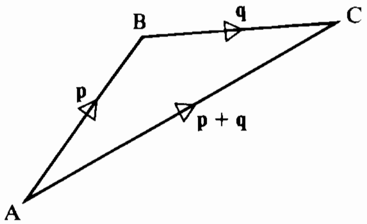
If λ is a positive real number, then $\lambda\mathbf{a}$ is a vector in the same direction as $\mathbf{a}$ and of magnitude $\lambda|\mathbf{a}|$

It follows that $-\lambda\mathbf{a}$ is a vector in the opposite direction, with magnitude $\lambda|\mathbf{a}|$



Addition of Vectors

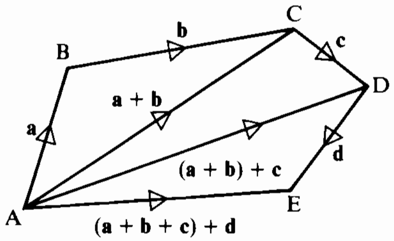
If the sides AB and BC of a triangle ABC represent the vectors $\mathbf{p}$ and $\mathbf{q}$ then the third side AC represents the vector sum, or resultant, of $\mathbf{p}$ and $\mathbf{q}$, which is denoted by $\mathbf{p}+\mathbf{q}$

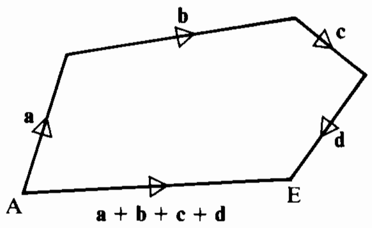


(This property was demonstrated for displacement vectors at the beginning of the chapter.)

Note that $\mathbf{p}$ and $\mathbf{q}$ follow each other round the triangle (in this case in the clockwise sense), whereas the resultant, $\mathbf{p}+\mathbf{q}$, goes the opposite way round (anticlockwise in the diagram).

This is known as *the triangle law* for addition of vectors. It can be extended to cover the addition of more than two vectors.
Let $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DE}$ represent the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively.





The triangle law gives $\overrightarrow{AB}+\overrightarrow{BC} = \mathbf{a}+\mathbf{b} = \overrightarrow{AC}$

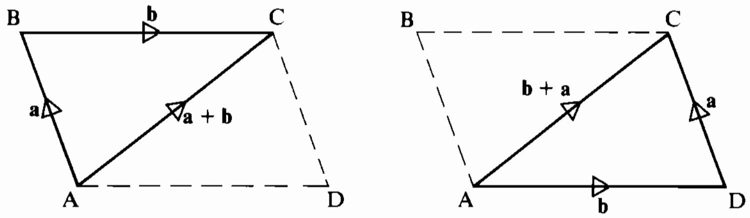
then $\overrightarrow{AC}+\overrightarrow{CD} = (\mathbf{a}+\mathbf{b})+\mathbf{c} = \overrightarrow{AD}$

and $\overrightarrow{AD}+\overrightarrow{DE} = (\mathbf{a}+\mathbf{b}+\mathbf{c})+\mathbf{d} = \overrightarrow{AE}$

Now AE completes the polygon of which AB, BC, CD and DE are four sides taken in order, (i.e. they follow each other round the polygon in the *same sense*). Note that, again, the side representing the resultant closes the polygon in the *opposite* sense.

Note that the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are not necessarily coplanar so the polygon may not be a plane figure.

The order in which the addition is performed does not matter as we can see by considering a parallelogram ABCD



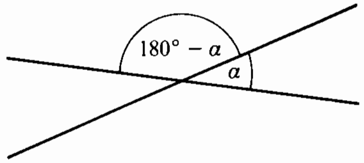
Because the opposite sides of a parallelogram are equal and parallel, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ both represent $\mathbf{a}$ and $\overrightarrow{BC}$ and $\overrightarrow{AD}$ both represent $\mathbf{b}$

In $\triangle ABC$ $\overrightarrow{AC} = \mathbf{a}+\mathbf{b}$ and in $\triangle ADC$ $\overrightarrow{AC} = \mathbf{b}+\mathbf{a}$

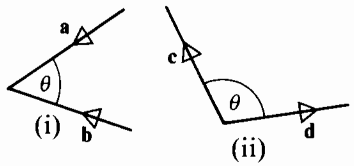
Therefore $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$

The Angle between Two Vectors

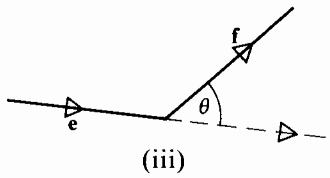
There are two angles between two lines i.e. α and $180° - \alpha$



The angle between two vectors, however, is defined uniquely.
It is the angle between their directions when the lines representing them *both converge* or *both diverge* (see diagrams i and ii).

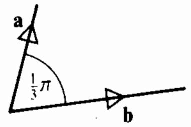


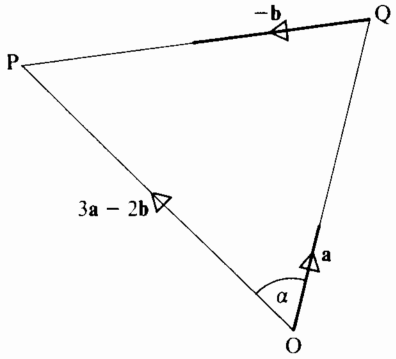
In some cases one of the lines may have to be produced in order to mark the correct angle (see diagram iii).



Examples 39a

1. Two vectors, **a** and **b**, are such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and the angle between **a** and **b** is $\frac{1}{3}\pi$. If the line OP represents the vector $3\mathbf{a} - 2\mathbf{b}$, find, correct to 1 d.p., the angle between $\overrightarrow{OP}$ and **a**





The line OP is found by drawing OQ parallel to **a** such that $\overrightarrow{OQ} = 3\mathbf{a}$, followed by QP parallel to **b** such that $\overrightarrow{QP} = -2\mathbf{b}$

Thus $\overrightarrow{OP} = 3\mathbf{a} - 2\mathbf{b}$

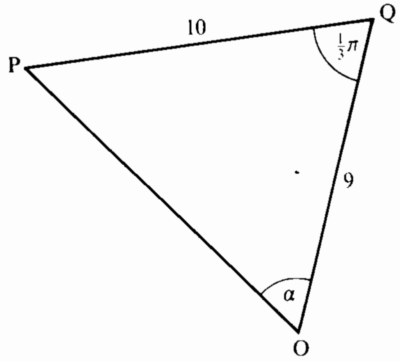
Now $OQ = 3|\mathbf{a}| = 9$

and $QP = 2|\mathbf{b}| = 10$

The angle between $\overrightarrow{OP}$ and **a** is α where

$$\frac{\sin\alpha}{10} = \frac{\sin\frac{1}{3}\pi}{OP}$$

So first we must find OP



Using the cosine formula in OPQ gives

$$OP^2 = 81 + 100 - (2)(9)(10)\cos\tfrac{1}{3}\pi = 91$$

$$\Rightarrow \qquad OP = \sqrt{91}$$

Then $\dfrac{\sin\alpha}{10} = \dfrac{\sin\frac{1}{3}\pi}{\sqrt{91}} \quad \Rightarrow \quad \sin\alpha = 0.9078$

$\therefore$ OP is inclined at 65.2° to **a**

---

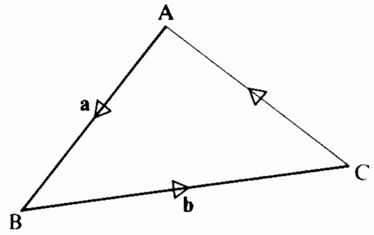
2. In a triangle ABC, $\overrightarrow{AB}$ represents **a** and $\overrightarrow{BC}$ represents **b**. If D is the midpoint of AB express in terms of **a** and **b** the vectors $\overrightarrow{CA}$ and $\overrightarrow{DC}$.

$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$

Now $\overrightarrow{CB} = -\overrightarrow{BC} = -\mathbf{b}$

and $\overrightarrow{BA} = -\overrightarrow{AB} = -\mathbf{a}$

$\therefore \quad \overrightarrow{CA} = -\mathbf{b} - \mathbf{a} = -(\mathbf{a} + \mathbf{b})$

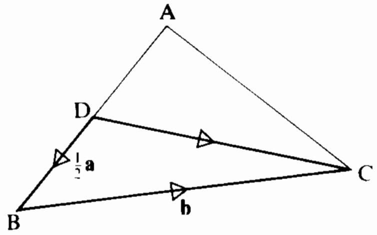


$\overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{BC}$

Now $DB = \frac{1}{2}AB$

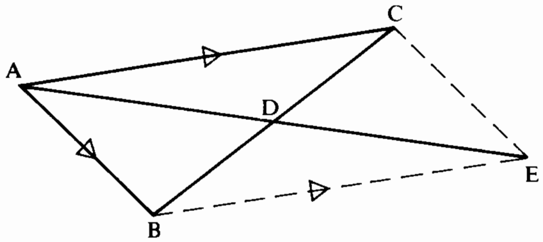
$\Rightarrow \quad \overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{a}$

$\therefore \quad \overrightarrow{DC} = \frac{1}{2}\mathbf{a} + \mathbf{b}$



---

3. If D is the midpoint of the side BC of a triangle ABC, show that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$



Completing the parallelogram ABEC we see that $\overrightarrow{BE} = \overrightarrow{AC}$

Therefore $\quad \overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$

The diagonals of a parallelogram bisect each other.

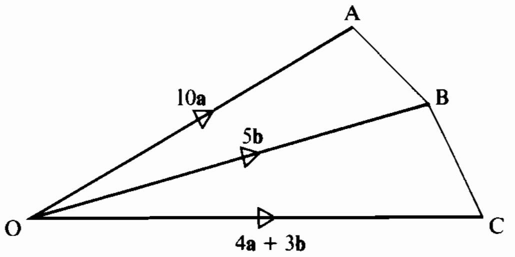
Therefore $\quad \overrightarrow{AE} = 2\overrightarrow{AD}$

$\Rightarrow \quad \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

---

4. Four points O, A, B and C are such that $\overrightarrow{OA} = 10\mathbf{a}$, $\overrightarrow{OB} = 5\mathbf{b}$ and $\overrightarrow{OC} = 4\mathbf{a} + 3\mathbf{b}$. Show that A, B and C are collinear.

If A, B and C are collinear then AB and BC have the same direction so this is what we must show.



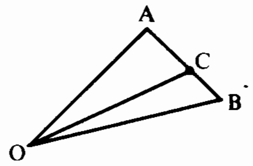
$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -10\mathbf{a} + 5\mathbf{b} = 5(\mathbf{b} - 2\mathbf{a})$$

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -5\mathbf{b} + 4\mathbf{a} + 3\mathbf{b}$$

$$= 4\mathbf{a} - 2\mathbf{b} = -2(\mathbf{b} - 2\mathbf{a})$$

AB and BC both have a direction given by $\lambda(\mathbf{b} - 2\mathbf{a})$ so they are parallel. Hence, since C is a common point, A, B and C are collinear.

Note that $\overrightarrow{BC} = -\frac{2}{5}\overrightarrow{AB}$ so, although $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel, they are in opposite directions, showing that the diagram really looks like this.



EXERCISE 39a

1. ABCD is a quadrilateral. Find the single vector which is equivalent to

(a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{BC} + \overrightarrow{CD}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ (d) $\overrightarrow{AB} + \overrightarrow{DA}$

2. ABCDEF is a regular hexagon in which $\overrightarrow{BC}$ represents $\mathbf{b}$ and $\overrightarrow{FC}$ represents $2\mathbf{a}$.
Express the vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{BE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

3. Draw diagrams representing the following vector equations.

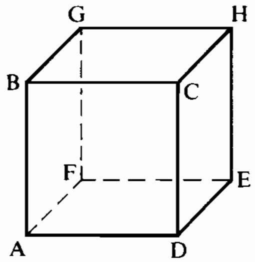
(a) $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC}$ (b) $\overrightarrow{AB} = 2\overrightarrow{PQ}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} = 3\overrightarrow{AD}$ (d) $2\overrightarrow{AB} + \overrightarrow{PQ} = 0$

4. If A, B, C, D are four points such that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} + \overrightarrow{DA} = \mathbf{0}$ prove that ABCD is a parallelogram.

5. O, A, B, C, D are five points such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{a} + 2\mathbf{b}$, $\overrightarrow{OD} = 2\mathbf{a} - \mathbf{b}$
Express $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

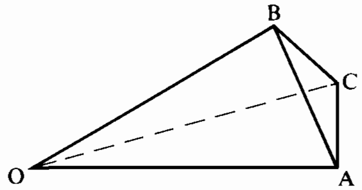
6.



If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are represented by the edges $\overrightarrow{AB}$, $\overrightarrow{AD}$, $\overrightarrow{AF}$ of the cube in the diagram, find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the vectors represented by the remaining edges.

7. If O, A, B, C are four points such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{a} - \mathbf{b}$, $\overrightarrow{OC} = \mathbf{b}$ show that A, B and C are collinear.

8.



If OABC is a tetrahedron and $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, find $\overrightarrow{AC}$, $\overrightarrow{AB}$, $\overrightarrow{CB}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

9. For the cube defined in Question 6, find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the vectors $\overrightarrow{BE}$, $\overrightarrow{GD}$, $\overrightarrow{AH}$, $\overrightarrow{FC}$.

10. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{3}\pi$, find the angle between

(a) $\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$ (b) $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ (c) $3\mathbf{a} - \mathbf{b}$ and $\mathbf{b}$

POSITION VECTORS

In general a vector has no specific location in space and is called a *free vector*. Some vectors, however, are constrained to a specific position, e.g. the vector $\overrightarrow{OA}$ where O is a fixed origin.

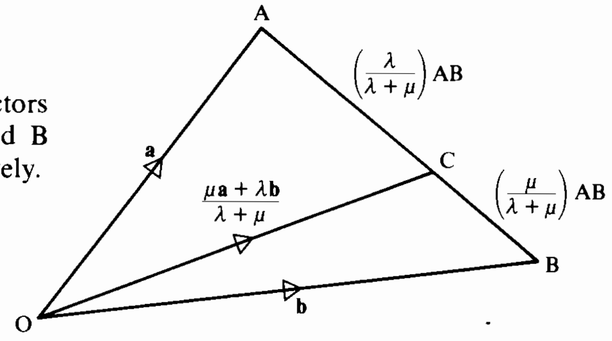
$\overrightarrow{OA}$ is called the position vector of A relative to O.

This displacement is unique and *cannot* be represented by any other line of equal length and direction.

Vectors such as $\overrightarrow{OA}$, representing quantities that have a specific location, are called *position* vectors or *tied* vectors.

The Position Vector of a Point Dividing a Given Line in a Given Ratio

Consider a line AB where the position vectors relative to O of A and B are **a** and **b** respectively.



If C divides AB in the ratio $\lambda:\mu$ then

$$\overrightarrow{AC} = \frac{\lambda}{\lambda+\mu}\overrightarrow{AB} = \frac{\lambda}{\lambda+\mu}(\overrightarrow{OB} - \overrightarrow{OA})$$

i.e. $$\overrightarrow{AC} = \frac{\lambda}{\lambda+\mu}(\mathbf{b} - \mathbf{a})$$

Now $$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{\lambda}{\lambda+\mu}(\mathbf{b} - \mathbf{a})$$

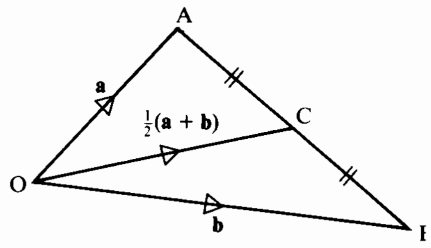
i.e. if A and B are points with position vectors **a** and **b**, and C divides AB in the ratio $\lambda:\mu$,

then the position vector of C is $\dfrac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$

A special case of this quotable result arises when $\lambda = \mu$ i.e. when C is the midpoint of AB.

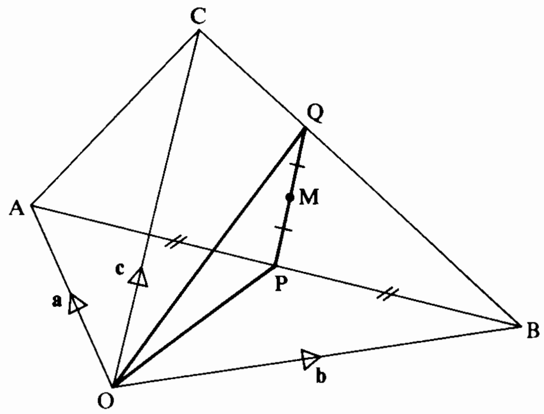
Then the position vector of C is

$\frac{1}{2}(\mathbf{a}+\mathbf{b})$



Examples 39b

1. In a triangle ABC, P is the midpoint of AB and Q divides BC in the ratio 2:1. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$, find in terms of **a**, **b** and **c**, the position vectors of P, Q and the midpoint of PQ.



AP:PB = 1:1

$\therefore$ the position vector of P is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$

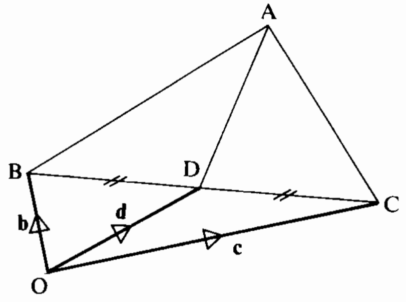
BQ:QC = 2:1

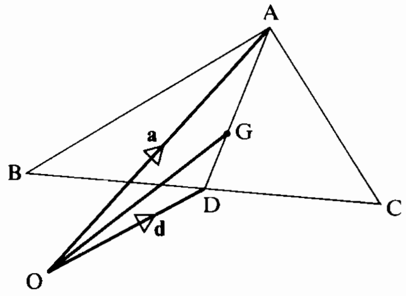
$\therefore$ the position vector of Q is $\dfrac{(1)(\mathbf{b})+(2)(\mathbf{c})}{2+1} = \frac{1}{3}(\mathbf{b}+2\mathbf{c})$

If M is the midpoint of PQ then the position vector of M is

$$\tfrac{1}{2}(\overrightarrow{OP}+\overrightarrow{OQ}) = \tfrac{1}{2}\{\tfrac{1}{2}(\mathbf{a}+\mathbf{b}) + \tfrac{1}{3}(\mathbf{b}+2\mathbf{c})\}$$
$$= \tfrac{1}{4}\mathbf{a} + \tfrac{5}{12}\mathbf{b} + \tfrac{1}{3}\mathbf{c}$$

2. Given that the centroid of a triangle divides each median in the ratio 2 : 1 from vertex to opposite side, find the position vector of the centroid of $\triangle ABC$ where the position vectors of A, B and C are **a**, **b** and **c** respectively





Considering the median AD we first find the position vector of D, the midpoint of BC.

The position vector of D is **d** where $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$
The centroid, G, divides AD in the ratio 2 : 1

So the position vector of G is $\dfrac{(1)(\mathbf{a}) + (2)(\mathbf{d})}{2 + 1} = \dfrac{\mathbf{a} + 2\{\frac{1}{2}(\mathbf{b} + \mathbf{c})\}}{3}$

$$= \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

Note that this result shows that the coordinates of the centroid of a triangle are the averages of the respective coordinates of the three vertices.

EXERCISE 39b

In this exercise the position vectors, relative to O, of A, B, C and D are **a**, **b**, **c** and **d** respectively. P, Q and R are the midpoints of AB, BC and CD respectively.

In Questions 1 to 4 find the position vector of each given point.

1. (a) The midpoint of AC (b) The midpoint of BD.
2. The point L which divides AD in the ratio 1 : 3
3. The point M which divides BC in the ratio 4 : 3
4. (a) The midpoint of PQ (b) The midpoint of QR.

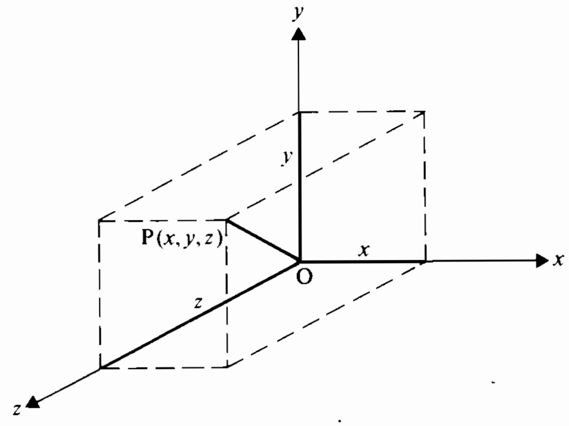
5. Show that PQ is parallel to AC
6. Find the vector represented by LD
7. Find the vector represented by AR
8. Find the position vector of the point which
   (a) divides CP in the ratio 2 : 1 (b) divides AQ in the ratio 2 : 1
   Say what you notice about your answers to (a) and (b) and explain this relationship.

THE LOCATION OF A POINT IN SPACE

We saw in Chapter 5 that any point P in a plane can be located by giving its distances from a fixed point O, in each of two perpendicular directions. These distances are the cartesian coordinates of the point.

Now we consider locating a point in three-dimensional space.

If we have a fixed point, O, then any other point can be located by giving its distances from O in each of *three* mutually perpendicular directions, i.e. we need *three* coordinates to locate a point in space. So we use the familiar x and y axes, together with a third axis Oz. Then any point has coordinates (x, y, z) relative to the origin O.

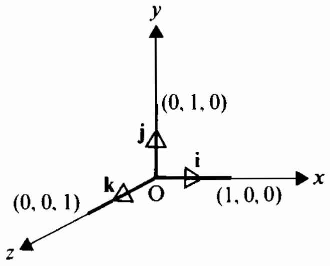


Cartesian Unit Vectors

A unit vector is a vector whose magnitude is one unit.

Now if

i is a unit vector in the direction of Ox
j is a unit vector in the direction of Oy
k is a unit vector in the direction of Oz

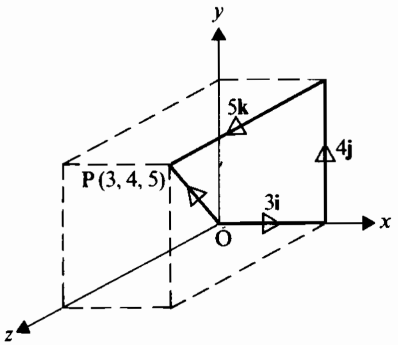


then the position vector, relative to O, of any point P can be given in terms of **i**, **j** and **k**.

e.g. the point distant
3 units from O in the direction Ox
4 units from O in the direction Oy
5 units from O in the direction Oz

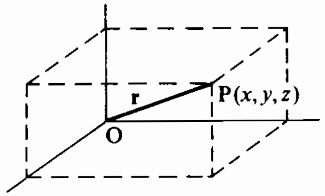
has coordinates $(3, 4, 5)$

and $\overrightarrow{OP} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$



In general, if P is a point, (x, y, z) and $\overrightarrow{OP} = \mathbf{r}$, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Free vectors can be given in the same form. For example, the vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ *can* represent the position vector of the point P(3, 4, 5) but it can equally well represent *any* vector of length and direction equal to those of OP.

Note that, unless a vector is *specified* as a position vector it is taken to be free.

OPERATIONS ON CARTESIAN VECTORS

Addition and Subtraction

To add or subtract vectors given in **i j k** form, the coefficients of **i**, **j** and **k** are collected separately,

e.g. if $\mathbf{v}_1 = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}_2 = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

then
$$\mathbf{v}_1 + \mathbf{v}_2 = (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$
$$= (3 + 1)\mathbf{i} + (2 + 2)\mathbf{j} + (2 - 3)\mathbf{k}$$
$$= 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

and
$$\mathbf{v}_1 - \mathbf{v}_2 = (3 - 1)\mathbf{i} + (2 - 2)\mathbf{j} + (2 - \{-3\})\mathbf{k}$$
$$= 2\mathbf{i} + 5\mathbf{k}$$

Modulus

The modulus of $\mathbf{v}$, where $\mathbf{v} = 12\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, is the length of OP where P is the point $(12, -3, 4)$.

Using Pythagoras twice we have

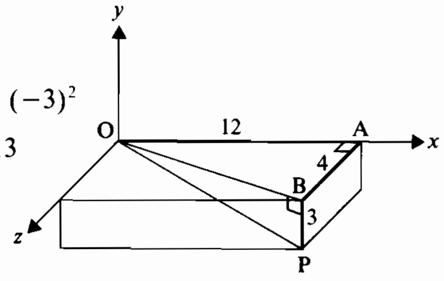
$OB^2 = OA^2 + AB^2 = 12^2 + 4^2$

$OP^2 = OB^2 + BP^2 = (12^2 + 4^2) + (-3)^2$

$\therefore \quad OP = \sqrt{(12^2 + 4^2 + 3^2)} = 13$

In general, if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$,

$$|\mathbf{v}| = \sqrt{(a^2 + b^2 + c^2)}$$



Parallel Vectors

Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel if $\mathbf{v}_1 = \lambda\mathbf{v}_2$

e.g. $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ is parallel to $4\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ $(\lambda = 2)$
and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is parallel to $-3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$ $(\lambda = -3)$

Equal Vectors

If two vectors $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are equal then

$$a_1 = a_2 \text{ and } b_1 = b_2 \text{ and } c_1 = c_2$$

Examples 39c

1. Given the vector $\mathbf{v}$ where $\mathbf{v} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, state whether each of the following vectors is parallel to $\mathbf{v}$, equal to $\mathbf{v}$ or neither.
   (a) $10\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ (b) $-\frac{1}{2}(-10\mathbf{i} + 4\mathbf{j} - 8\mathbf{k})$
   (c) $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ (d) $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

(a) $10\mathbf{i} - 4\mathbf{j} + 8\mathbf{k} = 2(5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ $(\lambda = 2)$

$\therefore$ $10\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ is parallel to $\mathbf{v}$

(b) $-\frac{1}{2}(-10\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}) = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\therefore$ $-\frac{1}{2}(-10\mathbf{i} + 4\mathbf{j} - 8\mathbf{k})$ is equal to $\mathbf{v}$

(c) $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} = -(5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ $(\lambda = -1)$

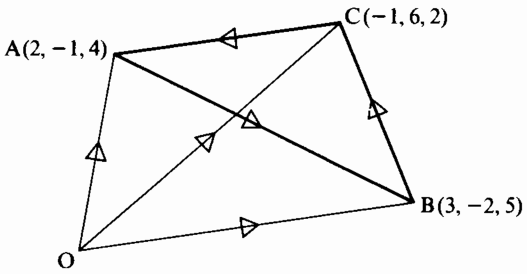
$\therefore$ $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is parallel to $\mathbf{v}$

(d) $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ is not a multiple of $5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\therefore$ $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ is not parallel to $\mathbf{v}$

---

2. A triangle ABC has its vertices at the points A(2, −1, 4), B(3, −2, 5) and C(−1, 6, 2). Find, in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ and hence find the lengths of the sides of the triangle.



The coordinate axes are not drawn in this diagram as they tend to cause confusion when two or more points are illustrated. The origin should always be included however as it provides a reference point.

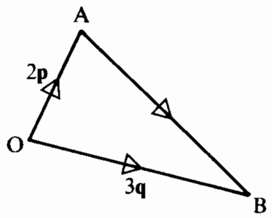
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$
$$= \mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= (-\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$$
$$= -4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$$
$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$$
$$= (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) - (-\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$$
$$= 3\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$$

Hence
$$AB = |\overrightarrow{AB}| = \sqrt{\{(1)^2 + (-1)^2 + (1)^2\}} = \sqrt{3}$$
$$BC = |\overrightarrow{BC}| = \sqrt{\{(-4)^2 + (8)^2 + (-3)^2\}} = \sqrt{89}$$
$$CA = |\overrightarrow{CA}| = \sqrt{\{(3)^2 + (-7)^2 + (2)^2\}} = \sqrt{62}$$

Two-dimensional problems can be solved by using the same principles as for three-dimensional cases but the working tends to be easier because it involves fewer terms.

---

3. Given that $\mathbf{p} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{OA} = 2\mathbf{p}$ and $\mathbf{OB} = 3\mathbf{q}$, find (a) $|\mathbf{OA}|$ (b) $|\mathbf{OB}|$ (c) $|\mathbf{AB}|$



(a) $|\mathbf{OA}| = 2|\mathbf{i} + 3\mathbf{j}| = 2\sqrt{(1^2 + 3^2)} = 2\sqrt{10}$

(b) $|\mathbf{OB}| = 3|4\mathbf{i} - 2\mathbf{j}| = 3\sqrt{\{4^2 + (-2)^2\}} = 6\sqrt{5}$

(c) $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = 3(4\mathbf{i} - 2\mathbf{j}) - 2(\mathbf{i} + 3\mathbf{j}) = 10\mathbf{i} - 12\mathbf{k}$

$|\mathbf{AB}| = \sqrt{(10^2 + 12^2)} = 2\sqrt{61}$

---

4. If P is a point with position vector $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, find the cartesian equation of the curve on which P lies.

If P is the point (x, y) and $\mathbf{OP} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ then

$$x = \cos\theta \quad \text{and} \quad y = \sin\theta$$

These are the parametric equations of the locus of P. We can find the cartesian equation by eliminating θ between them.

Using $\cos^2\theta + \sin^2\theta = 1$

gives $x^2 + y^2 = 1$

Therefore P lies on the curve $x^2 + y^2 = 1$ which can be recognised as a circle with radius 1 and centre at O.

EXERCISE 39c

1. Write down, in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the vector represented by $\overrightarrow{OP}$ if P is a point with coordinates

   (a) $(3, 6, 4)$ (b) $(1, -2, -7)$ (c) $(1, 0, -3)$

2. $\overrightarrow{OP}$ represents a vector $\mathbf{r}$. Write down the coordinates of P if

   (a) $\mathbf{r} = 5\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{r} = \mathbf{i} + 4\mathbf{j}$ (c) $\mathbf{r} = \mathbf{j} - \mathbf{k}$

3. Find the length of the line OP if P is the point

   (a) $(2, -1, 4)$ (b) $(3, 0, 4)$ (c) $(-2, -2, 1)$

4. Find the modulus of the vector $\mathbf{V}$ if

   (a) $\mathbf{V} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$ (b) $\mathbf{V} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

   (c) $\mathbf{V} = 11\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$

5. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ find

   (a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{c}$ (c) $\mathbf{a} + \mathbf{b} + \mathbf{c}$ (d) $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$

In Questions 6 to 8, $\overrightarrow{OA} = \mathbf{a} = 4\mathbf{i} - 12\mathbf{j}$ and $\overrightarrow{OB} = \mathbf{b} = \mathbf{i} + 6\mathbf{j}$.

6. Which of the following vectors are parallel to $\mathbf{a}$?

   (a) $\mathbf{i} + 3\mathbf{j}$ (b) $-\mathbf{i} + 3\mathbf{j}$ (c) $12\mathbf{i} - 4\mathbf{j}$ (d) $-4\mathbf{i} + 12\mathbf{j}$ (e) $\mathbf{i} - 3\mathbf{j}$

7. Which of the following vectors are equal to $\mathbf{b}$?

   (a) $2\mathbf{i} + 12\mathbf{j}$ (b) $-\mathbf{i} - 6\mathbf{j}$ (c) $\overrightarrow{AE}$ if E is $(5, -6)$

   (d) $\overrightarrow{AF}$ if F is $(6, 0)$

8. If $\overrightarrow{OD} = \lambda\overrightarrow{OA}$, find the value of λ for which $\overrightarrow{OD} + \overrightarrow{OB}$ is parallel to the x-axis

9. Which of the following vectors are parallel to $3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$?

   (a) $6\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ (b) $-9\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ (c) $-3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

   (d) $-2(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ (e) $\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$ (f) $-\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

10. Given that $\mathbf{a} = 4\mathbf{i} + \mathbf{j} - 6\mathbf{k}$, state whether each of the following vectors is parallel or equal to $\mathbf{a}$ or neither.

    (a) $8\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$ (b) $-4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ (c) $2(2\mathbf{i} + \frac{1}{2}\mathbf{j} - 3\mathbf{k})$

11. The triangle ABC has its vertices at the points $A(-1, 3, 0)$, $B(-3, 0, 7)$, $C(-1, 2, 3)$,. Find in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ the vectors representing

    (a) $\overrightarrow{AB}$ (b) $\overrightarrow{AC}$ (c) $\overrightarrow{CB}$

12. Find the lengths of the sides of the triangle described in Question 11.

13. Find $|\mathbf{a} - \mathbf{b}|$ where $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

14. A, B, C and D are the points $(0, 0, 2)$, $(-1, 3, 2)$, $(1, 0, 4)$ and $(-1, 2, -2)$ respectively. Find the vectors representing $\overrightarrow{AB}$, $\overrightarrow{BD}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$

15. If the position vector of P is $t^2\mathbf{i} + 2t\mathbf{j}$, find the cartesian equation of the locus of P and name this curve.

16. Show that, for all values of t, the point whose equation vector is $\mathbf{r} = t\mathbf{i} + (2t - 1)\mathbf{j}$ lies on the line $y = 2x - 1$

DIRECTION RATIOS

The vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ can be represented by $\overrightarrow{OP}$ where P is the point (a, b, c)

The coordinates of P determine the direction of OP relative to the axes and

the ratios $a:b:c$ are called the direction ratios of v

We often use the abbreviation d.r.s for direction ratios

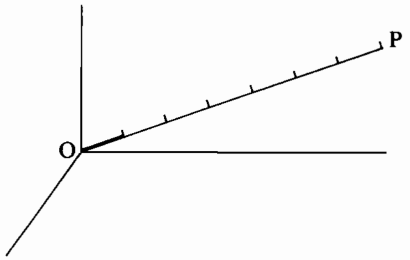
e.g. the d.r.s of $5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ are $5:-3:7$

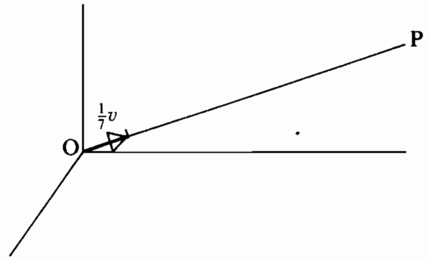
It follows that *the direction ratios of parallel vectors are equal*,

e.g. $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ has d.r.s $1:-2:3$ and is parallel to any vector whose d.r.s are $t:-2t:3t$, for instance $3:-6:9 = 1:-2:3$ and $-2:4:-6 = 1:-2:3$

FINDING A UNIT VECTOR

Consider the vector $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, represented by $\overrightarrow{OP}$ where P is the point $(6, 2, 3)$





Now $|\mathbf{v}|$ is $\sqrt{(6^2 + 2^2 + 3^2)}$, i.e. OP is 7 units long.
Therefore $\frac{1}{7}\mathbf{v}$ is a vector of unit magnitude and is denoted by $\hat{\mathbf{v}}$

i.e. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ or $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{v}}$

In general **any vector v is the product of its magnitude and a unit vector in the same direction**

and **a unit vector in the direction of v is given by $\dfrac{\mathbf{v}}{|\mathbf{v}|}$**

Using Direction Vectors

A vector which is used to specify the direction of another vector can be called a *direction* vector.
e.g. if we are told that a vector $\mathbf{v}$, of magnitude 14 units, is parallel to the vector $3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, then $3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ is a direction vector for $\mathbf{v}$

Therefore the unit vector in the direction of $\mathbf{v}$ is $\dfrac{3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{(3^2 + 6^2 + 2^2)}}$

i.e. $\hat{\mathbf{v}} = \frac{1}{7}(3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$

Then $\mathbf{v} = |\mathbf{v}|(\hat{\mathbf{v}}) = (14)\{\frac{1}{7}(3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})\} = 6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$

Examples 39d

1. Find the coordinates of P if OP is of length 5 units and is parallel to the vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

A vector parallel to $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ can be in either of the directions given by $\pm(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$\therefore$ the direction vector for **OP** is $\pm(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$$|2\mathbf{i} - \mathbf{j} + 4\mathbf{k}| = \sqrt{(2^2 + \{-1\}^2 + 4^2)} = \sqrt{21}$$

$\therefore$ the unit direction vector in the direction of **OP** is

$$\frac{\pm(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})}{\sqrt{21}}$$

Now $\mathbf{OP} = |\mathbf{OP}|(\widehat{\mathbf{OP}})$

therefore

$$\mathbf{OP} = (5)\left\{\frac{\pm(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})}{\sqrt{21}}\right\} = \pm\left(\frac{10}{\sqrt{21}}\mathbf{i} - \frac{5}{\sqrt{21}}\mathbf{j} + \frac{20}{\sqrt{21}}\mathbf{k}\right)$$

The coordinates of P are

$$\text{either } \left(\frac{10}{\sqrt{21}}, \frac{-5}{\sqrt{21}}, \frac{20}{\sqrt{21}}\right) \text{ or } \left(\frac{-10}{\sqrt{21}}, \frac{5}{\sqrt{21}}, \frac{-20}{\sqrt{21}}\right)$$

2. (a) Write down the direction ratios of the vectors
   (i) $\mathbf{i}+4\mathbf{j}-7\mathbf{k}$ (ii) $2\mathbf{i}-9\mathbf{k}$

   (b) Find a unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$

(a) (i) The d.r.s of $\mathbf{i}+4\mathbf{j}-7\mathbf{k}$ are $1:4:-7$
(ii) $2\mathbf{i}-9\mathbf{k} = 2\mathbf{i}+0\mathbf{j}-9\mathbf{k}$ so the d.r.s are $2:0:-9$

(b) If $\mathbf{v} = 2\mathbf{i}-\mathbf{j}-2\mathbf{k}$ then $|\mathbf{v}| = \sqrt{\{2^2+(-1)^2+(-2)^2\}} = 3$

$$\therefore \quad \hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{3} = \tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k}$$

EXERCISE 39d

1. Write down the direction ratios of the vectors
   (a) $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ (b) $6\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ (c) $3\mathbf{i}+4\mathbf{k}$ (d) $\mathbf{i}+8\mathbf{j}+4\mathbf{k}$

2. Find a unit vector in the direction of each of the vectors in Qu. 1.

3. Find the coordinates of Q if $|\mathbf{OQ}| = 1$ and $\mathbf{OQ}$ is in the direction of
   (a) $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ (b) $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ (c) $8\mathbf{i}-\mathbf{j}-4\mathbf{k}$ (d) $\mathbf{i}-\mathbf{j}-\mathbf{k}$

4. Find the coordinates of P if
   (a) $|\mathbf{OP}| = 6$ and $\mathbf{OP}$ is in the direction of $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$
   (b) $|\mathbf{OP}| = 2$ and $\mathbf{OP}$ is in the direction of $8\mathbf{i}+4\mathbf{j}-\mathbf{k}$

5. Find the vector $\mathbf{V}$ if
   (a) $\mathbf{V} = \mathbf{OP}$ where P is the point $(0, 4, 5)$
   (b) $|\mathbf{V}| = 24$ units and $\hat{\mathbf{V}} = \tfrac{2}{3}\mathbf{i}-\tfrac{2}{3}\mathbf{j}-\tfrac{1}{3}\mathbf{k}$
   (c) $\mathbf{V}$ is parallel to the vector $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and equal in magnitude to the vector $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

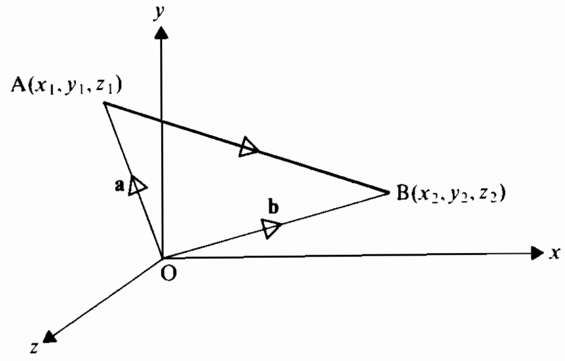
6. Find $\hat{\mathbf{r}}$ in the form $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ if
   (a) $\mathbf{r} = \mathbf{i}-\mathbf{j}+\mathbf{k}$ (b) $\mathbf{r} = 5\mathbf{j}-12\mathbf{k}$ (c) $\mathbf{r} = \mathbf{i}$

7. If $\mathbf{r}_1 = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{r}_2 = \mathbf{i}+3\mathbf{j}+2\mathbf{k}$, find the modulus and direction ratios of $\mathbf{r}_1+\mathbf{r}_2$ and $\mathbf{r}_1-\mathbf{r}_2$

8. If $\overrightarrow{OA} = 3\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i}-3\mathbf{j}+5\mathbf{k}$ show that $\overrightarrow{AB}$ is parallel to the vector with d.r.s $1:2:-3$

PROPERTIES OF A LINE JOINING TWO POINTS

Consider the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$



$$\overrightarrow{OA} = x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$$

and $\quad \overrightarrow{AB} = \overrightarrow{AO}+\overrightarrow{OB} = \overrightarrow{OB}-\overrightarrow{OA}$

hence $\quad \overrightarrow{AB} = (x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$

The Length of AB

$$AB = |(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}|$$

so the length of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

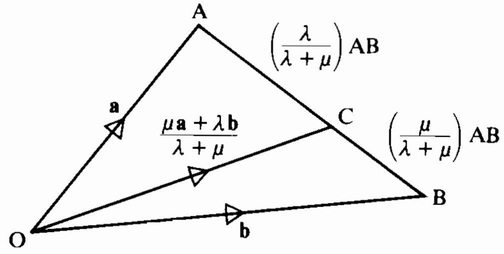
$$\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}$$

The Direction of AB

$$\overrightarrow{AB} = (x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$$

so $\quad$ the direction ratios of $\overrightarrow{AB}$ are $(x_2-x_1):(y_2-y_1):(z_2-z_1)$

The Position Vector of a Point Dividing AB in the Ratio $\lambda:\mu$



We saw on p. 726 that if C divides AB in the ratio $\lambda:\mu$ then the position vector of C is $\dfrac{\mu\overrightarrow{OA}+\lambda\overrightarrow{OB}}{\lambda+\mu}$

If A is the point (x_1, y_1, z_1) and B is (x_2, y_2, z_2) the coordinates of C are

$$\frac{\mu x_1+\lambda x_2}{\lambda+\mu},\ \frac{\mu y_1+\lambda y_2}{\lambda+\mu},\ \frac{\mu z_1+\lambda z_2}{\lambda+\mu}$$

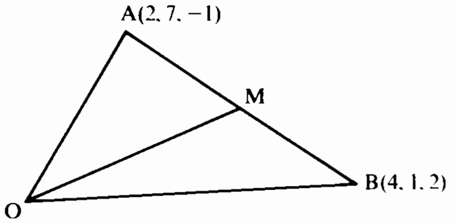
In particular, if $\lambda = \mu$ so that C bisects AB we see that the coordinates of the midpoint of AB are

$$\tfrac{1}{2}(x_1+x_2),\ \tfrac{1}{2}(y_1+y_2),\ \tfrac{1}{2}(z_1+z_2)$$

i.e. **the coordinates of the midpoint are the averages of the respective coordinates of the end points**

Examples 39e

1. Find the length of the median through O of the triangle OAB, where A is the point $(2, 7, -1)$ and B is the point $(4, 1, 2)$



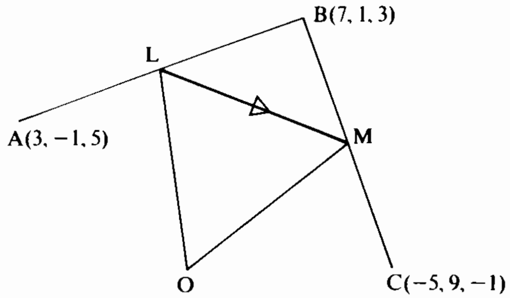
The coordinates of M, the midpoint of AB, are

$(\frac{1}{2}\{2+4\}, \frac{1}{2}\{7+1\}, \frac{1}{2}\{-1+2\})$

i.e. $(3, 4, \frac{1}{2})$

So the length of OM is $\sqrt{(3^2+4^2+\frac{1}{2}^2)} = \frac{1}{2}\sqrt{101}$

2. The points A, B and C have coordinates $(3, -1, 5)$, $(7, 1, 3)$ and $(-5, 9, -1)$ respectively. If L is the midpoint of AB and M is the midpoint of BC, find the length and direction ratios of LM.



L is the point $(\frac{1}{2}\{3+7\}, \frac{1}{2}\{-1+1\}, \frac{1}{2}\{5+3\})$, i.e. $(5, 0, 4)$

M is the point $(\frac{1}{2}\{7-5\}, \frac{1}{2}\{1+9\}, \frac{1}{2}\{3-1\})$, i.e. $(1, 5, 1)$

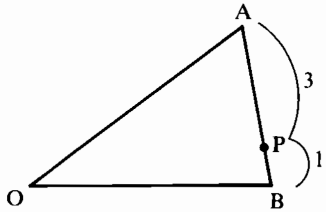
$\therefore \quad LM = \sqrt{(\{1-5\}^2+\{5-0\}^2+\{1-4\}^2)} = 5\sqrt{2}$

Now $\overrightarrow{LM} = (1-5)\mathbf{i}+(5-0)\mathbf{j}+(1-4)\mathbf{k}$

$\therefore$ the d.r.s of LM are $-4:5:-3$

3. A and B are two points with position vectors $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{i}-3\mathbf{j}-\mathbf{k}$ respectively. Find the position vectors of the points P and Q if
   (a) P divides AB internally in the ratio 3:1
   (b) Q divides AB externally in the ratio 3:1

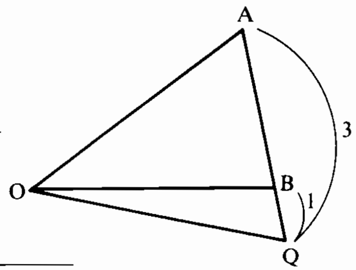
(a) $AP:PB = 3:1$



$$\therefore \quad \overrightarrow{OP} = \frac{1(3\mathbf{i}+\mathbf{j}-2\mathbf{k})+3(\mathbf{i}-3\mathbf{j}-\mathbf{k})}{3+1}$$

$$= \tfrac{3}{2}\mathbf{i}-2\mathbf{j}-\tfrac{5}{4}\mathbf{k}$$

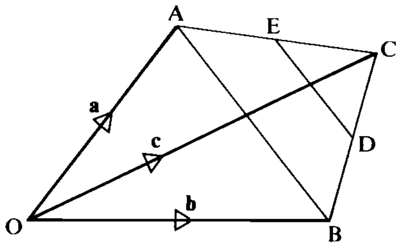
(b) $AQ:QB = 3:-1$



$$\therefore \quad \overrightarrow{OQ} = \frac{-1(3\mathbf{i}+\mathbf{j}-2\mathbf{k})+3(\mathbf{i}-3\mathbf{j}-\mathbf{k})}{3-1}$$

$$= -5\mathbf{j}-\tfrac{1}{2}\mathbf{k}$$

4. A, B and C are the points with position vectors $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively. If D and E are the respective midpoints of BC and AC, show that DE is parallel to AB.



Using $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ we have

$$\overrightarrow{OD} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c}) = \tfrac{1}{2}(4\mathbf{i} - \mathbf{j} - \mathbf{k})$$

and $$\overrightarrow{OE} = \tfrac{1}{2}(\mathbf{a} + \mathbf{c}) = \tfrac{1}{2}(5\mathbf{i} + 3\mathbf{k})$$

$\therefore$ $$\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \tfrac{1}{2}(\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

Also $$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -\mathbf{i} - \mathbf{j} - 4\mathbf{k}$$

The d.r.s of $\overrightarrow{DE}$ are $1:1:4$

and the d.r.s of $\overrightarrow{AB}$ are $-1:-1:-4 = 1:1:4$

$\therefore$ AB and DE are parallel.

EXERCISE 39e

In this exercise A, B, C and D are the points with position vectors $\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j}$ respectively.

1. Find $|\mathbf{AB}|$ and $|\mathbf{BD}|$.

2. Find the direction ratios of **CD** and **AC**.

3. If P divides BD in the ratio $1:2$, find the position vector of P.

4. Find the position vector of the point which
(a) divides BC internally in the ratio $3:2$
(b) divides AC externally in the ratio $3:2$

5. Determine whether any of the following pairs of lines are parallel.
(a) AB and CD (b) AC and BD (c) AD and BC.

6. If L and M are the position vectors of the midpoints of AD and BD respectively, show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$.

7. If H and K are the midpoints of AC and CD respectively show that $\overrightarrow{HK} = \tfrac{1}{2}\overrightarrow{AD}$.

8. If L, M, N and P are the midpoints of AD, BD, BC and AC respectively, show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{NP}$.

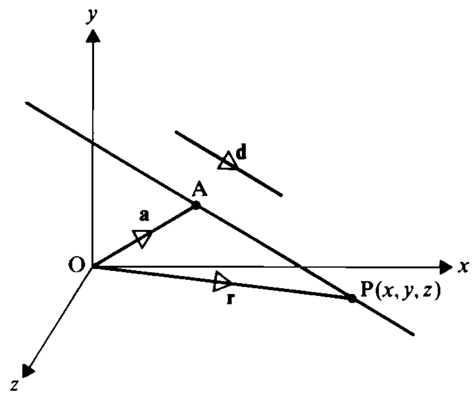
THE EQUATION OF A STRAIGHT LINE

A straight line is located uniquely in space if
either it passes through a known fixed point and has a known direction,
or it passes through two known fixed points.

In each of these cases we can find equations which describe the set of points on the line.

A Line with Known Direction and Passing through a Known Fixed Point

Consider a line for which $\mathbf{d}$ is a direction vector and which passes through a fixed point A with position vector $\mathbf{a}$



$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AP} = \lambda\mathbf{d}$ where λ is a scalar

If $P(x, y, z)$ is *any* point on the line and $\mathbf{r}$ is the position vector of P

then $\quad \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \quad \Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

For example, the vector equation of the line which passes through the point $(5, -2, 4)$ and is parallel to the vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, is

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \qquad [1]$$

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is called the *vector equation* of the line.

Each value of the parameter λ gives the position vector of one point on the line, e.g. taking $\lambda = 1$ in equation [1] above gives the position vector $7\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$, so $(7, -3, 7)$ is a point on the given line.

Now replacing $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in equation [1] gives

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (5 + 2\lambda)\mathbf{i} + (-2 - \lambda)\mathbf{j} + (4 + 3\lambda)\mathbf{k}$$

Hence $\quad x = 5 + 2\lambda, \quad y = -2 - \lambda, \quad z = 4 + 3\lambda \qquad [2]$

These three equations also define the set of points on the given line, and are called the *parametric equations* of the line.

If we now isolate λ in each of the parametric equations in [2] we get

$$\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3} \quad (= \lambda) \qquad [3]$$

These are the *cartesian equations* of the line.

Note that, as the direction of the line is given by $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, the direction ratios of the line are $2:-1:3$ and *these appear in the equation(s) of the line in all three forms.*

In general, the equation(s) of a line parallel to $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and passing through $A(x_1, y_1, z_1)$ can be written

$\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} + \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$ in vector form

or $\quad x = x_1 + \lambda a, \quad y = y_1 + \lambda b, \quad z = z_1 + \lambda c$ in parametric form

or $\quad \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ in cartesian form

Note that the point (x_1, y_1, z_1) is only one of an infinite set of fixed points on the line, so the equations above are *not* unique.

A Line Passing through Two Fixed Points

If a line passes through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then the direction vector for the line is $\mathbf{d}$ where

$$\mathbf{d} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

and the direction ratios of the line are

$$(x_2 - x_1):(y_2 - y_1):(z_2 - z_1)$$

These facts, together with either A or B as a fixed point on the line, allow the equations of the line to be written down in any of the three forms given above.

Examples 39f

1. A line passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and is in the direction of $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find equations for the line in vector and in cartesian form.

The equation of the line in vector form is

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

This shows that the coordinates of any point P on the line are

$$(\{2 + \lambda\}, \{-1 + \lambda\}, \{4 - 2\lambda\})$$

Hence the cartesian equations are

$$x - 2 = y + 1 = \frac{z-4}{-2} \quad (= \lambda)$$

2. Find a vector equation for the line through the points $A(3, 4, -7)$ and $B(1, -1, 6)$.

$$\overrightarrow{OA} = \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$$
$$\overrightarrow{OB} = \mathbf{b} = \mathbf{i} - \mathbf{j} + 6\mathbf{k}$$

For any point P on the line, $\overrightarrow{OP} = \mathbf{r}$

so $\quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\quad = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} + \lambda(-2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k})$

3. Show that the line through the points $\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $4\mathbf{i}+7\mathbf{j}+\mathbf{k}$ is parallel to the line $\mathbf{r} = \mathbf{i}-\mathbf{k}+\lambda(\frac{3}{2}\mathbf{i}+3\mathbf{j}+2\mathbf{k})$.

The line through the two given points has direction ratios

$$(1-4):(1-7):(-3-1) = 3:6:4$$

The given line has direction ratios $\frac{3}{2}:3:2 = 3:6:4$

Hence the two lines are parallel.

---

4. (a) Find a vector equation for the line whose cartesian equations are

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

(b) The vector equation of a line is $\mathbf{r} = \mathbf{i}-3\mathbf{j}+2\mathbf{k}+\lambda(5\mathbf{i}+2\mathbf{j}-\mathbf{k})$. Express the equation of this line in parametric form and hence find the coordinates of the points where the line crosses the xy plane.

(a) Comparing $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ with

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

shows that one point on the line has coordinates $(5, -4, 6)$ and its position vector is therefore $5\mathbf{i}-4\mathbf{j}+6\mathbf{k}$

We also see that the d.r.s of the line are $3:7:2$ so its direction vector is $3\mathbf{i}+7\mathbf{j}+2\mathbf{k}$

Hence a vector equation for the line is

$$\mathbf{r} = 5\mathbf{i}-4\mathbf{j}+6\mathbf{k}+\lambda(3\mathbf{i}+7\mathbf{j}+2\mathbf{k})$$

(b) $\mathbf{r} = \mathbf{i}-3\mathbf{j}+2\mathbf{k}+\lambda(5\mathbf{i}+2\mathbf{j}-\mathbf{k})$

gives $\mathbf{r} = (1+5\lambda)\mathbf{i}+(-3+2\lambda)\mathbf{j}+(2-\lambda)\mathbf{k}$

So, for any point on the line, the coordinates are given by

$$x = 1+5\lambda, \quad y = -3+2\lambda, \quad z = 2-\lambda$$

These are the parametric equations of the line.

At the point where the line crosses the xy plane, $z = 0$

i.e. $2-\lambda = 0 \Rightarrow \lambda = 2$

When $\lambda = 2$, $x = 11$ and $y = 1$

Therefore the line crosses the xy plane at the point $(11, 1, 0)$

EXERCISE 39f

1. Convert the following vector equations to cartesian form.
   (a) $\mathbf{r} = 2\mathbf{i}+3\mathbf{j}-\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})$
   (b) $\mathbf{r} = 4\mathbf{j}+\lambda(3\mathbf{i}+5\mathbf{k})$
   (c) $\mathbf{r} = \lambda(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$

2. Convert to vector form, the following equations,
   (a) $\frac{x-3}{4} = \frac{y-1}{2} = \frac{z-7}{6}$
   (b) $x = 3\lambda+2,\ y = \lambda-5,\ z = 4\lambda+1$
   (c) $\frac{1-x}{3} = \frac{y}{5} = z$

3. Write down equations, in vector and in cartesian form, for the line through a point A with position vector $\mathbf{a}$ and with a direction vector $\mathbf{b}$ if
   (a) $\mathbf{a} = \mathbf{i}-3\mathbf{j}+2\mathbf{k}$ $\quad \mathbf{b} = 5\mathbf{i}+4\mathbf{j}-\mathbf{k}$
   (b) $\mathbf{a} = 2\mathbf{i}+\mathbf{j}$ $\quad \mathbf{b} = 3\mathbf{j}-\mathbf{k}$
   (c) A is the origin $\quad \mathbf{b} = \mathbf{i}-\mathbf{j}-\mathbf{k}$

4. State whether or not the following pairs of lines are parallel.
   (a) $\mathbf{r} = \mathbf{i}+\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i}-4\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})$
   (b) $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z+1}{-4}$ and $\frac{x}{4} = \frac{y+5}{6} = \frac{3-z}{8}$
   (c) $\mathbf{r} = 2\mathbf{i}-\mathbf{j}+4\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+3\mathbf{k})$ and $x-4 = y+7 = \frac{z}{3}$
   (d) $\mathbf{r} = \lambda(3\mathbf{i}-3\mathbf{j}+6\mathbf{k})$ and $\mathbf{r} = 4\mathbf{j}+\lambda(-\mathbf{i}+\mathbf{j}-2\mathbf{k})$
   (e) $\mathbf{r} = 3\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}-2\mathbf{k})$ and $\frac{x-3}{1} = \frac{y}{1} = \frac{z-1}{2}$

5. The points A(4, 5, 10), B(2, 3, 4) and C(1, 2, −1) are three vertices of a parallelogram ABCD. Find vector and cartesian equations for the sides AB and BC and find the coordinates of D.

6. Write down a vector equation for the line through A and B if
   (a) $\overrightarrow{OA}$ is $3\mathbf{i}+\mathbf{j}-4\mathbf{k}$ and $\overrightarrow{OB}$ is $\mathbf{i}+7\mathbf{j}+8\mathbf{k}$
   (b) A and B have coordinates (1, 1, 7) and (3, 4, 1).
   Find, in each case, the coordinates of the points where the line crosses the *xy* plane, the *yz* plane and the *zx* plane.

7. A line has cartesian equations $\dfrac{x-1}{3}=\dfrac{y+2}{4}=\dfrac{z-4}{5}$
   Find a vector equation for a parallel line passing through the point with position vector $5\mathbf{i}-2\mathbf{j}-4\mathbf{k}$ and find the coordinates of the point on this line where $y=0$

8. The cartesian equations of a line are $x-2=2y+1=3z-2$
   Find the direction ratios of the line and write down the vector equation of the line through (2, −1, −1) which is parallel to the given line.

9. Given four points, A(1, 2, 4), B(2, 4, 2), C(6, 7, 1) and D(−3, 4, −2), find
   (a) parametric equations for the line, L, through A and B,
   (b) the coordinates of the point P where L cuts the *xy* plane.
   Show that P divides CD in the ratio 1 : 2

PAIRS OF LINES

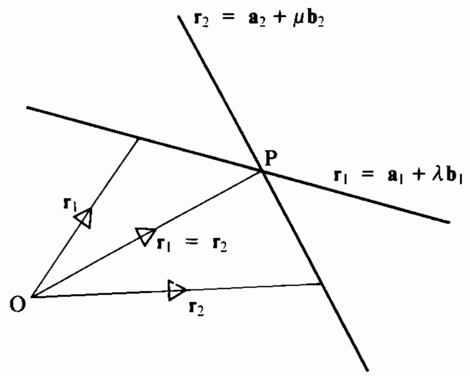
The location of two lines in space may be such that the lines

either are parallel
or are not parallel and intersect
or are not parallel and do not intersect (such lines are *skew*).

Parallel Lines

We have already seen that parallel lines have equal direction ratios. So if two lines are parallel, this property can be observed from their equations.

Non-parallel Lines



Consider two lines whose vector equations are

$$\mathbf{r}_1=\mathbf{a}_1+\lambda\mathbf{b}_1 \quad\text{and}\quad \mathbf{r}_2=\mathbf{a}_2+\mu\mathbf{b}_2$$

If these lines intersect it will be at a point where $\mathbf{r}_1=\mathbf{r}_2$

This is possible only if there are unique values for λ and μ for which

$$\mathbf{a}_1+\lambda\mathbf{b}_1=\mathbf{a}_2+\mu\mathbf{b}_2$$

If no such values can be found the lines do not intersect.

Example 39g

Find out whether the following pairs of lines are parallel, non-parallel and intersecting, or non-parallel and non-intersecting:

(a) $\mathbf{r}_1=\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{i}-2\mathbf{j}+4\mathbf{k})$
$\mathbf{r}_2=2\mathbf{i}-\mathbf{j}+3\mathbf{k}+\mu(-6\mathbf{i}+4\mathbf{j}-8\mathbf{k})$,

(b) $\mathbf{r}_1=\mathbf{i}-\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k})$
$\mathbf{r}_2=2\mathbf{i}+4\mathbf{j}+6\mathbf{k}+\mu(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$,

(c) $\mathbf{r}_1=\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}+4\mathbf{k})$
$\mathbf{r}_2=2\mathbf{i}+3\mathbf{j}+\mu(4\mathbf{i}-\mathbf{j}+\mathbf{k})$.

(a) Checking first whether the lines are parallel we compare the direction ratios of the two lines.

The first line has direction ratios 3 : −2 : 4
The second line has direction ratios −6 : 4 : −8 = 3 : −2 : 4

Therefore these two lines are parallel.

(b) In this case the two sets of direction ratios are $1:-1:1$ and $2:1:3$
These are not equal, so these two lines are not parallel.
Now if the lines intersect it will be at a point where $\mathbf{r}_1 = \mathbf{r}_2$, i.e. where

$$(1+\lambda)\mathbf{i}-(1+\lambda)\mathbf{j}+(3+\lambda)\mathbf{k} = 2(1+\mu)\mathbf{i}+(4+\mu)\mathbf{j}+(6+3\mu)\mathbf{k}$$

Equating the coefficients of **i** and **j**, we have

$$1+\lambda = 2(1+\mu)$$

and $$-(1+\lambda) = 4+\mu$$

Hence $$\mu = -2, \quad \lambda = -3$$

With these values for λ and μ, the coefficients of **k** become

first line $3+\lambda = 0$
second line $6+3\mu = 0$ } i.e. equal values.

So $\mathbf{r}_1 = \mathbf{r}_2$ when $\lambda = -3$ and $\mu = -2$

Therefore the lines *do* intersect at the point with position vector

$$(1-3)\mathbf{i}-(1-3)\mathbf{j}+(3-3)\mathbf{k} \qquad (\lambda = -3 \text{ in } \mathbf{r}_1)$$

i.e. $$-2\mathbf{i}+2\mathbf{j}$$

(c) The direction ratios of these two lines are not equal so the lines are not parallel.
If the lines intersect it will be where $\mathbf{r}_1 = \mathbf{r}_2$, i.e. where

$$(1+\lambda)\mathbf{i}+3\lambda\mathbf{j}+(1+4\lambda)\mathbf{k} = (2+4\mu)\mathbf{i}+(3-\mu)\mathbf{j}+\mu\mathbf{k}$$

Equating the coefficients of **i** and **j** we have

$$1+\lambda = 2+4\mu$$

and $$3\lambda = 3-\mu$$

Hence $$\mu = 0, \quad \lambda = 1$$

With these values of λ and μ, the coefficients of **k** become

for the first line $1+4\lambda = 5$
for the second line $\mu = 0$

As these are unequal there are no values of λ and μ for which $\mathbf{r}_1 = \mathbf{r}_2$

Therefore these two lines do not intersect and are skew.

EXERCISE 39g

In Questions 1 to 3 find whether the given two lines are parallel, intersecting or skew. If they intersect, state the position vector of the common point.

1. $\mathbf{r}_1 = \mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(3\mathbf{i}-4\mathbf{j}+\mathbf{k})$ and $\mathbf{r}_2 = \mu(-9\mathbf{i}+12\mathbf{j}-3\mathbf{k})$

2. $\dfrac{x-4}{1} = \dfrac{y-8}{2} = \dfrac{z-3}{1}$ and $\dfrac{x-7}{6} = \dfrac{y-6}{4} = \dfrac{z-5}{5}$

3. $\mathbf{r}_1 = \mathbf{i}+3\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ and $\mathbf{r}_2 = 2\mathbf{i}-\mathbf{j}+\mathbf{k}+\mu(\mathbf{i}-2\mathbf{j})$

4. Two lines have equations $\mathbf{r}_1 = 2\mathbf{i}+9\mathbf{j}+13\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ and $\mathbf{r}_2 = a\mathbf{i}+7\mathbf{j}-2\mathbf{k}+\mu(-\mathbf{i}+2\mathbf{j}-3\mathbf{k})$.
Given that they intersect, find the value of a and the position vector of the point of intersection.

5. If the line $\mathbf{r}_1 = 5\mathbf{j}+t\mathbf{k}+\lambda(-\mathbf{i}+3\mathbf{j}+4\mathbf{k})$ intersects the line $\mathbf{r}_2 = -4\mathbf{i}+\mathbf{j}+\mathbf{k}+\mu(5\mathbf{i}+\mathbf{j}+\mathbf{k})$, find the value of t and the coordinates of the point of intersection.

6. A line L has equation $\mathbf{r}_1 = -3\mathbf{i}+\mathbf{j}+4\mathbf{k}+\lambda(4\mathbf{i}+p\mathbf{j}-3\mathbf{k})$
Find the value of p if
(a) the line $\mathbf{r}_2 = 5\mathbf{i}-\mathbf{j}-2\mathbf{k}+\mu(4\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ is parallel to L,
(b) the line $\mathbf{r}_3 = -\mathbf{i}+3\mathbf{j}-3\mathbf{k}+\mu(\mathbf{i}-\mathbf{j}+2\mathbf{k})$ intersects L.

7. The equations of two lines are $\mathbf{r}_1 = 2\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$ and $\mathbf{r}_2 = \mathbf{i}-3\mathbf{j}+4\mathbf{k}+\mu(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})$. Show that these lines are skew.
$\overrightarrow{OQ}$ is the unit vector in the direction of the first line and $\overrightarrow{OR}$ is the unit vector in the direction of the second line. Write down the coordinates of Q and R .
By using the cosine formula in triangle OQR find the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ and hence the angle between the given lines.

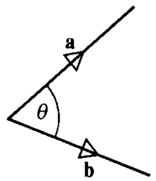
8. If $\overrightarrow{OA}$ is the unit vector $l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k}$ and $\overrightarrow{OB}$ is the unit vector $l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k}$, use the cosine formula in triangle OAB to show that θ, the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, is given by
$\cos\theta = l_1l_2+m_1m_2+n_1n_2$

THE SCALAR PRODUCT

We are now going to look at an operation involving two vectors and the angle between them. This operation is called a product but, because it involves vectors, it is in no way related to the product of real numbers.

The Definition of the Scalar Product

The scalar product of two vectors **a** and **b** is denoted by **a.b** and defined as $ab\cos\theta$ where θ is the angle between **a** and **b**



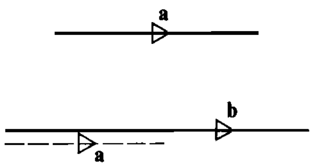
i.e. $\mathbf{a.b} = ab\cos\theta$

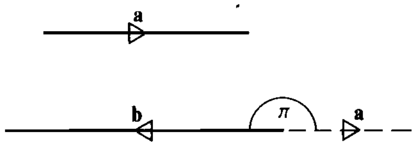
PROPERTIES OF THE SCALAR PRODUCT

Parallel Vectors

If **a** and **b** are parallel then

either $\mathbf{a.b} = ab\cos 0$ or $\mathbf{a.b} = ab\cos\pi$





i.e. for like parallel vectors $\mathbf{a.b} = ab$

and for unlike parallel vectors $\mathbf{a.b} = -ab$

In the special case when $\mathbf{a} = \mathbf{b}$

$$\mathbf{a.b} = \mathbf{a.a} = a^2 \quad \text{(sometimes } \mathbf{a.a} \text{ is written } \mathbf{a}^2\text{)}$$

In particular, for the cartesian unit vectors **i**, **j** and **k**

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$

Perpendicular Vectors

If **a** and **b** are perpendicular then $\theta = \frac{1}{2}\pi$, $\Rightarrow$ $\mathbf{a.b} = ab\cos\frac{1}{2}\pi = 0$

i.e. for perpendicular vectors $\mathbf{a.b} = 0$

For the unit vectors **i**, **j** and **k** we have

$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$

The Scalar Product is Commutative

This means that $\mathbf{a.b} = \mathbf{b.a}$ and this property is easy to prove as follows.

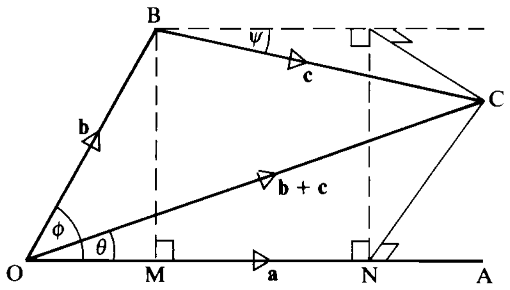
From the definition we have $\mathbf{a.b} = ab\cos\theta$ and $\mathbf{b.a} = ba\cos\theta$

Now $ab\cos\theta = ba\cos\theta$ therefore $\mathbf{a.b} = \mathbf{b.a}$

The Scalar Product is Distributive for Addition

This means that $\mathbf{a.(b+c)} = \mathbf{a.b} + \mathbf{a.c}$

It is not necessary to be able to prove this property but a proof is given below for readers who would like to see it.



$$\begin{aligned}\mathbf{a.(b+c)} &= |\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\theta \\ &= (\text{OA})(\text{OC})\cos\theta \\ &= (\text{OA})(\text{ON}) \\ &= (\text{OA})(\text{OM}+\text{MN}) \\ &= (\text{OA})(\text{OB}\cos\phi) + (\text{OA})(\text{BC}\cos\psi) \\ &= \mathbf{a.b} + \mathbf{a.c}\end{aligned}$$

CALCULATING a.b IN CARTESIAN FORM

If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ then we can find $\mathbf{a}.\mathbf{b}$ by using the properties given above.

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &= (x_1x_2\mathbf{i}.\mathbf{i} + y_1y_2\mathbf{j}.\mathbf{j} + z_1z_2\mathbf{k}.\mathbf{k}) \\ &\quad + (x_1y_2\mathbf{i}.\mathbf{j} + y_1z_2\mathbf{j}.\mathbf{k} + z_1x_2\mathbf{k}.\mathbf{i}) \\ &\quad + (y_1x_2\mathbf{j}.\mathbf{i} + z_1y_2\mathbf{k}.\mathbf{j} + x_1z_2\mathbf{i}.\mathbf{k}) \\ &= (x_1x_2 + y_1y_2 + z_1z_2) + (0) + (0),\end{aligned}$$

i.e. $$(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = x_1x_2 + y_1y_2 + z_1z_2$$

For example,

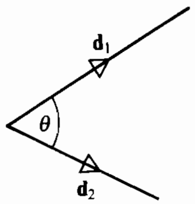
$$(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}).(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = (2)(1) + (-3)(3) + (4)(-2) = -15$$

Using the Scalar Product to Find the Angle Between Two Lines

The lines with equations $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{d}_2$ are in the directions of $\mathbf{d}_1$ and $\mathbf{d}_2$ respectively.
The angle between two lines is defined as the angle between their direction vectors and does not depend upon their positions. It does not even depend on whether the lines intersect, so we are looking for the angle between the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ in any convenient position.

Drawing $\mathbf{d}_1$ and $\mathbf{d}_2$ from a common point, the angle θ between them is given by $\mathbf{d}_1.\mathbf{d}_2 = d_1d_2\cos\theta$

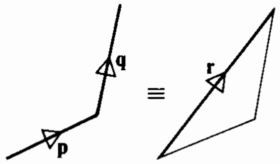
$$\therefore \quad \cos\theta = \frac{\mathbf{d}_1.\mathbf{d}_2}{d_1d_2}$$



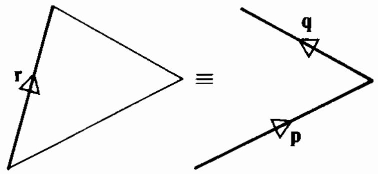
This confirms that, for perpendicular lines, $\mathbf{d}_1.\mathbf{d}_2 = 0$

Resolved Parts

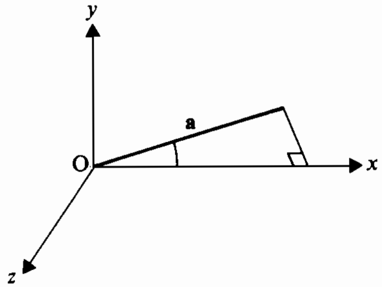
We have seen that the resultant of two vectors can be found by the triangle law of addition.
i.e. $\mathbf{p} + \mathbf{q} = \mathbf{r}$



Conversely, a single vector $\mathbf{r}$ can be replaced by two other vectors which, together with $\mathbf{r}$, form a closed triangle,
i.e. $\mathbf{r}$ is equivalent to $\mathbf{p} + \mathbf{q}$
The magnitudes of $\mathbf{p}$ and $\mathbf{q}$ are called the *resolved parts* or *components* of $\mathbf{r}$



Replacing a vector by its components is of particular importance when those components are at right angles to each other
In the diagram for example, the resolved part of $\mathbf{a}$ in the direction of Ox is $|\mathbf{a}|\cos\alpha$

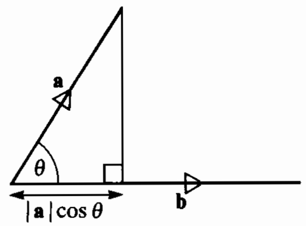


Now suppose that we have two vectors $\mathbf{a}$ and $\mathbf{b}$ which enclose an angle θ. The component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $|\mathbf{a}|\cos\theta$

But $$|\mathbf{a}||\mathbf{b}|\cos\theta = \mathbf{a}.\mathbf{b}$$

Therefore

$$|\mathbf{a}|\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|} \quad \text{or} \quad \mathbf{a}.\hat{\mathbf{b}}$$



In this way the resolved part of any vector in the direction of another vector can be found,
e.g. the component of $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the direction of $\mathbf{i} - 5\mathbf{k}$ is

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).\frac{(\mathbf{i} - 5\mathbf{k})}{\sqrt{26}} = \frac{7}{\sqrt{26}}$$

Examples 39h

1. Simplify $(\mathbf{a}-\mathbf{b}).(\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}+\mathbf{b}).\mathbf{c}-(\mathbf{a}+\mathbf{c}).\mathbf{b}$

$$(\mathbf{a}-\mathbf{b}).(\mathbf{a}+\mathbf{b}) = \mathbf{a}.\mathbf{a}-\mathbf{b}.\mathbf{a}+\mathbf{a}.\mathbf{b}-\mathbf{b}.\mathbf{b}$$

but $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$ hence $\mathbf{a}.\mathbf{b}-\mathbf{b}.\mathbf{a} = 0$

also $\mathbf{a}.\mathbf{a} = a^2$ and $\mathbf{b}.\mathbf{b} = b^2$

Therefore $(\mathbf{a}-\mathbf{b}).(\mathbf{a}+\mathbf{b}) = a^2-b^2$

Also
$$\begin{aligned}(\mathbf{a}+\mathbf{b}).\mathbf{c}-(\mathbf{a}+\mathbf{c}).\mathbf{b} &= \mathbf{a}.\mathbf{c}+\mathbf{b}.\mathbf{c}-\mathbf{a}.\mathbf{b}-\mathbf{c}.\mathbf{b}\\ &= \mathbf{a}.\mathbf{c}-\mathbf{a}.\mathbf{b}\\ &= \mathbf{a}.(\mathbf{c}-\mathbf{b})\end{aligned}$$

---

2. Find the scalar product of $\mathbf{a} = 2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-3\mathbf{j}+\mathbf{k}$ and hence find the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a}.\mathbf{b} = (2)(1)+(-3)(-3)+(5)(1) = 16$$

But $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$|\mathbf{a}| = \sqrt{(4+9+25)} = \sqrt{38}$$

$$|\mathbf{b}| = \sqrt{(1+9+1)} = \sqrt{11}$$

Hence
$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{16}{\sqrt{11}\sqrt{38}} = \frac{16}{\sqrt{418}}$$

---

3. If $\mathbf{a} = 10\mathbf{i}-3\mathbf{j}+5\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+6\mathbf{j}-3\mathbf{k}$ and $\mathbf{c} = \mathbf{i}+10\mathbf{j}-2\mathbf{k}$, verify that $\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c} = \mathbf{a}.(\mathbf{b}+\mathbf{c})$

$$\mathbf{a}.\mathbf{b} = (10)(2)+(-3)(6)+(5)(-3) = -13$$

$$\mathbf{a}.\mathbf{c} = (10)(1)+(-3)(10)+(5)(-2) = -30$$

$$\mathbf{b}+\mathbf{c} = 3\mathbf{i}+16\mathbf{j}-5\mathbf{k}$$

Hence $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = (10)(3)+(-3)(16)+(5)(-5) = -43$

But $\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c} = -13-30 = -43$

Therefore $\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c} = \mathbf{a}.(\mathbf{b}+\mathbf{c})$

---

4. Find the resolved part of the vector $3\mathbf{i}-\mathbf{j}+7\mathbf{k}$ in the direction of
(a) the vector $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ (b) the x-axis.

(a) If $\mathbf{a} = 3\mathbf{i}-\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-2\mathbf{j}-2\mathbf{k}$ then the resolved part of $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by $\mathbf{a}.\hat{\mathbf{b}}$

i.e.
$$(3\mathbf{i}-\mathbf{j}+7\mathbf{k}).\frac{(\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{3} = -3$$

(b) The direction vector of the x-axis is $\mathbf{i}$ which is a unit vector. Therefore the resolved part of $\mathbf{a}$ in the direction of the x-axis is

$$\mathbf{a}.\mathbf{i} = (3\mathbf{i}-\mathbf{j}+7\mathbf{k}).\mathbf{i} = 3$$

---

5. Find the angle between the lines

$$\mathbf{r}_1 = \mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}+6\mathbf{k}) \quad [1]$$

$$\mathbf{r}_2 = 2\mathbf{i}-7\mathbf{j}+10\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+2\mathbf{k}) \quad [2]$$

The angle between the lines depends only upon their directions.
Line [1] has a direction vector $\mathbf{d}_1 = 2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$
Line [2] has a direction vector $\mathbf{d}_2 = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$
The angle θ between the lines is given by

$$\cos\theta = \frac{\mathbf{d}_1.\mathbf{d}_2}{d_1 d_2} = \frac{2-6+12}{(7)(3)}$$

$$\Rightarrow \quad \theta = \arccos\tfrac{8}{21}$$

---

6. Find a unit vector which is perpendicular to AB and AC if $\overrightarrow{AB} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{AC} = 4\mathbf{i}-\mathbf{j}+2\mathbf{k}$

Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be a vector perpendicular to both AB and AC.

It is perpendicular to AB so $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 0$

It is perpendicular to AC so $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}).(4\mathbf{i}-\mathbf{j}+2\mathbf{k}) = 0$

Therefore
$$\begin{cases} a+2b+3c = 0\\ 4a-b+2c = 0\end{cases}$$

Eliminating b gives $a = -\frac{7}{9}c$

Eliminating a gives $b = -\frac{10}{9}c$

Hence
$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = -\tfrac{7}{9}c\mathbf{i} - \tfrac{10}{9}c\mathbf{j} + c\mathbf{k} = \tfrac{1}{9}c(-7\mathbf{i} - 10\mathbf{j} + 9\mathbf{k})$$

Thus $-7\mathbf{i} - 10\mathbf{j} + 9\mathbf{k}$ is perpendicular to both AB and AC.

A unit vector perpendicular to AB and AC is therefore
$$(-7\mathbf{i} - 10\mathbf{j} + 9\mathbf{k})/\sqrt{230}$$

EXERCISE 39h

1. Calculate $\mathbf{a}.\mathbf{b}$ if
   (a) $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$
   (b) $\mathbf{a} = 3\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j} - 4\mathbf{k}$
   (c) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
   What conclusion can you draw in (b)?

2. Find $\mathbf{p}.\mathbf{q}$ and the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$ if
   (a) $\mathbf{p} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
   (b) $\mathbf{p} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{q} = \mathbf{i} + \mathbf{j} - 6\mathbf{k}$
   (c) $\mathbf{p} = -2\mathbf{i} + 5\mathbf{j}$, $\mathbf{q} = \mathbf{i} + \mathbf{j}$
   (d) $\mathbf{p} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{q} = \mathbf{j} - 2\mathbf{k}$

3. Simplify
   (a) $(\mathbf{a} - \mathbf{b}).\mathbf{a}$ (b) $(\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$
   (c) $(\mathbf{a} + \mathbf{b}).\mathbf{b} - (\mathbf{a} + \mathbf{b}).\mathbf{a}$ (d) $(\mathbf{a} + \mathbf{b}).\mathbf{c} - (\mathbf{a} - \mathbf{b}).\mathbf{c}$

4. Given that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, simplify
   (a) $\mathbf{a}.\mathbf{b}$ (b) $(\mathbf{a} - \mathbf{b}).\mathbf{b}$ (c) $(\mathbf{a} + \mathbf{b}).\mathbf{a}$ (d) $(\mathbf{a} - \mathbf{b}).(2\mathbf{a} + \mathbf{b})$

5. The angle between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\arccos \frac{4}{21}$
   If $\mathbf{v}_1 = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}_2 = -2\mathbf{i} + \lambda\mathbf{j} - 4\mathbf{k}$, find the positive value of λ

6. If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, find
   (a) $\mathbf{a}.\mathbf{b}$ (b) $\mathbf{a}.\mathbf{c}$ (c) $\mathbf{a}.(\mathbf{b} + \mathbf{c})$ (d) $(2\mathbf{a} + 3\mathbf{b}).\mathbf{c}$ (e) $(\mathbf{a} - \mathbf{b}).\mathbf{c}$

7. In a triangle ABC, $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{BC} = -\mathbf{i} + 4\mathbf{j}$.
   Find the cosine of angle ABC.
   Find the vector $\overrightarrow{AC}$ and use it to calculate the angle BAC.

8. A, B and C are points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, relative to the origin O. AB is perpendicular to OC and BC is perpendicular to OA.
   Show that AC is perpendicular to OB.

9. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ ($\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$), show that
   (a) if $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular then $|\mathbf{a}| = |\mathbf{b}|$,
   (b) if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

10. Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c} \neq 0$.
    (a) If $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{b}.(\mathbf{a} - \mathbf{c})$ prove that $\mathbf{c}.(\mathbf{a} + \mathbf{b}) = 0$
    (b) If $(\mathbf{a}.\mathbf{b})\mathbf{c} = (\mathbf{b}.\mathbf{c})\mathbf{a}$ show that $\mathbf{c}$ and $\mathbf{a}$ are parallel.

11. Find the angle between each of the following pairs of lines.
    (a) $\mathbf{r}_1 = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and
    $\mathbf{r}_2 = 5\mathbf{j} - 2\mathbf{k} - \mu(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$
    (b) A line with direction ratios 2:2:1 and a line joining the points (3, 1, 4) and (7, 2, 12).
    (c) $\dfrac{x+4}{3} = \dfrac{y-1}{5} = \dfrac{z+3}{4}$ and $\dfrac{x+1}{1} = \dfrac{y-4}{1} = \dfrac{z-5}{2}$

12. Find the angle between the following pairs of lines.
    (a) $\dfrac{x-2}{3} = \dfrac{y+1}{-2}$, $z = 2$ and $\dfrac{x-1}{1} = \dfrac{2y+3}{3} = \dfrac{z+5}{2}$
    (b) $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ and
    $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} - \mu(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$

13. Show that $\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ is perpendicular to both $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

14. Show that $13\mathbf{i} + 23\mathbf{j} + 7\mathbf{k}$ is perpendicular to both $2\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

15. Find the resolved part of the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ in the direction of
    (a) $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{i}$ (c) $3\mathbf{j} - 4\mathbf{k}$

PLANES

There are a number of ways in which a plane can be specified. For example, one and only one plane can be drawn

1) through three non-collinear points.

2) to contain two concurrent lines.

3) perpendicular to a specified direction and at a given distance from a fixed origin.

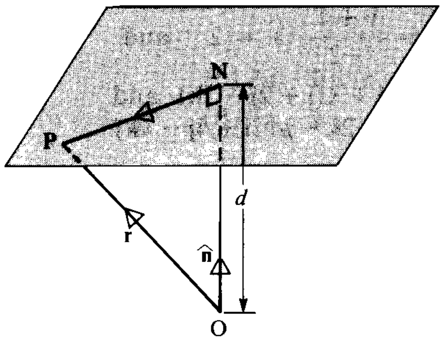
4) perpendicular to a given direction and through a fixed point.

The specifications described in (3) and (4) provide particularly simple methods for deriving the equation of a plane.

THE EQUATION OF A PLANE

The Vector Equation of a Plane

Consider a plane distant d from the origin O and perpendicular to a vector $\mathbf{n}$, where $\overrightarrow{ON} = \mathbf{n}$



For any point P on the plane, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$ so

$$\overrightarrow{NP}.\overrightarrow{ON} = 0$$

Now if $\mathbf{r}$ is the position vector of P. $\overrightarrow{NP} = \overrightarrow{NO} + \overrightarrow{OP} = \mathbf{r} - \mathbf{n}$

Therefore $\quad (\mathbf{r}-\mathbf{n}).\mathbf{n} = 0 \quad \Rightarrow \quad \mathbf{r}.\mathbf{n} - \mathbf{n}.\mathbf{n} = 0$

Now using $\mathbf{n} = d\hat{\mathbf{n}}$ and $\mathbf{n}.\mathbf{n} = d^2$ we have

$$\mathbf{r}.\mathbf{n} - \mathbf{n}.\mathbf{n} = 0 \quad \Rightarrow \quad d\mathbf{r}.\hat{\mathbf{n}} - d^2 = 0 \quad \Rightarrow \quad \mathbf{r}.\hat{\mathbf{n}} - d = 0$$

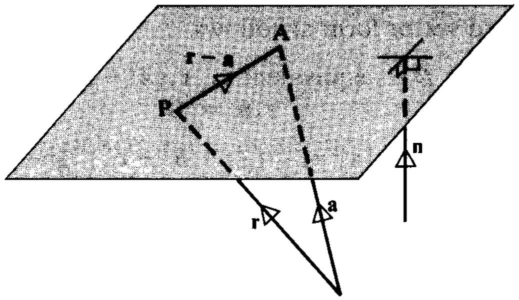
i.e. $$\mathbf{r}.\hat{\mathbf{n}} = d$$

This is the equation, in *standard form*, of a plane distant d from the origin and perpendicular to $\mathbf{n}$

This equation can be multiplied by any scalar quantity so, multiplying by $|\mathbf{n}|$, we have $\mathbf{r}.\mathbf{n} = D$, where $D = d|\mathbf{n}|$, i.e.

any equation of the form $\mathbf{r}.\mathbf{n} = D$ represents a plane perpendicular to $\mathbf{n}$ and distant $\dfrac{D}{|\mathbf{n}|}$ from the origin.

Now consider a plane which is perpendicular to a vector $\mathbf{n}$ and which passes through a fixed point A with position vector $\mathbf{a}$



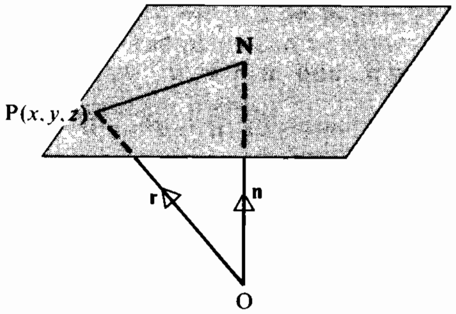
If $\mathbf{r}$ is the position vector of any point in the plane then

$$\overrightarrow{AP} = \mathbf{r} - \mathbf{a} \quad \text{and} \quad \overrightarrow{AP} \text{ is perpendicular to } \mathbf{n} \quad \Rightarrow \quad \overrightarrow{AP}.\mathbf{n} = 0$$

Therefore $\quad (\mathbf{r}-\mathbf{a}).\mathbf{n} = 0 \quad$ or $\quad \mathbf{r}.\mathbf{n} = \mathbf{r}.\mathbf{a}$

i.e. $\quad (\mathbf{r}-\mathbf{a}).\mathbf{n} = 0$ is the vector equation of a plane which is perpendicular to $\mathbf{n}$ and contains the point with position vector $\mathbf{a}$

The Cartesian Equation of a Plane



Consider again a plane distant d from the origin O and perpendicular to the vector $\mathbf{n}$, where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$

The vector equation of this plane can be given as $\mathbf{r}.\mathbf{n} = D$

Now for any point $P(x, y, z)$ on the plane, $\mathbf{r} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

i.e. $$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = D$$

$$\Rightarrow \quad Ax + By + Cz = D$$

which is the cartesian form for the equation of the plane.

When the cartesian equation of a plane is given it is easy to convert it to the standard vector form as follows.

$Ax + By + Cz = D$ is equivalent to $\mathbf{r}.(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = D$
or $\mathbf{r}.\mathbf{n} = D$

Therefore dividing by $|\mathbf{n}| = \sqrt{(A^2 + B^2 + C^2)}$ gives

$$\mathbf{r}.\hat{\mathbf{n}} = d$$

e.g. a plane whose cartesian equation is $2x + 3y + 6z = 21$ has

a vector equation $\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = 21$ which, converted to standard form is $\mathbf{r}.(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}) = 3$

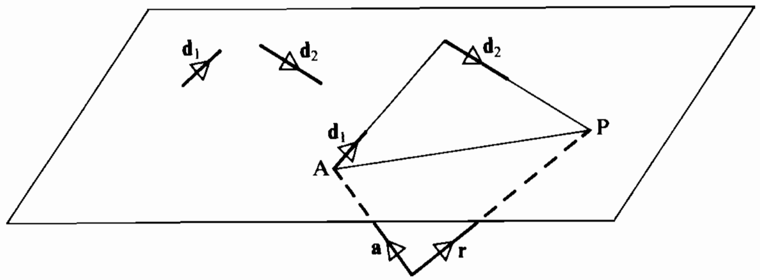
From the last equation we see that the plane is 3 units from the origin and is perpendicular to a vector with direction ratios 2:3:6.

Note that *a plane itself does not have direction ratios.*

When finding the equation of a plane, the given data sometimes suggests aiming at a vector equation but in other cases the cartesian equation is easier to find. Because these two forms are quickly interconvertible the choice of method can be based solely on convenience.

The Parametric Form for the Vector Equation of a Plane

A plane is defined uniquely if it contains two non-parallel direction vectors and a fixed point.
Suppose that the position vector of the fixed point is $\mathbf{a}$ and the two direction vectors are $\mathbf{d}_1$ and $\mathbf{d}_2$



Any vector $\overrightarrow{AP}$ in the plane can be expressed in the form $\lambda\mathbf{d}_1 + \mu\mathbf{d}_2$ where λ and μ are scalar parameters. So if $\mathbf{r}$ is the position vector of any point in the plane we have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$$

i.e. $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$$

This is the parametric form for the vector equation of a plane.

Examples 39l

1. (a) Find the vector equation, in standard form, of the plane whose cartesian equation is $x - 2y + 2z = 9$

   (b) Find the cartesian equation of the plane $\mathbf{r}.(3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}) = 8$

   In each case state the distance of the plane from the origin.

(a) The direction ratios of the normal to the plane $x - 2y + 2z = 9$ are $1:-2:2$
Hence a vector equation of the plane is $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 9$

To express this equation in standard form we convert $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ to a unit vector, i.e. we divide by $|\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}|$

giving $$\mathbf{r}.(\tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}) = 3$$

This plane is distant 3 units from the origin.

(b) The cartesian equation of the plane $\mathbf{r}.(3\mathbf{i}-2\mathbf{j}-6\mathbf{k}) = 8$ is

$$3x - 2y - 6z = 8$$

To express the vector equation in standard form we divide by the modulus of $3\mathbf{i}-2\mathbf{j}-6\mathbf{k}$ giving

$$\mathbf{r}.(\tfrac{3}{7}\mathbf{i}-\tfrac{2}{7}\mathbf{j}-\tfrac{6}{7}\mathbf{k}) = \tfrac{8}{7}$$

so the plane is distant $1\frac{1}{7}$ units from the origin.

---

2. Find the equation of the line passing through the point $(3, 1, 2)$ which is perpendicular to the plane $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4$. Find also the coordinates of the point where the line intersects the plane.

As the line is perpendicular to the plane it is parallel to the normal to the plane, i.e. the line is parallel to the vector $2\mathbf{i}-\mathbf{j}+\mathbf{k}$

The equation of the line through $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and in the direction $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is

$$\mathbf{r} = (3\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$$

To deal with a point of intersection we must find separate expressions for the coordinates of a point on the line, i.e. the parametric coordinates.

At a general point on the line $x = 3+2\lambda,\ y = 1-\lambda,\ z = 2+\lambda$

At the point of intersection of line and plane these coordinates also satisfy the equation of the plane

i.e. $(\{3+2\lambda\}\mathbf{i}+\{1-\lambda\}\mathbf{j}+\{2+\lambda\}\mathbf{k}).(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4$

$\Rightarrow \quad 2(3+2\lambda)-(1-\lambda)+(2+\lambda) = 4$

$\Rightarrow \quad \lambda = -\frac{1}{2}$

$\Rightarrow \quad x = 2,\ y = \frac{3}{2}$ and $z = \frac{3}{2}$

$\therefore$ the line intersects the plane at the point $(2, \frac{3}{2}, \frac{3}{2})$

---

3. Show that the plane whose vector equation is $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = 3$ contains the line with vector equation $\mathbf{r} = \mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j}+4\mathbf{k})$

A line is contained in a plane if any two points on the line are also in the plane. In the equation of the line, each value of λ gives one point on the line.

Taking $\lambda = 0$ and $\lambda = 1$ we see that $\mathbf{i}+\mathbf{j}$ and $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ are the position vectors of two points on the given line.

If $\mathbf{r} = \mathbf{i}+\mathbf{j}$ then $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = (\mathbf{i}+\mathbf{j}).(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$= 1+2+0 = 3$

$\therefore$ $\mathbf{i}+\mathbf{j}$ is a point on the given plane.

If $\mathbf{r} = 3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ then $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = (3\mathbf{i}+2\mathbf{j}+4\mathbf{k}).(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$= 3+4-4 = 3$

$\therefore$ $3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ is also a point on the given plane.

Two points on the line are also in the plane so the line is contained in the plane.

---

4. Find a vector equation of the plane through the origin and the points $(1, 2, 0)$ and $(-3, 0, 5)$

The plane passes through the origin so its distance from the O is zero.

The cartesian equation of the plane is

$$Ax+By+Cz = 0 \qquad [1]$$

$(1, 2, 0)$ and $(-3, 0, 5)$ are both on the plane

so $\quad A(1)+B(2)+C(0) = 0$ and $A(-3)+B(0)+C(5) = 0$

$\Rightarrow \quad A+2B = 0$ and $-3A+5C = 0$

Hence $B = -\frac{1}{2}A$ and $C = \frac{3}{5}A$

Substituting these values in equation [1] gives

$$Ax-\tfrac{1}{2}Ay+\tfrac{3}{5}Az = 0 \quad \Rightarrow \quad 10x-5y+6z = 0$$

This is the cartesian equation of the required plane and the corresponding vector equation is

$$\mathbf{r}.(10\mathbf{i}-5\mathbf{j}+6\mathbf{k}) = 0$$

---

5. Find, in parametric form, the equation of the plane which contains the line $\mathbf{r} = \mathbf{i}+\mathbf{j}-\mathbf{k}+\lambda(3\mathbf{i}+5\mathbf{j}-2\mathbf{k}) = 6$ and the point A(4, 3, 2). Show that the point (8, 9, −1) is on the plane.

To find the parametric equation of the plane we need a fixed point, i.e. A and *two* direction vectors, one of which is $(3\mathbf{i}+5\mathbf{j}-2\mathbf{k})$. The second direction vector can be obtained by taking a point on the given line and finding the direction of the vector joining it to A.

One point on the given line is $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and the direction of the line joining this point to A is $(4\mathbf{i}+3\mathbf{j}+2\mathbf{k})-(\mathbf{i}+\mathbf{j}-\mathbf{k}) = 3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

The required plane contains the direction vectors $3\mathbf{i}+5\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and passes through $4\mathbf{i}+3\mathbf{j}+2\mathbf{k}$

Therefore its parametric equation is

$$\mathbf{r} = 4\mathbf{i}+3\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{i}+5\mathbf{j}-2\mathbf{k})+\mu(\mathbf{i}+\mathbf{j}-\mathbf{k})$$

Any pair of values for λ and μ give one point on the plane and we can see that when $\lambda = 1$ and $\mu = 1$ we get $\mathbf{r} = 8\mathbf{i}+9\mathbf{j}-\mathbf{k}$

Therefore the point (8, 9, −1) is on the plane.

---

6. L is a line whose vector equation is $\mathbf{r} = 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+4\mathbf{k})$.
Π is a plane with vector equation $\mathbf{r}.(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = 5$
Show that L is parallel to Π and find the distance between them.

L is parallel to $\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and Π is perpendicular to $\mathbf{i}+5\mathbf{j}+\mathbf{k}$

If L is parallel to Π, it must be perpendicular to the normal to Π

Now $(\mathbf{i}-\mathbf{j}+4\mathbf{k}).(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = 1-5+4 = 0$

$\therefore$ $\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $\mathbf{i}+5\mathbf{j}+\mathbf{k}$ are perpendicular

i.e. L is perpendicular to the normal to Π

$\therefore$ L is parallel to Π

If we find the equation of a plane containing L and parallel to Π we can find the distance from the origin to this plane, and to the plane Π, and hence the distance between them.

As $2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ is a point on L, the equation of a plane containing L must contain this point.

The equation of the plane Π_1, parallel to Π and containing L, can now be obtained in the form $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

i.e. $\mathbf{r}.(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = (2\mathbf{i}-2\mathbf{j}+3\mathbf{k}).(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = -5$

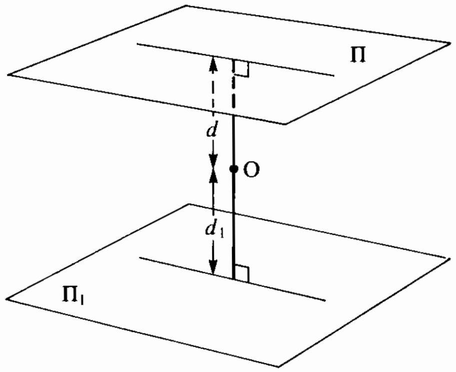
Now if d is the distance from the origin to Π then

$$d = \frac{5}{\sqrt{(1^2+5^2+1^2)}} = \frac{5}{\sqrt{27}}$$

and if d_1 is the distance from the origin to Π_1 then

$$d_1 = \frac{-5}{\sqrt{(1^2+5^2+1^2)}} = \frac{-5}{\sqrt{27}}$$

The opposite signs obtained for d and d_1 indicate that Π and Π_1 are on opposite sides of the origin.



Therefore the distance between L and Π is $\frac{5}{\sqrt{27}}+\frac{5}{\sqrt{27}} = \frac{10}{\sqrt{27}}$

EXERCISE 39I

1. Write down the cartesian equation of the plane whose vector equation is
(a) $\mathbf{r}.(\mathbf{i}-\mathbf{j}-\mathbf{k}) = 2$ (b) $\mathbf{r}.(2\mathbf{i}+3\mathbf{j}-4\mathbf{k}) = 0$

2. Find a vector equation for the plane whose cartesian equation is
(a) $3x - 4y + 5x = 6$ (b) $x - 7y - z = 5$

3. Find the cartesian equations of the following planes.
(a) $\mathbf{r}.(\mathbf{i}+\mathbf{j}-\mathbf{k}) = 2$ (b) $\mathbf{r}.(2\mathbf{i}+3\mathbf{j}-4\mathbf{k}) = 1$

4. Find the equation of the plane containing the points, A, B and C, in cartesian form and in scalar product form.
(a) A, B and C are the points $(0, 0, -1)$, $(1, 3, 0)$, $(0, 2, 1)$ respectively.
(b) The position vectors of A, B and C are $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}-3\mathbf{k}$

5. A plane goes through the three points whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ where,
$$\mathbf{a} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$$
and $$\mathbf{c} = -\mathbf{i}+2\mathbf{j}-2\mathbf{k}$$
Find the vector equation of this plane in scalar product form and hence find the distance of the plane from the origin.

6. A plane goes through the points whose position vectors are $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ and is parallel to the line with equation $\mathbf{r} = \mathbf{i}-\mathbf{j}+\lambda(3\mathbf{i}+\mathbf{j}-2\mathbf{k})$. Find the distance of this plane from the origin.

7. Two planes Π_1 and Π_2 have vector equations $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 3$ and $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 9$. Explain why Π_1 and Π_2 are parallel and hence find the distance between them.

8. Find the vector equation of the line through the origin which is perpendicular to the plane $\mathbf{r}.(\mathbf{i}-2\mathbf{j}+\mathbf{k}) = 3$

9. Find the vector equation of the line through the point $(2, 1, 1)$ which is perpendicular to the plane $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = 6$

10. Find the vector equation of the plane which goes through the point $(0, 1, 6)$ and is parallel to the plane $\mathbf{r}.(\mathbf{i}-2\mathbf{j}) = 3$

11. Find the vector equation of the plane which goes through the origin and which contains the line $\mathbf{r} = 2\mathbf{i}+\lambda(\mathbf{j}+\mathbf{k})$

12. Find the point of intersection of the line with equation $\mathbf{r} = (\mathbf{i}+\mathbf{j}-2\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k})$ and the plane $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k}) = 2$

13. Find the point of intersection of the line $x - 2 = 2y + 1 = 3 - z$ and the plane $x + 2y + z = 3$

14. Show that the line $x + 1 = y = \frac{1}{2}(z - 3)$ is parallel to the plane $\mathbf{r}.(\mathbf{i}+\mathbf{j}-\mathbf{k}) = 3$ and find the distance between them.

15. Find the vector equation of the following planes in scalar product form.
(a) $\mathbf{r} = \mathbf{i}-\mathbf{j}+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})+\mu(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$
(b) $\mathbf{r} = 2\mathbf{i}-\mathbf{k}+\lambda(\mathbf{i})+\mu(\mathbf{i}-2\mathbf{j}-\mathbf{k})$
(c) $\mathbf{r} = (1+s-t)\mathbf{i}+(2-s)\mathbf{j}+(3-2s+2t)\mathbf{k}$

16. Find the vector equation in parametric form of the plane that contains the lines
$$\mathbf{r} = -3\mathbf{i}-2\mathbf{j}+t(\mathbf{i}-2\mathbf{j}+\mathbf{k})$$
$$\mathbf{r} = \mathbf{i}-11\mathbf{j}+4\mathbf{k}+s(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

17. Find the vector equation in parametric form of the plane that goes through the point with position vector $\mathbf{i}+\mathbf{j}$ and which is parallel to the lines
$\mathbf{r}_1 = 2\mathbf{i}-\mathbf{j}+\lambda(\mathbf{i}+\mathbf{k})$ and $\mathbf{r}_2 = 2\mathbf{j}-\mathbf{k}+\mu(\mathbf{i}-\mathbf{j}+\mathbf{k})$.
Is either of these lines contained in the plane?

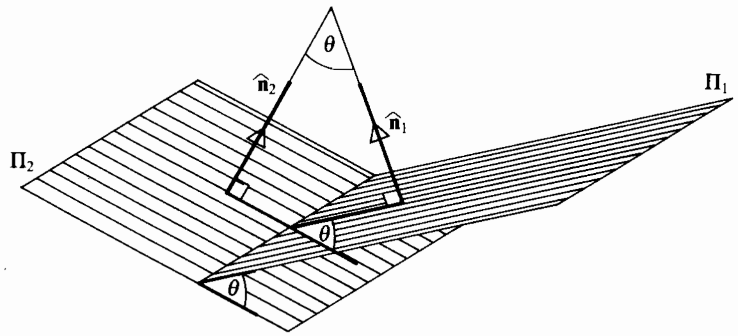
18. Determine whether the given lines are parallel to, contained in, or intersect the plane $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k}) = 5$
(a) $\mathbf{r} = 3\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(-2\mathbf{i}+\mathbf{j}-3\mathbf{k})$
(b) $\mathbf{r} = \mathbf{i}-\mathbf{j}+\mu(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$
(c) $x = y = z$
(d) $\mathbf{r} = (2\mathbf{i}+\mathbf{j})+s(3\mathbf{i}+2\mathbf{k})$

THE ANGLE BETWEEN TWO PLANES

Consider two planes Π_1 and Π_2 whose equations are

$$\mathbf{r}.\mathbf{n}_1 = d_1 \quad \text{and} \quad \mathbf{r}.\mathbf{n}_2 = d_2$$

i.e. $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to Π_1 and Π_2 respectively.



The angle between the planes is equal to the angle between their normals. Therefore

the angle θ between Π_1 and Π_2 is given by

$$\cos\theta = \hat{\mathbf{n}}_1 . \hat{\mathbf{n}}_2$$

e.g. the angle between the planes whose equations are

$\mathbf{r}.(\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 3$ and $\mathbf{r}.(2\mathbf{i}-2\mathbf{j}+\mathbf{k}) = 2$ is given by

$$\cos\ \theta = \frac{(\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{6}}.\frac{(2\mathbf{i}-2\mathbf{j}+\mathbf{k})}{3} = -\frac{\sqrt{6}}{9}$$

The negative sign shows that we have found the obtuse angle between the planes. The acute angle between them is $\arccos\frac{\sqrt{6}}{9}$

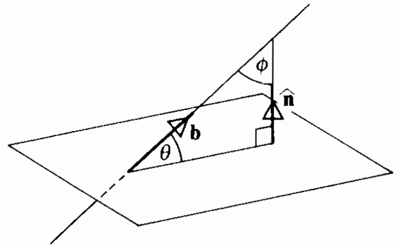
There are two special cases, i.e.

if $\hat{\mathbf{n}}_1 . \hat{\mathbf{n}}_2 = 0$ the two planes are perpendicular

and if $\hat{\mathbf{n}}_1 = \pm\hat{\mathbf{n}}_2$ the two planes are parallel

THE ANGLE BETWEEN A LINE AND A PLANE

Consider the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r}.\mathbf{n} = D$



If ϕ is the angle between the line and the normal to the plane and θ is the angle between the line and the plane then

$$\theta = \tfrac{1}{2}\pi - \phi \quad \Rightarrow \quad \sin\theta = \cos\phi$$

Now $\cos\phi = \hat{\mathbf{b}}.\hat{\mathbf{n}} = \dfrac{\mathbf{b}.\mathbf{n}}{bn}$ therefore

the angle between the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane $\mathbf{r}.\mathbf{n} = D$ is given by $\sin\theta = \dfrac{\mathbf{b}.\mathbf{n}}{bn}$

e.g. the angle between the line $\mathbf{r} = (\mathbf{i}+2\mathbf{j}-\mathbf{k}) + \lambda(\mathbf{i}-\mathbf{j}+\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4$ is given by

$$\sin\theta = \frac{(\mathbf{i}-\mathbf{j}+\mathbf{k})}{\sqrt{3}}.\frac{(2\mathbf{i}-\mathbf{j}+\mathbf{k})}{\sqrt{6}} = \frac{2\sqrt{2}}{3}$$

If the exact value of $\cos\theta$ should be required, it can be found by using $\cos^2\theta + \sin^2\theta = 1$

EXERCISE 39j

In Questions 1 to 4 find the cosine of the angle between the two given planes.

1. The planes whose equations are
   $\mathbf{r}.(\mathbf{i}-\mathbf{j}+3\mathbf{k}) = 3$ and $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+2\mathbf{k}) = 5$
2. The cartesian equations of the two planes are
   $2x + 2y - 3z = 3$ and $x + 3y - 4z = 6$

3. One plane passes through the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and the other plane is perpendicular to the line with equation
$\mathbf{r} = -2\mathbf{j} + 5\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

4. One plane has equations

$$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-4}{6}$$

and the other is parallel to the xy plane.

In Questions 5 to 7 find the sine of the angle between the line and the plane whose equations are given.

5. $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 4$

6. $(x-2) = \frac{1}{2}(y+1) = \frac{1}{2}(z+3)$ and $2x - y - 2z = 4$

7. The equation of a plane is $\mathbf{r}.(\mathbf{i} + \mathbf{j}) = 7$ and the equation of a line is $\mathbf{r} = 3\mathbf{i} - \mathbf{k} + \lambda(a\mathbf{i} + \mathbf{j} - \mathbf{k})$ where a is a constant.
Find the value of a if the angle between the line and the plane is
(a) 30° (b) 45°

8. Show that the angle between the line $\frac{1}{2}(x-2) = \frac{1}{2}(y+2) = z - 5$ and the plane $3x + 2y - 6z = 8$ is $\arccos\dfrac{5\sqrt{17}}{21}$

9. $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j})$ and $\mathbf{r} = \mathbf{i} - \mathbf{j} + s(\mathbf{i} + \mathbf{k}) + t(\mathbf{j} - \mathbf{k})$

CHAPTER 40

COMPLEX NUMBERS

IMAGINARY NUMBERS

The numbers we have worked with up to now have been such that, when squared, the result is either positive or zero,
i.e. for a number k, $k^2 \geqslant 0$

Such numbers are *real* numbers.

However, the roots of an equation such as $x^2 = -1$ are clearly not real since they give -1 when squared.
If we are to work with equations of this type we need another category of numbers, i.e. the set of numbers whose squares are negative real numbers.

Members of this set are called *imaginary* numbers and some examples are

$$\sqrt{-1}, \quad \sqrt{-7}, \quad \sqrt{-20}$$

In general any imaginary number can be represented by $\sqrt{(-n^2)}$ where n is real.

Now $\qquad \sqrt{(-n^2)} = \sqrt{[(n^2)(-1)]} = \sqrt{(n^2)} \times \sqrt{(-1)}$

i.e. $\qquad \sqrt{(-n^2)} = n\mathrm{i}$ where $\mathrm{i} = \sqrt{(-1)}$

So any imaginary number can be written $n\mathrm{i}$ where n is real and

$$\mathrm{i} = \sqrt{(-1)}$$

e.g. $\qquad \sqrt{-16} = 4\mathrm{i}$ and $\sqrt{-3} = \mathrm{i}\sqrt{3}$

Usually the real number is placed before i but we write $\mathrm{i}\sqrt{3}$ rather than $\sqrt{3}\mathrm{i}$ in order to avoid ambiguity.
Note that j is sometimes used instead of i for $\sqrt{-1}$ but in this book we use i.

Imaginary numbers can be added to, or subtracted from, each other.

For example $3i + 9i = 12i$ and $i\sqrt{7} - 2i = (\sqrt{7} - 2)i$

The product of two imaginary numbers is always real.

For example $2i \times 5i = 10i^2 = 10(\sqrt{-1})^2 = 10(-1)$

i.e. $2i \times 5i = -10$

The quotient of two imaginary numbers is always real.

For example $\frac{3i}{7i} = \frac{3}{7}$

Powers of i can be simplified.

For example $i^3 = (i^2)(i) = -i$ $\quad i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^4)(i) = i$ $\quad i^{-1} = i/i^2 = i/-1 = -i$

COMPLEX NUMBERS

When a real number and an imaginary number are added or subtracted, the expression which is obtained cannot be simplified and is called a *complex number*,

e.g. $2 + 3i$, $4 - 7i$, and $-1 + 4i$ are complex numbers.

A general complex number can be represented by $a + bi$ where a and b can have any real value including zero.

If $a = 0$ we have numbers of the form bi, i.e. imaginary numbers.
If $b = 0$ we have numbers of the form a, i.e. real numbers.

Therefore

the set of complex numbers includes all real numbers and all imaginary numbers.

OPERATIONS ON COMPLEX NUMBERS

Addition and Subtraction

Real terms and imaginary terms are collected separately in two groups,

e.g. $(2 + 3i) + (4 - i) = (2 + 4) + (3i - i) = 6 + 2i$

and $(4 - 2i) - (3 + 5i) = (4 - 3) - (2i + 5i) = 1 - 7i$

Multiplication

The distributive law of multiplication applied to two complex numbers gives their product,

e.g.
$$\begin{aligned} i(5 - 2i) &= 5i - 2i^2 \\ &= 5i - 2(-1) \\ &= 2 + 5i \end{aligned}$$

and
$$\begin{aligned} (2 + 3i)(4 - i) &= 8 - 2i + 12i - 3i^2 \\ &= 8 + 10i - 3(-1) \\ &= 11 + 10i \end{aligned}$$

and
$$\begin{aligned} (2 + 3i)(2 - 3i) &= 4 - 6i + 6i - 9i^2 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

Conjugate Complex Numbers

Notice that the product in the last example above is a real number. This is because of the special form of the given complex numbers, $2 \pm 3i$, which are the factors of a 'difference of two squares'

Any pair of complex numbers of the form $a \pm bi$ have a product which is real, since

$$\begin{aligned} (a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \end{aligned}$$

Such complex numbers are said to be *conjugate* and each is the conjugate of the other. Thus $4 + 5i$ and $4 - 5i$ are conjugate complex numbers and $4 + 5i$ is the conjugate of $4 - 5i$

If $a + bi$ is denoted by z then its conjugate, $a - bi$, is denoted by $\bar{z}$ or z^*

Division

Division by a complex number cannot be carried out directly because the denominator is made up of two independent terms. This problem can be overcome by making the denominator real, a process called 'realising the denominator'. This is done by using the property that the product of conjugate complex numbers is real.

e.g. if we wish to divide $2 + 9i$ by $5 - 2i$ we multiply both numerator and denominator by the conjugate of the denominator, which is $5 + 2i$ in this case, giving

$$\frac{2+9i}{5-2i} = \frac{(2+9i)(5+2i)}{(5-2i)(5+2i)}$$

$$= \frac{10 + 49i + 18i^2}{25 - 4i^2}$$

$$= \frac{-8+49i}{29} = -\frac{8}{29} + \frac{49}{29}i$$

Note that the real term is given first, even when it is negative.

THE ZERO COMPLEX NUMBER

A complex number is zero if, and only if, the real term and the imaginary term are each zero.

i.e. $$a + bi = 0 \iff a = 0 \text{ and } b = 0$$

EQUAL COMPLEX NUMBERS

If $\quad a + bi = c + di \quad$ then $\quad (a + bi) - (c + di) = 0$

$\Rightarrow \quad (a - c) + (b - d)i = 0$

Hence $\quad a - c = 0 \quad$ and $\quad b - d = 0$

$\Rightarrow \quad a = c \quad$ and $\quad b = d$

i.e. two complex numbers are equal if, and only if, the real terms are equal and the imaginary terms are equal.

Denoting the real part of $a + bi$ by $\text{Re}(a + bi)$ and the imaginary part by $\text{Im}(a + bi)$ we have

$$a + bi = c + di \iff \begin{cases} \text{Re}(a+bi) = \text{Re}(c+di) \\ \text{Im}(a+bi) = \text{Im}(c+di) \end{cases}$$

A complex equation is therefore equivalent to two separate equations.

This property provides an alternative (but not better) method for division by a complex number.

e.g. to divide $3 - 2i$ by $5 + i$ we can represent the quotient by $p + qi$, where p and q are real,

i.e. $$\frac{3-2i}{5+i} = p + qi$$

Hence $$3 - 2i = (p + qi)(5 + i)$$
$$= 5p + pi + 5qi + qi^2$$
$$= (5p - q) + (p + 5q)i$$

Equating real and imaginary parts gives

$$3 = 5p - q \quad \text{and} \quad -2 = p + 5q$$

Solving these two equations gives $p = \frac{1}{2}$ and $q = -\frac{1}{2}$

$\therefore \quad (3 - 2i) \div (5 + i) = \frac{1}{2} - \frac{1}{2}i$

FINDING THE SQUARE ROOTS OF A COMPLEX NUMBER

Equating the real parts and the imaginary parts of a complex equation provides one method for determining the square roots of a complex number.

For example, if we assume that a square root of a complex number is itself a complex number then, using $\sqrt{(15 + 8i)}$ as an example we can say

$$\sqrt{(15+8i)} = a + bi \qquad \text{where } a \text{ and } b \text{ are real}$$

$\Rightarrow \quad 15 + 8i = (a + bi)^2$

$\quad = a^2 - b^2 + 2abi$

Equating real and imaginary parts gives

$$a^2 - b^2 = 15 \qquad [1]$$

and $$2ab = 8 \qquad [2]$$

Using $b = \frac{4}{a}$ in [1] gives $a^2 - \frac{16}{a^2} = 15$

$\Rightarrow \quad a^4 - 15a^2 - 16 = 0$

$\Rightarrow \quad (a^2 - 16)(a^2 + 1) = 0$

Thus $a^2 - 16 = 0$ or $a^2 + 1 = 0$

Now a is real, so $a^2 + 1 = 0$ gives no suitable values

therefore the only values for a are $a = \pm 4$

Then from equation [2] we have

$a = 4 \Rightarrow b = 1$

and $a = -4 \Rightarrow b = -1$

Note. It is not correct to say $a = \pm 4$ therefore $b = \pm 1$ as this offers four different pairs of values for a and b (i.e. 4, 1; 4, −1; −4, 1; −4, −1) two of which are invalid.

Hence $\sqrt{(15 + 8i)} = 4 + i$ or $-4 - i$

$= \pm(4 + i)$

This result justifies our original assumption that the square root of a complex number is another complex number.

Sometimes it is possible to find the square roots of a complex number simply by observation.

In the example above, equation [2] shows that the product of a and b is half the coefficient of i in the given complex number.

Suitable integral values for a and b can then be checked quite quickly, e.g. to find $\sqrt{(8 - 6i)}$ we note that $ab = -3$ so possible values for a and b are: 1, −3; 3, −1

Checking: $(1 - 3i)^2 = -8 - 6i$ which is not correct

$(3 - i)^2 = 8 - 6i$ which is correct

Hence one square root of $8 - 6i$ is $3 - i$ and the other is $-(3 - i)$

i.e. $\sqrt{(8 - 6i)} = \pm(3 - i)$

Note. Unless a and b are integers, this method is unlikely to be useful.

EXERCISE 40a

1. Simplify: i^7, i^{-3}, i^9, i^{-5}, i^{4n}, i^{4n+1}

2. Add the following pairs of complex numbers.
   (a) $3 + 5i$ and $7 - i$ (b) $4 - i$ and $3 + 3i$
   (c) $2 + 7i$ and $4 - 9i$ (d) $a + bi$ and $c + di$

3. Subtract the second number from the first in each part of Question 2.

4. Simplify
   (a) $(2 + i)(3 - 4i)$ (b) $(5 + 4i)(7 - i)$ (c) $(3 - i)(4 - i)$
   (d) $(3 + 4i)(3 - 4i)$ (e) $(2 - i)^2$ (f) $(1 + i)^3$
   (g) $i(3 + 4i)$ (h) $(x + yi)(x - yi)$ (i) $i(1 + i)(2 + i)$
   (j) $(a + bi)^2$

5. Realise the denominator of each of the following fractions and hence express each in the form $a + bi$
   (a) $\frac{2}{1 - i}$ (b) $\frac{3 + i}{4 - 3i}$ (c) $\frac{4i}{4 + i}$ (d) $\frac{1 + i}{1 - i}$
   (e) $\frac{7 - i}{1 + 7i}$ (f) $\frac{x + yi}{x - yi}$ (g) $\frac{3 + i}{i}$ (h) $\frac{-2 + 3i}{-i}$

6. Solve the following equations for x and y
   (a) $x + yi = (3 + i)(2 - 3i)$ (b) $\frac{2 + 5i}{1 - i} = x + yi$
   (c) $3 + 4i = (x + yi)(1 + i)$ (d) $x + yi = 2$
   (e) $x + yi = (3 + 2i)(3 - 2i)$ (f) $x + yi = (4 + i)^2$
   (g) $\frac{x + yi}{2 + i} = 5 - i$ (h) $(x + yi)^2 = 3 + 4i$

7. Find the real and imaginary parts of
   (a) $(2 - i)(3 + i)$ (b) $(1 - i)^3$ (c) $(3 + 4i)(3 - 4i)$
   (d) $\frac{3 + 2i}{4 - i}$ (e) $\frac{2}{3 + i} + \frac{3}{2 + i}$ (f) $\frac{1}{x + yi} - \frac{1}{x - yi}$

8. Find the square roots of
   (a) $3 - 4i$ (b) $21 - 20i$ (c) $2i$ $15 + 8i$ (e) $-24 + 10i$

COMPLEX ROOTS OF QUADRATIC EQUATIONS

Consider the quadratic equation $x^2 + 2x + 2 = 0$

The formula $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ gives $x = \dfrac{-2 \pm \sqrt{-4}}{2}$

Previously we dismissed solutions of this type, in which $b^2 - 4ac < 0$, as not being real. But now, because $\sqrt{-4} = 2\mathrm{i}$, we see that the roots of this equation are the complex numbers $-1 + \mathrm{i}$ and $-1 - \mathrm{i}$.

Further, the roots are conjugate complex numbers.

We can show that if $b^2 - 4ac < 0$, the roots of the general quadratic equation $ax^2 + bx + c = 0$ are *always* conjugate complex numbers.

If $\quad ax^2 + bx + c = 0$ and $b^2 - 4ac < 0$

then $\quad x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} = \dfrac{-b}{2a} \pm \dfrac{\mathrm{i}\sqrt{(4ac - b^2)}}{2a}$

$\therefore\ x = p \pm q\mathrm{i}$ where $p = -b/2a$ and $q = \sqrt{(4ac - b^2)}/2a$

i.e. when $b^2 - 4ac < 0$, the roots of the equation $ax^2 + bx + c = 0$ are

$$p + q\mathrm{i} \quad \text{and} \quad p - q\mathrm{i}$$

and these are conjugate complex numbers.

So, if one root of a quadratic equation with real coefficients is known to be complex, the other root must also be complex and the conjugate of the first.

When the quadratic equation $ax^2 + bx + c = 0$ has real roots α and β we know that

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

We now show that these relationships are valid also when the roots are complex.

If $\alpha = p + q\mathrm{i}$ and $\beta = p - q\mathrm{i}$ then

$$\alpha + \beta = 2p = 2\left(\frac{-b}{2a}\right) = \frac{-b}{a}$$

and $\quad \alpha\beta = (p + q\mathrm{i})(p - q\mathrm{i}) = p^2 + q^2 = \dfrac{b^2}{4a^2} + \dfrac{(4ac - b^2)}{4a^2} = \dfrac{c}{a}$

Hence, for *any* quadratic equation $ax^2 + bx + c = 0$, the roots α and β satisfy the relationships

$$\alpha + \beta = -b/a \quad \text{and} \quad \alpha\beta = c/a$$

Examples 40b

1. One root of the equation $x^2 + px + q = 0$ is $2 - 3\mathrm{i}$. Find the values of p and q

If one root is $2 - 3\mathrm{i}$ the other must be $2 + 3\mathrm{i}$

Then $\quad \alpha + \beta = 4 \quad$ and $\quad \alpha\beta = (2 - 3\mathrm{i})(2 + 3\mathrm{i}) = 13$

Now any quadratic equation can be written in the form

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

So the equation with roots $2 \pm 3\mathrm{i}$ is

$$x^2 - 4x + 13 = 0$$

Therefore $p = -4$ and $q = 13$

---

2. Find the complex roots of the equation $2x^2 + 3x + 5 = 0$. If these roots are α and β, confirm the relationships

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

If $\quad 2x^2 + 3x + 5 = 0$

then $\quad x = \dfrac{-3 \pm \sqrt{(9 - 40)}}{4}$

$\Rightarrow \quad \alpha = -\dfrac{3}{4} + \dfrac{\sqrt{31}}{4}\mathrm{i} \quad \beta = -\dfrac{3}{4} - \dfrac{\sqrt{31}}{4}\mathrm{i}$

Hence $\quad \alpha + \beta = \left(-\dfrac{3}{4} + \dfrac{\sqrt{31}}{4}\mathrm{i}\right) + \left(-\dfrac{3}{4} - \dfrac{\sqrt{31}}{4}\mathrm{i}\right)$

$$= -\frac{3}{2} = -\frac{b}{a}$$

and $$\alpha\beta = \left(-\frac{3}{4} + \frac{\sqrt{31}}{4}\text{i}\right)\left(-\frac{3}{4} - \frac{\sqrt{31}}{4}\text{i}\right)$$

$$= \frac{9}{16} - \frac{31}{16}\text{i}^2$$

$$= \frac{40}{16} = \frac{5}{2} = \frac{c}{a}$$

EXERCISE 40b

1. Solve the following equations.
   (a) $x^2 + x + 1 = 0$ (b) $2x^2 + 7x + 1 = 0$
   (c) $x^2 + 9 = 0$ (d) $x^2 + x + 3 = 0$
   (e) $x^4 - 1 = 0$ (f) $3x^2 + x + 3 = 0$

2. Form the equation whose roots are
   (a) $\text{i}, -\text{i}$ (b) $2 + \text{i}, 2 - \text{i}$
   (c) $1 - 3\text{i}, 1 + 3\text{i}$ (d) $1 + \text{i}, 1 - \text{i}, 2$

3. Without calculating a, b and c, evaluate $-b/a$ and c/a if one root of the equation $ax^2 + bx + c = 0$ is
   (a) $2 + \text{i}$ (b) $3 - 4\text{i}$ (c) i (d) $5\text{i} - 12$

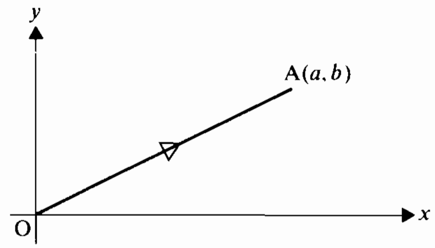
4. Find the equation, one of whose roots is
   (a) $-1 - \text{i}$ (b) $-5 + \text{i}$ (c) $1 - 3\text{i}$ (d) $4 + \text{i}$
   Explain why this question cannot be answered if the given root is 2

THE ARGAND DIAGRAM

In the complex number $a + b\text{i}$, a and b are both real numbers so $a + b\text{i}$ can be represented by the ordered pair $\begin{pmatrix} a \\ b \end{pmatrix}$.

This suggests that a and b could be used as the coordinates of a point $\text{A}(a, b)$ in the xy plane.

Then the vector $\overrightarrow{\text{OA}}$ provides a visual representation of the complex number.



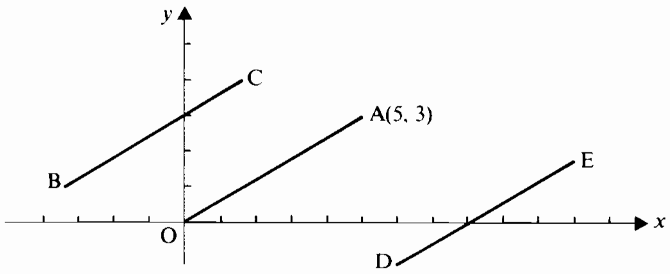
This idea was introduced by a French mathematician and his name, Argand, is given to the diagram which illustrates a complex number in this way. On an Argand diagram, the real part of a complex number is taken as the x-coordinate, and the coefficient of i is the y-coordinate. For this reason the x and y axes are often called the real and imaginary axes. It must be remembered however that the y-coordinate is the *real* number b.

A general complex number $x + y\text{i}$ is represented by the vector $\overrightarrow{\text{OP}}$ where P is the point (x, y)

In an Argand diagram, the magnitude and direction of a line are used to represent a complex number in the same way that a section of line can be used to represent a vector quantity. The techniques and operations used in vector analysis can therefore be applied equally well to complex number analysis.

A COMPLEX NUMBER AS A VECTOR

On an Argand diagram a complex number such as $5 + 3\text{i}$ can be represented by the vector $\overrightarrow{\text{OA}}$ where A is the point (5, 3). However it can equally well be represented by any other vector with the same length and direction as $\overrightarrow{\text{OA}}$, e.g. by $\overrightarrow{\text{BC}}$ or $\overrightarrow{\text{DE}}$ in the diagram below.



A complex number can be treated in this way only when it behaves as a *free* vector. If, on the other hand, $5 + 3i$ is regarded as a *position* vector, then *only* the vector $\overrightarrow{OA}$ represents $5 + 3i$. In this case the *actual point* A(5, 3) is sometimes taken to represent $5 + 3i$

A vector representing a complex number is usually denoted by the symbol z, so

for a general complex number we have

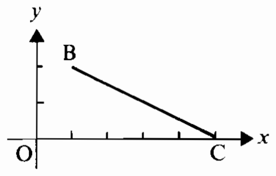
$$z = x + yi$$

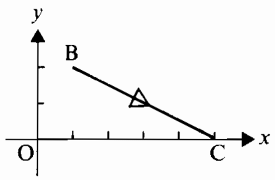
and for unique complex numbers we use z_1, z_2, etc.,

e.g. $z_1 = 5 + 3i,\quad z_2 = 7 - 4i$

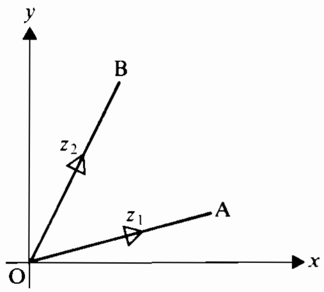
When z is used on an Argand diagram, an arrow is needed to indicate the direction of the line representing the complex number.

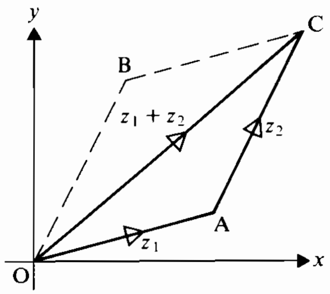
For example, in the diagram below the line BC without an arrow could represent *either* $4 - 2i$ *or* $-4 + 2i$
Whereas the arrow shows that this line represents the complex number $4 - 2i$ only





Graphical Addition and Subtraction of Complex Numbers

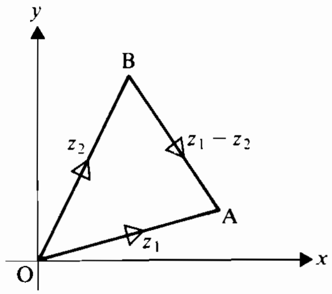


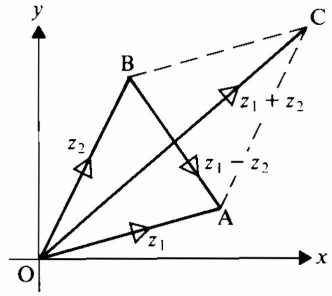


Consider two complex numbers, z_1 and z_2, represented on an Argand diagram by $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

We know that the sum of the two *vectors* $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $\overrightarrow{OC}$ where AC is equal and parallel to OB. Therefore, as z_1 and z_2 behave as vectors, we have

> $z_1 + z_2$ is represented by the diagonal $\overrightarrow{OC}$ of the parallelogram OACB.





Now considering the vector triangle OAB we see that

$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$$

But $\overrightarrow{OA}$ represents z_1 and $\overrightarrow{BO} = -\overrightarrow{OB}$ represents $-z_2$ therefore

> $z_1 - z_2$ is represented by the diagonal joining B to A in parallelogram OACB.

Note carefully the direction of the vector represented by this diagonal.

Hence the two diagonals of the parallelogram OACB represent the sum and the difference of z_1 and z_2

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ then

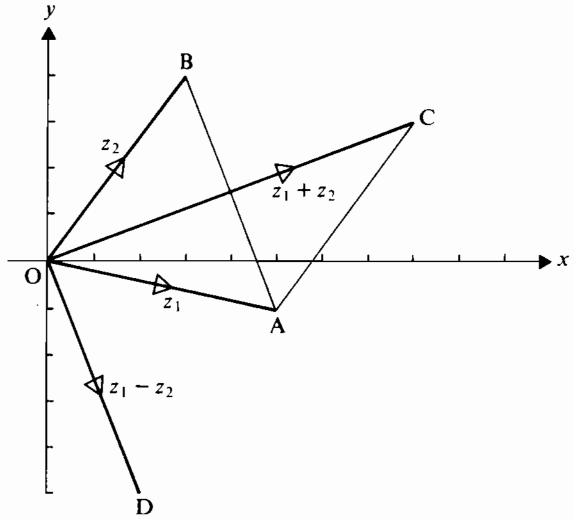
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

Therefore C is the point $(\{x_1 + x_2\}, \{y_1 + y_2\})$,

e.g. if $z_1 = 3 + 5i$ and $z_2 = 7 - 2i$ then $z_1 + z_2$ can be represented on the Argand diagram by $\overrightarrow{OC}$ where C is the point (10, 3)

Example 40c

If $z_1 = 5 - \mathrm{i}$ and $z_2 = 3 + 4\mathrm{i}$, represent on one Argand diagram the complex numbers z_1, z_2, $z_1 + z_2$, $z_1 - z_2$



$\overrightarrow{OA}$ represents z_1 and $\overrightarrow{OB}$ represents z_2
$\overrightarrow{OC}$, the diagonal of the parallelogram OACB, represents $z_1 + z_2$
$z_1 - z_2$ is represented by the diagonal $\overrightarrow{BA}$ and also by $\overrightarrow{OD}$ which is equal and parallel to BA

EXERCISE 40c

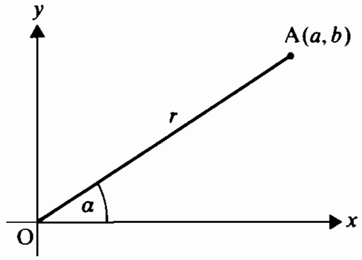
1. Represent each complex number on an Argand diagram.
   (a) $5 + \mathrm{i}$ (b) $-2 + 6\mathrm{i}$ (c) $3 - 5\mathrm{i}$ (d) $-4 - 4\mathrm{i}$
   (e) $4 + 4\mathrm{i}$ (f) -3 (g) $3\mathrm{i}$ (h) $-7\mathrm{i}$

2. If $z_1 = 3 - \mathrm{i}$, $z_2 = 1 + 4\mathrm{i}$, $z_3 = -4 + \mathrm{i}$, $z_4 = -2 - 5\mathrm{i}$, represent the following by lines on Argand diagrams, showing the direction of each line by an arrow.
   (a) $z_1 + z_2$ (b) $z_2 - z_3$ (c) $z_1 - z_3$ (d) $z_2 + z_4$
   (e) $z_4 - z_1$ (f) $z_3 - z_4$ (g) z_1
   (h) z_4 (i) $z_2 - z_1$ (j) $z_1 + z_3$

3. If $z_1 = x_1 + y_1\mathrm{i}$ and $z_2 = x_2 + y_2\mathrm{i}$ show on an Argand diagram the position of the points representing $\frac{1}{2}(z_1 + z_2)$ and $\frac{1}{3}(2z_1 + z_2)$

MODULUS AND ARGUMENT

Consider the point $A(a, b)$, representing the complex number $a + b\mathrm{i}$



The length of the line OA is called the *modulus* of $a + b\mathrm{i}$ and is denoted by r,

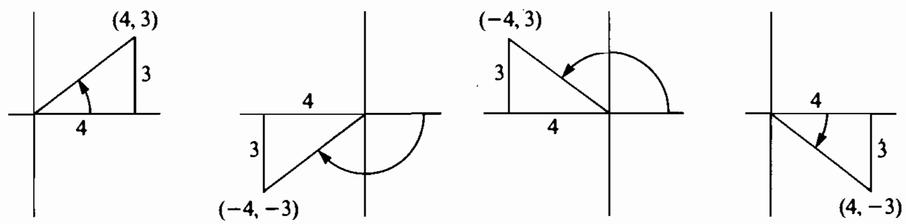
i.e. $$|a + b\mathrm{i}| = r = \sqrt{(a^2 + b^2)}$$

The angle between the positive x-axis and the line OA is called the *argument* or *amplitude* of $a + b\mathrm{i}$ and is denoted by α, where α is within the range $-\pi < \alpha \leqslant \pi$

i.e. $$\arg(a + b\mathrm{i}) = \alpha = \arctan b/a$$

Even within this range there are usually two angles with the same tangent so, when giving the argument of a complex number in the form $\arctan b/a$, it is often necessary to include a diagram. Consider, for example, the complex numbers

$$4 + 3\mathrm{i}, \quad -4 - 3\mathrm{i}, \quad -4 + 3\mathrm{i} \quad \text{and} \quad 4 - 3\mathrm{i}$$



For $4 + 3\mathrm{i}$, $\alpha = \arctan \frac{3}{4}$ and is positive and acute
$\Rightarrow \quad \alpha = 0.644$ rad

For $-4 - 3\mathrm{i}$, $\alpha = \arctan \frac{3}{4}$ and is negative and obtuse
$\Rightarrow \quad \alpha = -2.498$ rad

For $-4+3i$, $\alpha = \arctan -\frac{3}{4}$ and is positive and obtuse

$\Rightarrow \quad \alpha = 2.498$ rad

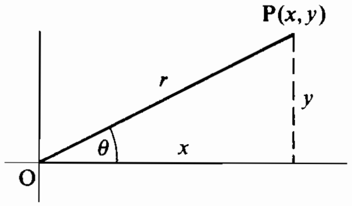
For $4-3i$, $\alpha = \arctan -\frac{3}{4}$ and is negative and acute

$\Rightarrow \quad \alpha = -0.644$ rad

We see that the argument of each of the first two complex numbers is given by $\arctan \frac{3}{4}$ but the values of α are not the same: a similar observation can be made for the last two complex numbers.
This example shows that stating the value of $\arctan \alpha$ is ambiguous and that more information is required to define α uniquely, e.g. a diagram of the type used on page 787.

THE MODULUS/ARGUMENT FORM FOR A COMPLEX NUMBER

Consider a general complex number $x+yi$ represented on an Argand diagram by $\overrightarrow{OP}$ where P is the point (x, y) and OP is inclined at an angle θ to Ox.



From the diagram we see that $x = r\cos\theta$ and $y = r\sin\theta$

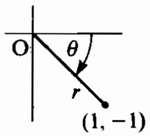
Therefore $x+yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

Hence a complex number can be changed from the form $x+yi$ into the form $r(\cos\theta + i\sin\theta)$ by finding the modulus, r, and the argument, θ

To convert the complex number $1-i$, for example, we have

$r = |1-i| = \sqrt{(1^2+\{-1\}^2)} = \sqrt{2}$

$\theta = \arg(1-i) = \arctan(-1) = -\frac{1}{4}\pi$



Hence, in modulus/argument form,

$1-i = \sqrt{2}(\cos\{-\frac{1}{4}\pi\} + i\sin\{-\frac{1}{4}\pi\})$

Conversely, a complex number given in modulus/argument form can be changed directly into cartesian form,

e.g. $4(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi) = 4(\{-\frac{1}{2}\} + i\{\frac{1}{2}\sqrt{3}\}) = -2+2i\sqrt{3}$

Note that the position where i is written varies according to the form of the complex term; the aim is always to make the term clear and unambiguous,
e.g. we write $6i$, $i\sin\theta$, $2i\sqrt{3}$, $3i\cos\frac{1}{2}\pi$, etc.

Examples 40d

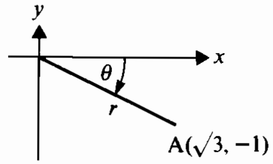
1. Express in the form $r(\cos\theta + i\sin\theta)$
(a) $\sqrt{3}-i$ (b) -2 (c) $-5i$ (d) $-2+2i$

(a) For $\sqrt{3}-i$

$r = \sqrt{(\{\sqrt{3}\}^2+\{-1\}^2)} = 2$

$\tan\theta = -1/\sqrt{3} \Rightarrow \theta = -\frac{1}{6}\pi$

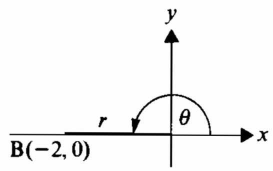
$\therefore \sqrt{3}-i = 2(\cos\{-\frac{1}{6}\pi\} + \sin\{-\frac{1}{6}\pi\})$



(b) For -2, $r = 2$ and $\theta = \pi$

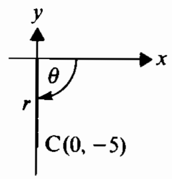
$(\theta \neq -\pi$ as $-\pi < \theta < \pi)$

$\therefore \quad -2 = 2(\cos\pi + i\sin\pi)$



(c) For $-5i$, $r = 5$ and $\theta = -\frac{1}{2}\pi$

$\therefore \quad -5i = 5(\cos\{-\frac{1}{2}\pi\} + i\sin\{-\frac{1}{2}\pi\})$

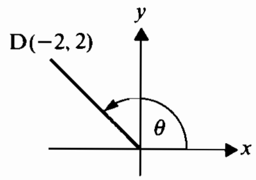


(d) For $-2+2i$

$r = \sqrt{(\{-2\}^2+2^2)} = 2\sqrt{2}$

$\tan\theta = 2/-2 = -1 \Rightarrow \theta = \frac{3}{4}\pi$

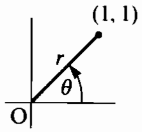
$\therefore \quad -2+2i = 2\sqrt{2}(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$



2. Find the modulus and argument of $\dfrac{7-i}{3-4i}$

First we express $\dfrac{7-i}{3-4i}$ in the form $a+bi$

$$\frac{7-i}{3-4i} = \frac{(7-i)(3+4i)}{(3-4i)(3+4i)} = \frac{25+25i}{25} = 1+i$$



Then $|1+i| = \sqrt{2}$

and $\arg(1+i)$ is the positive acute angle $\arctan 1$, i.e. $\frac{1}{4}\pi$

$\therefore$ the modulus and argument of $\dfrac{7-i}{3-4i}$ are $\sqrt{2}$ and $\frac{1}{4}\pi$

EXERCISE 40d

1. Represent each of the following complex numbers by a line on an Argand diagram. Find the modulus and argument of each complex number.

(a) $3-2i$ (b) $-4+i$ (c) $-3-4i$ (d) $5+12i$
(e) $1-i$ (f) $-1+i$ (g) 6 (h) $-4i$
(i) $(1+i)^2$ (j) $i(1-i)$ (k) $i^2(1-i)$ (l) $(3+i)(4+i)$
(m) $\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi$ (n) $2(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi)$
(p) $\cos(-\frac{5}{6}\pi) + i\sin(-\frac{5}{6}\pi)$ (q) $a+bi$

2. Express in the form $r(\cos\theta + i\sin\theta)$

(a) $1+i$ (b) $\sqrt{3}-i$ (c) $-3-4i$ (d) $-5+12i$
(e) $2-i$ (f) 6 (g) -3 (h) $4i$
(i) $-3-i\sqrt{3}$ (j) $24+7i$

3. The modulus, r, and argument, θ, of a complex number are given. Express the complex number in the form $x+yi$ when r and θ are

(a) $2, \frac{1}{6}\pi$ (b) $3, -\frac{1}{4}\pi$ (c) $1, \frac{2}{3}\pi$ (d) $3, 0$
(e) $4, \pi$ (f) $1, -\frac{3}{4}\pi$ (g) $2, \frac{1}{2}\pi$ (h) $2, -\frac{1}{2}\pi$

PRODUCTS AND QUOTIENTS

The product and quotient of two complex numbers can already be found algebraically when the numbers are given in the form $a+bi$

Now we are going to investigate what happens when we multiply or divide complex numbers in the form $r(\cos\theta + i\sin\theta)$

Taking $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i^2\sin\theta_1\sin\theta_2] \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \end{aligned}$$

Hence

$z_1 z_2$ gives a complex number with modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$

i.e. $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$

Similarly it can be shown that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

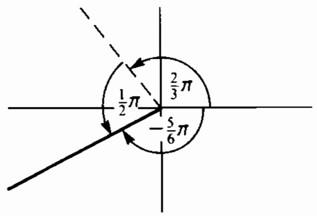
i.e. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = (\theta_1 - \theta_2)$

Note that when the argument of a product or a quotient is found in this way, the angle obtained may not lie between $-\pi$ and π. In such cases the corresponding argument within this range should be stated,

e.g. if $\theta_1 = \frac{2}{3}\pi$ and $\theta_2 = \frac{1}{2}\pi$

then $\theta_1 + \theta_2 = \frac{7}{6}\pi$

but $\arg(z_1 z_2) = -\frac{5}{6}\pi$



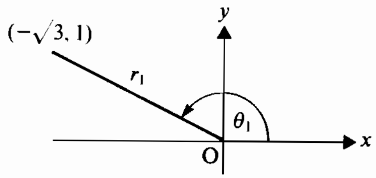
Example 40e

Write down the modulus and argument of $-\sqrt{3}+i$ and of $4+4i$. Hence express in the form $r(\cos\theta + i\sin\theta)$ the complex numbers

$(-\sqrt{3}+i)(4+4i)$ and $\dfrac{(-\sqrt{3}+i)}{4+4i}$

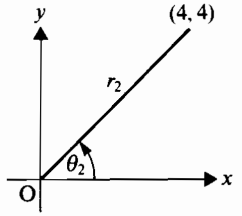
For $-\sqrt{3} + \mathrm{i}$

$r_1 = 2$ and $\theta_1 = \frac{5}{6}\pi$



For $4 + 4\mathrm{i}$

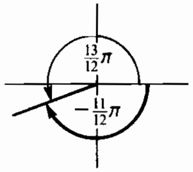
$r_2 = 4\sqrt{2}$ and $\theta_2 = \frac{1}{4}\pi$



If $(-\sqrt{3} + \mathrm{i})(4 + 4\mathrm{i}) = r_3(\cos\theta_3 + \mathrm{i}\sin\theta_3)$

then $r_3 = r_1 r_2 = 8\sqrt{2}$ and $\theta_3 = \theta_1 + \theta_2 = \frac{13}{12}\pi$

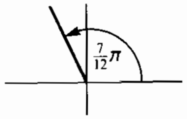
As θ_3 is not between $-\pi$ and π we refer to the diagram to find that the required argument is $-\frac{11}{12}\pi$



$\therefore \quad (-\sqrt{3} + \mathrm{i})(4 + 4\mathrm{i}) = 8\sqrt{2}(\cos\{-\frac{11}{12}\pi\} + \mathrm{i}\sin\{-\frac{11}{12}\pi\})$

If $\dfrac{(-\sqrt{3} + \mathrm{i})}{(4 + 4\mathrm{i})} = r_4(\cos\theta_4 + \mathrm{i}\sin\theta_4)$

then $r_4 = \dfrac{r_1}{r_2} = \dfrac{\sqrt{4}}{2}$ and $\theta_4 = \theta_1 - \theta_2 = \frac{7}{12}\pi$



this time θ_4 is between $-\pi$ and π so it is the required argument.

$$\therefore \quad \frac{(-\sqrt{3} + \mathrm{i})}{(4 + 4\mathrm{i})} = \frac{\sqrt{2}}{4}(\cos\tfrac{7}{12}\pi + \mathrm{i}\sin\tfrac{7}{12}\pi)$$

EXERCISE 40e

In each Question from 1 to 4

(a) find the modulus and argument of the given complex number, without first expressing it in the form $a + b\mathrm{i}$ and illustrate your result on an Argand diagram.

(b) express the given complex number in the form $a + b\mathrm{i}$, find the modulus and argument and check the results against the answers to part (a).

1. $(3 - \sqrt{3}\mathrm{i})(1 - \mathrm{i})$ **2.** $(1 + \mathrm{i})(1 - \mathrm{i})$

3. $\dfrac{(-2 - \mathrm{i}\sqrt{3})}{(-2 + \mathrm{i}\sqrt{3})}$ **4.** $\dfrac{2}{(1 + \mathrm{i})}$

5. Illustrate on an Argand diagram lines representing z, $1/z$, z^2 and $z - z^2$ when z is

(a) $2 + \mathrm{i}$ (b) $\frac{1}{2} - \frac{1}{2}\mathrm{i}$ (c) $3 + 4\mathrm{i}$ (d) $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$ (e) $5 - 12\mathrm{i}$

MIXED EXERCISE 40

1. Find, in the form $a + b\mathrm{i}$,

(a) $\dfrac{1}{7 + 5\mathrm{i}}$ (b) $(2 - \mathrm{i})(4 + 3\mathrm{i})$ (c) $\dfrac{2 - \mathrm{i}}{3 + \mathrm{i}}$ (d) $(2 + 5\mathrm{i})^2$

2. If $z = 3 - \mathrm{i}$ express $z + 1/z$ in the form $a + b\mathrm{i}$ where a and b are real.

3. Find the square roots of z where $z = 12 + 5\mathrm{i}$. Illustrate z and its square roots on an Argand diagram.

4. Find the modulus and argument of each of the following complex numbers.

(a) $-1 + \mathrm{i}$ (b) $3 + 4\mathrm{i}$ (c) $(-1 + \mathrm{i})(3 + 4\mathrm{i})$

Represent each of these numbers by points on an Argand diagram and label these points A, B and C. Find the area of triangle ABC.

5. If $z_1 = 1 + \mathrm{i}$ and $z_2 = 7 - \mathrm{i}$, find the modulus of

(a) $z_1 - z_2$ (b) $z_1 z_2$ (c) $\dfrac{z_1}{z_2}$ (d) $\dfrac{z_1 - z_2}{z_1 z_2}$

6. Given that $(2 + 3\mathrm{i})\lambda - 2\mu = 3 + 6\mathrm{i}$ find the values of λ and μ.

7. Two complex numbers z_1 and z_2 are such that $z_1 + z_2 = 1$. If $z_1 = \dfrac{a}{1+i}$ and $z_2 = \dfrac{b}{1+2i}$ find a and b

8. If $z = x + yi$ find the real and imaginary parts of $z - 1/z$

9. One root of the equation $x^2 - \lambda x - \mu = 0$ is $2 - i$. Find λ and μ.

10. Find the modulus and argument of each root of the equation $y^2 + 4y + 8 = 0$

11. If α and β are the roots of the equation $3x^2 + x + 2 = 0$, find the value of
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $3\alpha^2 + \alpha + 2$

12. Given that z^* is the conjugate of z and $z = a + bi$ where a and b are real, find the possible values of z if $zz^* - 2iz = 7 - 4i$

CONSOLIDATION G

SUMMARY

SOLUTION OF EQUATIONS

Trigonometric Equations

Listed below are some of the trig identities useful in solving equations.

$$\left.\begin{aligned}\cos^2\theta + \sin^2\theta &\equiv 1\\ \tan^2\theta + 1 &\equiv \sec^2\theta\\ \cot^2\theta + 1 &\equiv \operatorname{cosec}^2\theta\end{aligned}\right\}$$

Use in an equation containing two ratios of one angle, at least one ratio being squared.

$$\left.\begin{aligned}\cos 2\theta &\equiv 2\cos^2\theta - 1\\ &\equiv 1 - 2\sin^2\theta\end{aligned}\right\}$$

Use to express an equation in terms of trig ratios of θ only.

$\sin A \pm \sin B \equiv 2\sin\frac{1}{2}(A \pm B)\cos\frac{1}{2}(A \mp B)$

$\cos A + \cos B \equiv 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$

$\cos A - \cos B \equiv -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$

Use to factorise an expression, e.g.

$$\begin{aligned}&\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta\\ &= 2\sin 2\theta\cos\theta + 2\sin 6\theta\cos\theta\\ &= 2\cos\theta(\sin 2\theta + \sin 6\theta)\end{aligned}$$

$a\sin\theta + b\cos\theta \equiv r\sin(\theta + \alpha)$ and variations of this form.

Use to reduce $a\sin\theta + b\cos\theta = c$ to $\sin(\theta + \alpha) = k$

$$\left.\begin{aligned}\sin A &\equiv \frac{2t}{1+t^2}\\ \cos A &\equiv \frac{1-t^2}{1+t^2}\\ \tan A &\equiv \frac{2t}{1-t^2}\end{aligned}\right\}\ t = \tan\tfrac{1}{2}A$$

Use in equations, containing assorted trig ratios, which have not responded to other approaches.

$\sin A \equiv \cos(\frac{1}{2}\pi - A)$

Use for general solution of equations of the type $\sin 2\theta = \cos 5\theta$

Exponential Equations

If there is only one term on each side, take logs.

If a sum or difference of terms is involved, suspect a disguised quadratic equation, e.g.

$$e^{2x} - 2e^x - 3 = 0 \quad \Rightarrow \quad y^2 - 2y - 3 = 0 \quad \text{where } y = e^x$$

Logarithmic Equations

In general make all the bases the same and use the laws of logarithms.

Polynomial Equations

If $f(a) = 0$ and $f'(a) = 0$ then $y = f(x)$ has a repeated root $x = a$

Approximate Solutions

Using graphs

$f(x) = g(x)$ where the curves $y = f(x)$ and $y = g(x)$ cut.
$f(x) = 0$ between two points on the x-axis where $f(x) < 0$ and $f(x) > 0$

Newton-Raphson Method

If $x = a$ is an approximate solution of the equation $f(x) = 0$ then a better approximation is $x = b$ where $b = a - \dfrac{f(a)}{f'(a)}$

COMPLEX NUMBERS

A complex number is of the form $a + bi$ where a and b are real and

$$i = \sqrt{(-1)} \quad \text{and} \quad \sqrt{(-n^2)} = ni$$

$(a + bi) \pm (c + di) \equiv (a \pm c) + (b \pm d)i$

$(a + bi)(c + di) \equiv (ac - bd) + (ad + bc)i$

$\dfrac{(a + bi)}{(c + di)} \equiv \dfrac{(a + bi)(c - di)}{(c + di)(c - di)} \equiv \dfrac{(a + bi)(c - di)}{c^2 + d^2}$

$x + yi = a + bi \quad \Rightarrow \quad x = a$ and $y = b$

$x + yi = 0 \quad \Rightarrow \quad x = 0$ and $y = 0$

$|a + bi| = \sqrt{(a^2 + b^2)}$

$\arg(a + bi) = \alpha$ where $\tan\alpha = \dfrac{b}{a}$ and $-\pi < \alpha \leqslant \pi$

$x + yi = r(\cos\theta + i\sin\theta)$ where $r = |x + yi|$ and $\theta = \arg(x + yi)$

If $z = a + bi$, $\bar{z} = a - bi$, where $\bar{z}$ and z are conjugate.

If a quadratic or cubic equation has any complex roots, they occur in conjugate pairs.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $|z_1 z_2| = r_1 r_2$ and $\arg z_1 z_2 = \theta_1 + \theta_2$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg\frac{z_1}{z_2} = \theta_1 - \theta_2$$

VECTORS

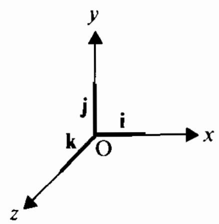
A vector is a quantity with both magnitude and direction and can be represented by a line segment.

If lines representing several vectors are drawn 'head to tail' in order, then the line which completes a closed polygon represents the sum of the vectors (or the resultant vector).

A position vector has a fixed location in space.

The position vector of a point C which divides AB in the ratio $\lambda:\mu$ is given by $\dfrac{\lambda b + \mu a}{\lambda + \mu}$ where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

Cartesian Unit Vectors



i, **j** and **k** are unit vectors in the directions of Ox, Oy and Oz respectively,

Any vector can be given in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \pm (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = (a_1 \pm a_2)\mathbf{i} + (b_1 \pm b_2)\mathbf{j} + (c_1 \pm c_2)\mathbf{k}$

$$|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{(a^2 + b^2 + c^2)}$$

For two vectors $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel if $\mathbf{v}_1 = \lambda\mathbf{v}_2$,
i.e. $a_1 = \lambda a_2$, $b_1 = \lambda b_2$, $c_1 = \lambda c_2$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are equal if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

The direction ratios of $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are $a:b:c$

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then the unit vector in the direction of $\mathbf{v}$ is $\hat{\mathbf{v}}$

where $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

If a vector $\mathbf{v}$ is in the direction of a vector $\mathbf{d}$ then $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{d}}$

Equations for a Line

For a line in the direction of the vector $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and passing through a point with position vector $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$.

a vector equation in standard form is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

parametric equations are $x = x_1 + \lambda a,\ y = y_1 + \lambda b,\ z = z_1 + \lambda c$

cartesian equations are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

Two lines with equations $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{d}_2$

are parallel if $\mathbf{d}_1$ is a multiple of $\mathbf{d}_2$

intersect if there are values of λ and μ for which $\mathbf{r}_1 = \mathbf{r}_2$

are skew in all other cases

The Scalar Product of Two Vectors

If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ then

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$\mathbf{a}$ and $\mathbf{b}$ are perpendicular $\Rightarrow$ $\mathbf{a}.\mathbf{b} = 0$

If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$$

Equations of a Plane

A plane perpendicular to a vector $\mathbf{n}$ has a vector equation in standard form of $\mathbf{r}.\mathbf{n} = D$ where D/n is the distance of the plane from O.

Taking $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ gives the cartesian equation for the plane

i.e. $$Ax + By + Cz = D$$

A plane perpendicular to $\mathbf{n}$ and containing the point with position vector $\mathbf{a}$ has the equation $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$

A plane containing two direction vectors, $\mathbf{d}_1$ and $\mathbf{d}_2$, and a point with position vector $\mathbf{a}$, has a vector equation in parametric form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$$

The acute angle θ between the two planes $\mathbf{r}_1.\mathbf{n}_1 = \mathbf{d}_1$ and $\mathbf{r}_2.\mathbf{n}_2 = \mathbf{d}_2$ is equal to the acute angle between the two normals.

i.e. $$\cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{n_1 n_2}$$

The acute angle ϕ between the plane $\mathbf{r}.\mathbf{n} = D$ and the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is given by $\sin\phi = \dfrac{\mathbf{d}.\mathbf{n}}{dn}$

MULTIPLE CHOICE EXERCISE G

TYPE I

1. The modulus of the vector $6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ is

   A $\sqrt{23}$ **B** 7 **C** 1 **D** 49 **E** $\sqrt{11}$

2. $\overrightarrow{OP} = 2\mathbf{i} - 6\mathbf{j}$ and $\overrightarrow{OQ} = 4\mathbf{i} - 3\mathbf{k}$. The vector of magnitude 7 in the direction of $\overrightarrow{PQ}$ is

   A $2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$
   B $7(-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$
   C $-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$
   D $7(2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$
   E $7(6\mathbf{i} - 6\mathbf{j} - 3\mathbf{k})$

3. If $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ then $\mathbf{a}.\mathbf{b}$ is

   A $2\mathbf{i}$ **B** -1 **C** $-2\mathbf{j} + 2\mathbf{k}$ **D** 2 **E** 3

4. The line with equation $\mathbf{r} = \mathbf{j} + \lambda(2\mathbf{i} - 3\mathbf{k})$ has direction ratios

   A $2:1:-3$
   B $0:1:0$
   C $2:3$
   D $2\lambda:1:-3\lambda$
   E $2:0:-3$

5. The plane whose equation is $\mathbf{r}.(\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2$ contains the point

   A $(1, -1, 1)$
   B $(-1, 1, 0)$
   C $(0, 1, 1)$
   D $(2, 0, 0)$
   E $(x, -y, z)$

6. The angle between the lines whose equations are $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{b}_2$ is

   A $\arccos\dfrac{\mathbf{b}_1.\mathbf{b}_2}{b_1b_2}$
   B $\mathbf{b}_1.\mathbf{b}_2$
   C $\arccos\dfrac{\mathbf{a}_1.\mathbf{a}_2}{a_1a_2}$
   D $\mathbf{r}_1.\mathbf{r}_2$
   E $\arccos\lambda.\mu$

7. The equation of the plane normal to $4\mathbf{i} + 3\mathbf{j}$ and 5 units from 0 is

A $\mathbf{r}.(4\mathbf{i} + 3\mathbf{j}) = 5$ C $\mathbf{r}.(4\mathbf{i} + 3\mathbf{j}) = 1$ E $\mathbf{r}.5\mathbf{k} = 0$
B $\mathbf{r}.5\mathbf{k} = 5$ D $\mathbf{r}.(4\mathbf{i} + 3\mathbf{j}) = 25$

8. The points A, B and C are collinear and $\overrightarrow{OA} = \mathbf{i} + \mathbf{j}$, $\overrightarrow{OB} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\overrightarrow{OC} = 3\mathbf{i} + a\mathbf{j} + b\mathbf{k}$

A $a = -3,\ b = 2$ D $a = -1,\ b = 0$
B $a = 3,\ b = -2$ E $a = 6,\ b = -1$
C $a = 0,\ b = 1$

9. The modulus of $12 - 5\mathrm{i}$ is

A 119 B 7 C 13 D $\sqrt{119}$ E $\sqrt{7}$

10. On an Argand diagram OP represents a complex number z. The conjugate of z is $\bar{z}$
If P and Q are the points $(3, 5)$ and $(5, -3)$ then OQ represents

A $-\bar{z}$ B $\mathrm{i}\bar{z}$ C $-z$ D $\mathrm{i}z$ E $-\mathrm{i}z$

11. $\dfrac{3 + 2\mathrm{i}}{3 - 2\mathrm{i}}$ is equal to

A $\dfrac{5 + 12\mathrm{i}}{13}$ C $\dfrac{5 + 6\mathrm{i}}{13}$ E $\dfrac{13 + 12\mathrm{i}}{5}$

B $\dfrac{13 + 12\mathrm{i}}{13}$ D $\dfrac{5 + 6\mathrm{i}}{5}$

12. When $\sqrt{3} - \mathrm{i}$ is divided by $-1 - \mathrm{i}$ the modulus and argument of the quotient are respectively

A $2\sqrt{2},\ \frac{7}{12}\pi$ C $\sqrt{2},\ \frac{7}{12}\pi$ E $\sqrt{2},\ \frac{11}{12}\pi$
B $\sqrt{2},\ -\frac{11}{12}\pi$ D $2\sqrt{2},\ -\frac{11}{12}\pi$

13. The equation $x^2 + 3x + 1 = 0$ has

A no roots D two real roots
B one real and one complex root E two complex roots
C two imaginary roots

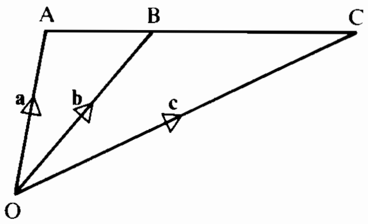
14. $\arg\left(\dfrac{1 - \mathrm{i}}{1 + \mathrm{i}}\right) =$

A $\frac{1}{2}\pi$ B 0 C $-\frac{1}{4}\pi$ D $-\frac{1}{2}\pi$ E π

15. The general solution of the equation $\tan 2x = 1$ is

A $x = n\pi + \frac{1}{4}\pi$ C $x = \frac{1}{2}n\pi + \frac{1}{8}\pi$ E $x = \frac{1}{8}\pi \pm 2n\pi$
B $x = n\pi \pm \frac{1}{4}\pi$ D $x = n\pi + \frac{1}{8}\pi$

16.



Given that $\overrightarrow{BC} = 2\overrightarrow{AB}$ then

A $\mathbf{a} - 3\mathbf{b} + 2\mathbf{c} = 0$ D $2\mathbf{a} + 3\mathbf{b} - \mathbf{c} = 0$
B $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c} = 0$ E $2\mathbf{a} - 3\mathbf{b} + \mathbf{c} = 0$
C $\mathbf{a} + \mathbf{b} - 2\mathbf{c} = 0$

17. Given that $z_1 = 5 - 2\mathrm{i}$ and $z_2 = 2 - \mathrm{i}$, then if $\theta = \arg\left(\dfrac{z_1}{z_2}\right)$

A $\tan\theta = \frac{3}{4}$ C $\tan\theta = -\frac{3}{4}$ E $\tan\theta = -\frac{1}{12}$
B $\tan\theta = \frac{1}{12}$ D $\tan\theta = \frac{1}{8}$

18. A line has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(4\mathbf{i} - \mathbf{j} + 7\mathbf{k})$

A The line has direction ratios $4:-1:7$
B The length of the line is $\sqrt{14}$
C The line is parallel to the line $\mathbf{r} = \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

19. Given that $\mathrm{e}^{2x} + \mathrm{e}^{x} = 2$,

A $2x + x = \ln 2$
B $\mathrm{e}^{3x} = 2$
C $\mathrm{e}^{x} = 1$ satisfies the equation

20. $\ln x + 3\ln y = 4$

A $\ln xy^3 = 4$
B $3xy = \mathrm{e}^4$
C $y = 1$ when $x = 1$

TYPE II

21. The point P represents the complex number z on an Argand diagram and $|z-1| = 4$

A The locus of P is a circle
B The locus of P is a straight line
C $z = 3$

22. $\mathbf{V} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

A $\hat{\mathbf{V}} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
B $\mathbf{V}$ makes equal angles with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$
C $\mathbf{V}$ is perpendiculoar to $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

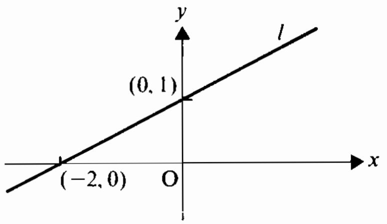
23. $-\frac{3}{4}\pi$ is an argument of

A $1 - \mathrm{i}$ **B** $\cos\frac{3}{4}\pi - \mathrm{i}\sin\frac{3}{4}\pi$ **C** $-1 - \mathrm{i}$

24. The plane $x + 2y + 3z = 4$

A is 4 units from the origin
B has a vector equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4$
C passes through the point $(-1, 1, 1)$

25.



The line l

A is perpendicular to the vector $-\mathbf{i} + 2\mathbf{j}$
B passes through the point with position vector $2\mathbf{i} + \mathbf{j}$
C is parallel to the vector $\mathbf{i} + 2\mathbf{j}$

26. Given that $z_1 = 2 + 3\mathrm{i}$ and $z_2 = 3 - 2\mathrm{i}$,

A $|z_2| = \sqrt{5}$
B $|z_1| = \sqrt{13}$
C z_1 and z_2 are conjugate complex numbers

27. If $f(x) = x^3 + 2x - 4$,

A the roots of $f(x) = 0$ are the values of x where $y = 2x$ cuts $y = x^3 - 4$
B $f(x) = 0$ has a root between 1 and 2
C $f(x)$ has no maximum or minimum values

TYPE III

28. If two lines do not intersect they are parallel.

29. $\mathbf{a}.\mathbf{b} = 0 \quad \Rightarrow \quad \mathbf{a} = 0$ or $\mathbf{b} = 0$

30. Any complex number whose modulus is unity can be expressed as $\cos\theta + \mathrm{i}\sin\theta$

31. A complex number $a + b\mathrm{i}$ is zero if $a = -b$

32. If the equation $f(x) = 0$ has a root near to $x = a$ then a better approximation to that root is $x = a - \dfrac{f'(a)}{f(a)}$

MISCELLANEOUS EXERCISE G

1. Find all values of θ, such that $0° \leqslant \theta \leqslant 180°$, which satisfy the equation $2\sin 2\theta = \tan\theta$ (C 87)

2. (a) Find, in radians, the general solution of the equation $4\sin\theta = \sec\theta$

(b) If $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$, show that θ is either a multiple of $\frac{1}{2}\pi$ or a multiple of $\frac{2}{5}\pi$

3. (a) Find the values of x, for angles between 0° and 360° inclusive, for which $3\sin 2x = 2\tan x$

(b) Solve the equation $\sec\theta\tan\theta = 2$, giving solutions for $0° \leqslant \theta < 360°$

4. (a) Find, in radians, the general solution of the equation $2\sin\theta = \sqrt{3}\tan\theta$

(b) Solve the equation $2\cos\theta\cos 2\theta + \sin 2\theta = 2(3\cos^3\theta - \cos\theta)$ for values of θ within the range $0 < \theta < 2\pi$

5. Solve the equation

$$2\log_3 x = 1 + \log_3(18 - x)$$ (AEB 86)

6. Solve the equation $3^{2x} = 4^{2-x}$, giving your answer to three significant figures. (C 87)

7. Given that $\log_9 xy = 6$, prove that $\log_3 x + \log_3 y = 12$
Hence solve for x and y the simultaneous equations

$$\log_9 xy = 6$$
$$(\log_3 x)(\log_3 y) = 20$$

8. (a) Find the values of x which satisfy the equation

$$\log_4 x = 9 \log_x 4$$

(b) By taking $\log_{10} 5 \approx 0.7$, obtain an estimate of the root of the equation

$$10^{y-5} = 5^{y+2}$$

giving your answer to the nearest integer.

9. Given that $f(x) = x^3 - 12x + 16$ show that $f(2) = f'(2) = 0$
Hence factorise $f(x)$ completely.
Sketch the graph of the curve

$$y = x^3 - 12x + 16$$

showing clearly the coordinates of the points where the curve cuts the coordinate axes.

10. Show that the equation $e^x \cos 2x - 1 = 0$ has a root between 0.4 and 0.45
Taking 0.45 as a first approximation to this root, apply the Newton–Raphson process once to obtain a second approximation, giving your answer to 3 significant figures. (U of L 85)

11. Show that $x^4 + 2x^2 - 100 = 0$ has a root near $x = 3$
Taking $x = 3$ as the first approximation, use Newton's Method twice to obtain a better approximation.

12. By using sketch graphs, or otherwise, show that the equation

$$x = 2 \sin x$$

where the angle is measured in radians, has only one positive root.
Verify, with the use of a calculator that this root lies between 1.8 and 1.9
Determine the root correct to two places of decimals.

13. Show that the equation $\sin x - \ln x = 0$ has a root lying between $x = 2$ and $x = 3$
Given that this root lies between $a/10$ and $(a + 1)/10$, where a is an integer, find the value of a
Estimate the value of the root to 3 significant figures. (U of L 85)

14. Given that $f(x) \equiv 3 + 4x - x^4$, show that the equation $f(x) = 0$ has a root $x = a$, where a is in the interval $1 \leqslant a \leqslant 2$
It may be assumed that if x_n is an approximation to a, then a better approximation is given by x_{n+1}, where

$$x_{n+1} = (3 + 4x_n)^{1/4}$$

Starting with $x_0 = 1.75$, use this result twice to obtain the value of a to 2 decimal places. (U of L 88)

15. By investigating the turning values of

$$f(x) = x^3 + 3x^2 + 6x - 14$$

or otherwise, show that the equation $f(x) = 0$ has only one real root.
If this root lies between n and $n + 1$, find the value of the integer n.
Taking n as a first approximation to the root, calculate two further successive approximations, giving your answers correct to 3 s.f.

16. Given that $z = 2 + 2i$, express z in the form $r(\cos\theta + i\sin\theta)$, where r is a positive real number and $-\pi < \theta \leqslant \pi$
On the same Argand diagram, display and label clearly the numbers

$$z,\ z^2 \text{ and } 4/z$$

Find the values of $|z + z^2|$ and $\arg(z + 4/z)$ (AEB 86)

17. (a) Find all possible values of the real numbers a and b which satisfy

$$2 + ai = \frac{6 - 2i}{b + i}$$

(b) Given that $w = -\frac{1}{2} + \frac{1}{2}i$, find the modulus and the argument of $\dfrac{1}{1 + w}$, giving the argument in radians between $-\pi$ and π

(AEB 88)

18. Given that $z = 1 + i$, show that $z^3 = -2 + 2i$. For this value of z, the real numbers p and q are such that

$$\frac{p}{1 + z} + \frac{q}{1 + z^3} = 2i$$

Find the values of p and q (JMB 84)

19. Given that $z_1 = -1 + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, find $\arg z_1$ and $\arg z_2$

Express z_1/z_2 in the form $a + ib$, where a and b are real, and hence find $\arg(z_1/z_2)$

Verify that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (U of L 88)

20. Find the values of the real numbers a and b so that

$$(a + bi)^2 = 16 - 30i$$

Write down the two square roots of $16 - 30i$ (AEB 87)

21. Find the roots, z_1 and z_2, of the equation

$$z^2 - 5 + 12i = 0$$

in the form $a + bi$, where a and b area real, and give the value of $z_1 z_2$

Draw, on graph paper, an Argand diagram to illustrate the points representing (a) $z_1 z_2$ (b) $\dfrac{1}{z_1 z_2}$ (c) $z_1^* z_2^*$ where z^* denotes the conjugate of z. (U of L 85)

22. Find, in terms of π, the argument of the complex number

(a) $(1 + i)^2$

(b) $\dfrac{3 + i}{1 + 2i}$

(c) $(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)(\cos \frac{1}{4}\pi - i \sin \frac{1}{4}\pi)$ (AEB 86)

23. Given that the real and imaginary parts of the complex number $z = x + iy$ satisfy the equation

$$(2 - i)x - (1 + 3i)y - 7 = 0$$

find x and y

State the values of

(a) $|z|$ (b) $\arg z$ (U of L 88)

24. Given that p and q are real and that $1 + 2i$ is a root of the equation

$$z^2 + (p + 5i)z + q(2 - i) = 0$$

determine (a) the values of p and q

(b) the other root of the equation. (AEB 87)

25. The complex number z satisfies the equation

$$2zz^* - 4z = 3 - 6i$$

where z^* is the complex conjugate of z. Find, in the form $x + iy$, the two possible values of z (JMB 86)

26. Given that $z_1 = 1 + i\sqrt{3}$ and $z_2 = 4 + 3i$, calculate the moduli and arguments of $z_1 z_2$ and z_2/z_1, giving the arguments in degrees to one decimal place.

Illustrate in one Argand diagram

(a) $z_1 - z_2$ (b) $z_1 z_2$ (c) z_2/z_1 (U of L 85)

27. The points A and B have coordinates $(2, 3, -1)$ and $(5, -2, 2)$ respectively. Calculate the acute angle between AB and the line with equation

$$r = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

giving your answer correct to the nearest degree.

[$r = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$] (C 87)

28. The magnitudes of the vectors **a** and **b** are 8 and 3, respectively, and the angle between the vectors is 60°. Sketch a diagram showing these vectors and the vector $\mathbf{a} - \mathbf{b}$. Calculate

(a) the magnitude $\mathbf{a} - \mathbf{b}$

(b) the resolved part of **a** in the direction of **b** (JMB 84)

29. With respect to a fixed origin O, the straight lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$
$$l_2 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(-3\mathbf{i} + 4\mathbf{k})$$

where λ and μ are scalar parameters.

(a) Show that the lines intersect.

(b) Find the position vector of their point of intersection.

(c) Find the cosine of the acute angle contained between the lines.

(d) Find a vector equation of the plane containing the lines. (U of L 88)

30. Relative to an origin O, the points A and B have position vectors $3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ respectively. Find the position vector of the point C in AB such that OC is perpendicular to AB.

Given that $\overrightarrow{OQ} = \frac{3}{4}\overrightarrow{OC}$, find

(a) the position vector of the point P where AQ produced intersects OB

(b) the angle OPA

(c) a unit vector perpendicular to the plane OAB. (AEB 87)

31. Referred to a fixed origin O, the lines l_1 and l_2 have equations

$$l_1: \mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k}) + \lambda(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$$
$$l_2: \mathbf{r} = (4\mathbf{i}-4\mathbf{j}+9\mathbf{k}) + \mu(\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

where λ and μ are scalar parameters.

Show that the lines l_1 and l_2 are perpendicular and that they intersect at the point P whose position vector is $2\mathbf{i}-6\mathbf{j}+3\mathbf{k}$

Determine, to the nearest half-degree, the acute angle between $\overrightarrow{OP}$ and l_2 (AEB 87)

32. The position vectors of the points A and B are given by

$$\overrightarrow{OA} = 3\mathbf{i}-2\mathbf{j}+2\mathbf{k} \quad \overrightarrow{OB} = -\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

where O is the origin. Find a vector equation of the straight line passing through A and B. Given that this line is perpendicular to the vector $\mathbf{i}+2\mathbf{j}+p\mathbf{k}$, find the value of p (C 86)

33. With respect to a fixed origin O, the points L and M have position vectors $6\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ respectively.

(a) Form the scalar product $\overrightarrow{OL}.\overrightarrow{OM}$ and hence find the cosine of angle LOM.

(b) The point N is on the line LM produced such that angle MON is 90°. Find an equation for the line LM in the form $\mathbf{r} = \mathbf{a}+\mathbf{b}t$ and hence calculate the position vector of N. (AEB 88)

34. The points A and B have position vectors $4\mathbf{i}+\mathbf{j}-7\mathbf{k}$ and $2\mathbf{i}+6\mathbf{j}+2\mathbf{k}$ respectively relative to the origin O. Show that the angle AOB is a right angle.

Find a vector equation for the median AM of the triangle OAB.

Find also, in the form $\mathbf{r}.\mathbf{n} = p$, a vector equation of the plane OAB. (U of L 85)

35. Relative to an origin O, the points P and Q have position vectors

$$\mathbf{p} = 4\mathbf{i}-3\mathbf{j}+4\mathbf{k}, \quad \mathbf{q} = 6\mathbf{i}+\mathbf{j}-2\mathbf{k} \quad \text{respectively.}$$

(a) Prove that triangle OPQ is isosceles.

(b) Hence, or otherwise, find a unit vector in the plane OPQ which is perpendicular to the line PQ.

(c) Evaluate $(\mathbf{q}-\mathbf{p}).\mathbf{q}$ and deduce the value of angle PQO, to the nearest 0.1°. (AEB 86)

36. Shade in an Argand diagram the region in which *both* of the following inequalities are satisfied

$$|z-2\mathrm{i}| \leqslant 2$$
$$\tfrac{1}{6}\pi \leqslant \arg z \leqslant \tfrac{1}{2}\pi$$

(JMB 84)

37. With respect to the origin O the points A, B, C have position vectors

$$a(5\mathbf{i}-\mathbf{j}-3\mathbf{k}) \quad a(-4\mathbf{i}+4\mathbf{j}-\mathbf{k}) \quad a(5\mathbf{i}-2\mathbf{j}+11\mathbf{k})$$

respectively, where a is a non-zero constant.

Find (a) a vector equation for the line BC

(b) a vector equation for the plane OAB

(c) the cosine of the acute angle between the lines OA and OB.

Obtain, in the form $\mathbf{r}.\mathbf{n} = p$, a vector equation for Π, the plane which passes through A and is perpendicular to BC.

Find cartesian equations for

(d) the plane Π,

(e) the line BC. (U of L 85)

38. Relative to the origin O, the position vectors of the points A, B and C are $\mathbf{j}-4\mathbf{k}$, $6\mathbf{i}-5\mathbf{j}-\mathbf{k}$ and $4\mathbf{i}+7\mathbf{j}-9\mathbf{k}$ respectively, the unit of length being the metre.

(a) Show that, for all values of the scalar parameter t, the point P with position vector $2t\mathbf{i}+(1-2t)\mathbf{j}+(t-4)\mathbf{k}$ lies on the straight line passing through A and B.

(b) Use the scalar product $\overrightarrow{AB}.\overrightarrow{CP}$ to determine the value of t for which CP is perpendicular to AB.

(c) Hence find the shortest distance from C to AB. (AEB 86)

ANSWERS

Answers to questions taken from past examination papers are the sole responsibility of the authors and have not been approved by the Examining Boards.

CHAPTER 1

Exercise 1a – p. 2

1. $15x$
2. $2x^2$
3. $4x^2$
4. $10pq$
5. $8x^2$
6. $10p^2qr$
7. $9a^2$
8. $63ab$
9. $24st^2$
10. $8a^3$
11. $\frac{5}{3}x$
12. $2m$
13. $4ab^3$
14. $5xy$
15. $196p^4q^2$
16. $2a$
17. $6ax$
18. $2x$
19. $9b/5a$
20. $\frac{2}{5}x$
21. x^3/y^2

Exercise 1b – p. 3

1. $3x^2 - 4x$
2. $a - 12$
3. $2y - xy + y^2$
4. $5pq - 9p^2$
5. $3xy + y^2$
6. $x^3 - x^2 + x + 7$
7. $5 + t - t^2$
8. $a^2 - ab - 2b$
9. $7 - x$
10. $4x - 9$
11. $3x^2 + 18x - 20$
12. $ab - 2ac + cb$
13. $11cT - 2cT^2 - 55T^2$
14. $-x^3 + 7x^2 - 7x$
15. $-4y^2 + 24y - 10$
16. $5RS + 5RF - R^2$

Exercise 1c – p. 4

1. -7
2. 2
3. (a) 1 (b) -5 (c) -1
4. (a) 1 (b) 0 (c) -3

Exercise 1d – p. 4

1. $x^2 + 6x + 8$
2. $x^2 + 8x + 15$
3. $a^2 + 13a + 42$
4. $t^2 + 15t + 56$
5. $s^2 + 17s + 66$
6. $2x^2 + 11x + 5$
7. $5y^2 + 28y + 15$
8. $6a^2 + 17a + 12$
9. $35t^2 + 86t + 48$
10. $99s^2 + 49s + 6$
11. $x^2 - 5x + 6$
12. $y^2 - 5y + 4$
13. $a^2 - 11a + 24$
14. $b^2 - 17b + 72$
15. $p^2 - 15p + 36$
16. $2y^2 - 13y + 15$
17. $3x^2 - 13x + 4$
18. $6r^2 - 25r + 14$
19. $20x^2 - 19x + 3$
20. $6a^2 - 7ab + 2b^2$
21. $x^2 - x - 6$
22. $a^2 + a - 56$
23. $y^2 + 2y - 63$
24. $s^2 + s - 30$
25. $q^2 + 8q - 65$
26. $2t^2 + 3t - 20$
27. $4x^2 + 11x - 3$

28. $6q^2 - q - 15$
29. $x^2 - xy - 2y^2$
30. $2s^2 + st - 6t^2$

Exercise 1e – p. 5

1. $x^2 - 4$
2. $25 - x^2$
3. $x^2 - 9$
4. $4x^2 - 1$
5. $x^2 - 64$
6. $x^2 - a^2$
7. $x^2 - 1$
8. $9b^2 - 16$
9. $4y^2 - 9$
10. $a^2b^2 - 36$
11. $25x^2 - 1$
12. $x^2y^2 - 16$

Exercise 1f – p. 6

1. $x^2 + 8x + 16$
2. $x^2 + 4x + 4$
3. $4x^2 + 4x + 1$
4. $9x^2 + 30x + 25$
5. $4x^2 + 28x + 49$
6. $x^2 - 2x + 1$
7. $x^2 - 6x + 9$
8. $4x^2 - 4x + 1$
9. $16x^2 - 24x + 9$
10. $25x^2 - 20x + 4$
11. $9t^2 - 42t + 49$
12. $x^2 + 2xy + y^2$
13. $4p^2 + 36p + 81$
14. $9q^2 - 66q + 121$
15. $4x^2 - 20xy + 25y^2$

Exercise 1g – p. 7

1. $11x - 2x^2 - 12$
2. $x^2 - 49$
3. $6 - 25x + 4x^2$
4. $14p^2 - 3p - 2$
5. $9p^2 - 6p + 1$
6. $15t^2 + t - 2$
7. $16 - 8p + p^2$
8. $14t - 3 - 8t^2$
9. $x^2 + 4xy + 4y^2$
10. $16x^2 - 9$
11. $9x^2 + 42x + 49$
12. $15 - R - 2R^2$
13. $a^2 - 6ab + 9b^2$
14. $4x^2 - 20x + 25$
15. $49a^2 - 4b^2$
16. $9a^2 + 30ab + 25b^2$
17. (a) $6, -22$ (b) $15, 31$ (c) $14, -31$ (d) $81, 18$

Exercise 1h – p. 9

1. $(x + 5)(x + 3)$
2. $(x + 7)(x + 4)$
3. $(x + 6)(x + 1)$
4. $(x + 4)(x + 3)$
1. $(x - 1)(x - 9)$
6. $(x - 3)^2$
7. $(x + 6)(x + 2)$
8. $(x - 8)(x - 1)$
9. $(x + 7)(x - 2)$
10. $(x + 4)(x - 3)$
11. $(x - 5)(x + 1)$
12. $(x - 12)(x + 2)$
13. $(x + 7)(x + 2)$
14. $(x - 1)^2$
15. $(x - 3)(x + 3)$
16. $(x + 8)(x - 3)$
17. $(x + 2)^2$
18. $(x - 1)(x + 1)$
19. $(x - 6)(x + 3)$
20. $(x + 5)^2$
21. $(x - 4)(x + 4)$
22. $(4 + x)(1 + x)$
23. $(2x - 1)(x - 1)$
24. $(3x + 1)(x + 1)$
25. $(3x - 1)^2$
26. $(3x + 1)(2x - 1)$
27. $(3 + x)^2$
28. $(2x - 3)(2x + 3)$
29. $(x + a)^2$
30. $(xy - 1)^2$

Exercise 1i

1. $(3x - 4)(2x + 3)$
2. $(4x - 3)(x - 2)$
3. $(4x - 1)(x + 1)$
4. $(3x - 2)(x - 5)$
5. $(2x - 3)^2$
6. $(1 - 2x)(3 + x)$
7. $(5x - 4)(5x + 4)$
8. $(3 + x)(1 - x)$
9. $(5x - 1)(x - 12)$
10. $(3x + 5)^2$
11. $(3 - x)(1 + x)$
12. $(3 + 4x)(4 - 3x)$
13. $(1 + x)(1 - x)$
14. $(3x + 2)^2$
15. $(x + y)^2$
16. $(1 - 2x)(1 + 2x)$
17. $(2x - y)^2$
18. $(3 - 2x)(3 + 2x)$
19. $(6 + x)^2$
20. $(5x - 4)(8x + 3)$

21. $(7x + 30)(x - 5)$
22. $(6 - 5x)(6 + 5x)$
23. $(x - y)(x + y)$
24. $(9x - 2y)^2$
25. $(7 - 6x)^2$
26. $(5x - 2y)(5x + 2y)$
27. $(6x + 5y)^2$
28. $(2x - 3y)(2x + y)$
29. $(3x + 4y)(2x + y)$
30. $(7pq - 2)^2$

Exercise 1j – p. 11

1. not possible
2. $2(x + 1)^2$
3. $(x + 2)(x + 1)$
4. $3(x + 5)(x - 1)$
5. not possible
6. not possible
7. not possible
8. $2(x - 2)^2$
9. $3(x - 2)(x + 1)$
10. $2(x^2 - 3x + 4)$
11. $3(x - 4)(x + 2)$
12. $(x - 6)(x + 2)$
13. not possible
14. $4(x - 5)(x + 5)$
15. $5(x^2 - 5)$
16. not possible
17. not possible
18. not possible

Exercise 1k – p. 12

1. $x^3 - x^2 - x - 2$
2. $3x^3 - 5x^2 - x + 2$
3. $4x^3 - 8x^2 + 13x - 5$
4. $x^3 - 2x^2 + 1$
5. $2x^3 - 9x^2 - 24x - 9$
6. $x^3 + 6x^2 + 11x + 6$
7. $x^3 + 4x^2 - x - 4$
8. $x^3 - 4x^2 + x + 6$
9. $2x^3 + 7x^2 + 7x + 2$
10. $x^3 + 4x^2 + 5x + 2$
11. $4x^3 + 4x^2 - 7x + 2$
12. $27x^3 - 27x^2 + 9x - 1$
13. $4x^3 - 9x^2 - 25x - 12$
14. $4x^3 - 4x^2 - x + 1$
15. $6x^3 + 13x^2 + x - 2$
16. $x^3 + 3x^2 + 3x + 1$
17. $x^3 + x^2 - 4x - 4$
18. $2x^3 + 7x^2 - 9$
19. $24x^3 + 38x^2 - 51x + 10$
20. $4x^3 - 42x^2 + 68x + 210$
21. $3x^3 - 16x^2 + 28x - 16$; $-16, 28$
22. $6, -17$
23. $x^3 + 3x^2y + 3xy^2 + y^3$
24. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Exercise 1l – p. 14

1. $x^3 + 9x^2 + 27x + 27$
2. $x^4 - 8x^3 + 24x^2 - 32x + 16$
3. $x^4 + 4x^3 + 6x^2 + 4x + 1$
4. $8x^3 + 12x^2 + 6x + 1$
5. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$
6. $p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$
7. $8x^3 + 36x^2 + 54x + 27$
8. $x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1024$
9. $81x^4 - 108x^3 + 54x^2 - 12x + 1$
10. $1 + 20a + 150a^2 + 500a^3 + 625a^4$
11. $64a^6 - 192a^5b + 240a^4b^2 - 160a^3b^3 + 60a^2b^4 - 12ab^5 + b^6$
12. $8x^3 - 60x^2 + 150x - 125$

Mixed Exercise 1 – p. 15

1. -23
2. $115x - 105x^2 - 30$
3. 108
4. $3(x - 2)(x - 1)$
5. 250
6. $4(x - 3)(x + 3)$
7. $4x^3 - 4x^2 + x$
8. $(x - 5)^2$
9. 7

CHAPTER 2

Exercise 2a – p. 17

1. $\frac{1}{4}$
2. $\dfrac{2(x + 2)}{3(x - 2)}$
3. $\frac{2}{3}$
4. $\frac{3}{5}$
5. $\dfrac{x}{y}$
6. not possible
7. not possible
8. $\dfrac{5x(x + y)}{5y + 2x}$
9. $\dfrac{2a - 6b}{6a + b}$
10. $\dfrac{b - 4}{3x(b + 4)}$
11. $\dfrac{x - 3}{x + 4}$
12. $\dfrac{4y^2 + 3}{(y + 3)(y - 3)}$
13. $\dfrac{1}{3(x + 3)}$
14. $\dfrac{x + 2}{2x + 1}$
15. $\dfrac{x - 2}{x - 1}$
16. $\dfrac{1}{2(a - 5)}$
17. $\dfrac{3}{p + 3q}$
18. $\dfrac{a^2 + 2a + 4}{(a + 5)(a + 2)}$; not possible
19. $\dfrac{x + 1}{3(x + 3)}$
20. $\dfrac{4(x - 3)}{(x + 1)^2}$

Exercise 2b – p. 18

1. $\dfrac{2x^2}{3y^2}$
2. $\dfrac{6t^2}{s}$
3. $\dfrac{8v^2}{3}$
4. $\dfrac{2r}{3}$
5. $3x$
6. $\dfrac{9x}{4y^2}$
7. $\dfrac{x^2}{24}$
8. $\dfrac{2}{a(a + b)}$
9. $\dfrac{1}{x + 1}$
10. $\dfrac{1}{2a}$
11. $\dfrac{1}{x - 1}$
12. $\dfrac{3}{2(x + 3)}$
13. $\dfrac{a^4}{27}$
14. $\frac{2}{3}$
15. $\dfrac{2r}{3s^2}$
16. $\dfrac{3x^2}{2(y - 2)}$
17. $\dfrac{b^2}{c^2}$
18. $2(x + 3)$
19. $\dfrac{x - 3}{3}$
20. $\dfrac{x(2x - 3)}{x - 1}$

Exercise 2c – p. 19

1. $\dfrac{b - a}{ab}$
2. $\dfrac{8}{15x}$
3. $\dfrac{q - p}{pq}$
4. $\dfrac{11}{10x}$
5. $\dfrac{x^2 + 1}{x}$
6. $\dfrac{x^2 - y^2}{xy}$
7. $\dfrac{2p^2 - 1}{p}$
8. $\dfrac{7x + 3}{12}$
9. $\dfrac{5x - 1}{6}$
10. $\dfrac{11 - 7x}{15}$
11. $\dfrac{\sin B + \sin A}{\sin A \sin B}$
12. $\dfrac{\sin A + \cos A}{\cos A \sin A}$
13. $\dfrac{12x^2 + 1}{4x}$
14. $\dfrac{2x^2 + x - 2}{2x + 1}$
15. $\dfrac{x^2 + 2x + 2}{x + 1}$
16. $\dfrac{2x + 3}{2x}$
17. $\dfrac{1 + x - x^2}{x}$
18. $\dfrac{n + 1}{n^2}$

19. $\dfrac{x(b^2 + a^2)}{a^2b^2}$

20. $\dfrac{a^2 + 3a + 1}{a + 1}$

Exercise 2d – p. 21

1. $\dfrac{2x}{(x + 1)(x - 1)}$
2. $\dfrac{2x - 1}{(x + 1)(x - 2)}$
3. $\dfrac{7x + 18}{(x + 2)(x + 3)}$
4. $\dfrac{x}{(x - 1)(x + 1)}$
5. $\dfrac{-1 - 3a}{(a - 1)(a + 1)} = \dfrac{1 + 3a}{(1 - a)(1 + a)}$
6. $\dfrac{x + 2}{(x + 1)^2}$
7. $\dfrac{1 - 4x}{(2x + 1)^2}$
8. $\dfrac{-3x - 10}{(x + 1)(x + 4)} = -\dfrac{3x + 10}{(x + 1)(x + 4)}$
9. $\dfrac{2x + 6}{(x + 1)^2}$
10. $\dfrac{8 - x - x^2}{(x + 2)^2(x + 4)}$
11. $\dfrac{7x + 8}{6(x - 1)(x + 4)}$
12. $\dfrac{8 - 3x}{5(x + 2)(x + 4)}$
13. $\dfrac{15x - 58}{6(x + 2)(3x - 5)}$
14. $\dfrac{5x^2 - 9x - 32}{(x + 1)(x - 2)(x + 3)}$
15. $\dfrac{2x^2 + 6x + 6}{(x + 1)(x + 2)(x + 3)} = \dfrac{2(x^2 + 3x + 3)}{(x + 1)(x + 2)(x + 3)}$
16. $-\dfrac{1}{x(x + 1)^2}$
17. $\dfrac{7t + 3}{(t + 1)^2}$
18. $\dfrac{-t^4 + 2t^3 - 2t^2 - 2t - 1}{(t^2 + 1)(t^2 - 1)}$
19. $\dfrac{1 + 3y - 3x}{(y - x)(y + x)}$
20. $\dfrac{n^3 + 6n^2 + 8n + 2}{n(n + 1)(n + 2)}$

Exercise 2e – p. 23

1. $\dfrac{3}{2(x + 1)} - \dfrac{1}{2(x - 1)}$
2. $\dfrac{9}{8(x - 5)} - \dfrac{1}{8(x + 3)}$
3. $\dfrac{13}{6(x - 7)} - \dfrac{1}{6(x - 1)}$
4. $\dfrac{4}{(x - 1)} - \dfrac{5}{(2x - 1)}$
5. $\dfrac{4}{5(x - 2)} - \dfrac{4}{5(x + 3)}$
6. $\dfrac{1}{5(x - 2)} + \dfrac{6}{5(4x - 3)}$
7. $\dfrac{7}{9(2x - 1)} + \dfrac{28}{9(x + 4)}$
8. $\dfrac{5}{3(2x - 1)} - \dfrac{4}{3(x + 1)}$
9. $\dfrac{1}{x - 2} - \dfrac{1}{x}$
10. $\dfrac{3}{x} - \dfrac{6}{2x + 1}$
11. $\dfrac{3}{x - 2} - \dfrac{1}{x - 1}$
12. $\dfrac{4}{9(x - 8)} - \dfrac{4}{9(x + 1)}$
13. $\dfrac{1}{2(x - 3)} - \dfrac{1}{2(x + 3)}$
14. $\dfrac{1}{2x - 3} + \dfrac{1}{2x + 3}$
15. $\dfrac{7}{3x} - \dfrac{1}{3(x + 1)}$
16. $\dfrac{5}{x + 2} - \dfrac{1}{x}$
17. $\dfrac{9}{x} - \dfrac{18}{2x + 1}$
18. $\dfrac{1}{(x - 2)} + \dfrac{1}{2(x + 1)}$
19. $\dfrac{2}{5(x - 1)} - \dfrac{1}{5(3x + 2)}$
20. $\dfrac{2}{(x - 2)} - \dfrac{1}{2x}$

Mixed Exercise 2 – p. 23

1. (a) $\dfrac{x^2 - 9}{2(x - 3)} = \dfrac{x + 3}{2}$ (b) $\dfrac{1}{x - 3}$
2. (a) $\dfrac{4a}{rp}$ (b) $\dfrac{2p^2 - 3r}{pr}$
3. (a) $\dfrac{2}{3(n + 2)}$ (b) $\dfrac{5x^2 + x - 1}{x(x + 1)(2x - 1)}$

4. $\dfrac{3}{2(x - 1)} - \dfrac{3}{2(x + 1)}$
5. $\dfrac{4}{7(x - 3)} - \dfrac{8}{7(2x + 1)}$
6. $\dfrac{5}{x - 1} - \dfrac{5}{x}$
7. (a) $\dfrac{2x - 5}{2x + 5}$ (b) $\dfrac{2t}{t^2 - 1}$
8. (a) $\dfrac{(x - 1)^3}{x + 1}$ (b) $\dfrac{ab + bc + ac}{abc}$
9. $\dfrac{5}{7(x + 1)} + \dfrac{1}{7(4x - 3)}$
10. $\dfrac{1}{t - 1} + \dfrac{1}{t + 1}$

CHAPTER 3

Exercise 3a – p. 26

1. $2\sqrt{3}$
2. $4\sqrt{2}$
3. $3\sqrt{3}$
4. $5\sqrt{2}$
5. $10\sqrt{2}$
6. $6\sqrt{2}$
7. $9\sqrt{2}$
8. $12\sqrt{2}$
9. $5\sqrt{3}$
10. $4\sqrt{3}$
11. $10\sqrt{5}$
12. $2\sqrt{5}$

Exercise 3b – p. 28

1. $2\sqrt{3} - 3$
2. $5\sqrt{2} + 8$
3. $2\sqrt{5} + 5\sqrt{15}$
4. 4
5. $\sqrt{6} + \sqrt{2} - \sqrt{3} - 1$
6. $13 + 7\sqrt{3}$
7. 4
8. $5 - 3\sqrt{2}$
9. $22 - 10\sqrt{5}$
10. 9
11. $10 - 4\sqrt{6}$
12. $31 + 12\sqrt{3}$
13. $(4 + \sqrt{5})$
14. $(\sqrt{11} - 3)$
15. $(2\sqrt{3} + 4)$
16. $(\sqrt{6} + \sqrt{5})$
17. $(3 + 2\sqrt{3})$
18. $(2\sqrt{5} + \sqrt{2})$

Exercise 3c – p. 29

1. $\frac{3}{2}\sqrt{2}$
2. $\frac{1}{7}\sqrt{7}$
3. $\frac{2}{11}\sqrt{11}$
4. $\frac{3}{5}\sqrt{10}$
5. $\frac{1}{9}\sqrt{3}$
6. $\frac{1}{2}\sqrt{2}$
7. $\sqrt{2} + 1$
8. $\frac{1}{23}(15\sqrt{2} - 6)$
9. $\frac{1}{3}(4\sqrt{3} + 6)$
10. $-5(2 + \sqrt{5})$
11. $\frac{1}{4}(\sqrt{7} + \sqrt{3})$
12. $4(2 + \sqrt{3})$
13. $\sqrt{5} - 2$
14. $\frac{1}{13}(7\sqrt{3} + 2)$
15. $3 + \sqrt{5}$
16. $3(\sqrt{3} + \sqrt{2})$
17. $\frac{3}{19}(10 - \sqrt{5})$
18. $3 + 2\sqrt{2}$
19. $\frac{2}{3}(7 - 2\sqrt{7})$
20. $\frac{1}{2}(1 + \sqrt{5})$
21. $\frac{1}{4}(\sqrt{11} + \sqrt{7})$
22. $\frac{1}{6}(9 + \sqrt{3})$
23. $\frac{1}{14}(9\sqrt{2} - 20)$
24. $\frac{1}{6}(3\sqrt{2} + 2\sqrt{3})$
25. $\frac{1}{2}(2 + \sqrt{2})$
26. $\frac{1}{42}(3\sqrt{7} - \sqrt{21})$
27. $\frac{1}{9}(\sqrt{30} + 2\sqrt{3})$

Exercise 3d – p. 33

1. $\dfrac{1}{2^4}$
2. $\dfrac{1}{2^2}$
3. 3^2
4. x^2
5. 1
6. t^4
7. 1
8. 2
9. $y^{3/2}$
10. x^5
11. $\dfrac{1}{y^{3/4}}$
12. p
13. 3
14. $\frac{1}{32}$
15. $\frac{1}{2}$

16. 2
17. 27
18. $\frac{9}{4}$
19. 1
20. 16
21. $\frac{5}{4}$
22. -5
23. 1331
24. $\frac{3}{5}$
25. 6
26. $\frac{16}{27}$
27. 8
28. 5
29. 1
30. 1

Exercise 3e – p. 35

1. $\log_{10} 1000 = 3$
2. $\log_2 16 = 4$
3. $\log_{10} 10\,000 = 4$
4. $\log_3 9 = 2$
5. $\log_4 16 = 2$
6. $\log_5 25 = 2$
7. $\log_{10} 0.01 = -2$
8. $\log_9 3 = \frac{1}{2}$
9. $\log_5 1 = 0$
10. $\log_4 2 = \frac{1}{2}$
11. $\log_{12} 1 = 0$
12. $\log_8 2 = \frac{1}{3}$
13. $\log_q p = 2$
14. $\log_x 2 = y$
15. $\log_p r = q$
16. $10^5 = 100\,000$
17. $4^3 = 64$
18. $10^1 = 10$
19. $2^2 = 4$
20. $2^5 = 32$
21. $10^3 = 1000$
22. $5^0 = 1$
23. $3^2 = 9$
24. $4^2 = 16$
25. $3^3 = 27$
26. $36^{1/2} = 6$
27. $a^0 = 1$
28. $x^z = y$
29. $a^b = 5$
30. $p^r = q$

Exercise 3f – p. 36

1. 2
2. 6
3. 6
4. 4
5. 2
6. 3
7. $\frac{1}{2}$
8. -2
9. -1
10. $\frac{1}{2}$
11. 0
12. 1
13. $\frac{1}{3}$
14. 0
15. $\frac{1}{3}$
16. 3

Exercise 3g – p. 38

1. $\log p + \log q$
2. $\log p + \log q + \log r$
3. $\log p - \log q$
4. $\log p + \log q - \log r$
5. $\log p - \log q - \log r$
6. $2 \log p + \log q$
7. $\log q - 2 \log r$
8. $\log p + \frac{1}{2} \log q$
9. $2 \log p + 3 \log q - \log r$
10. $\frac{1}{2} \log q - \frac{1}{2} \log r$
11. $n \log q$
12. $n \log p + m \log q$
13. $\log pq$
14. $\log p^2q$
15. $\log q/r$
16. $\log q^3p^4$
17. $\log p^n/q$
18. $\log pq^2/r^3$

Mixed Exercise 3 – p. 38

1. (a) $2\sqrt{21}$ (b) $10\sqrt{3}$ (c) $3\sqrt{5}$
2. (a) $8 - 2\sqrt{2}$ (b) $7 - 2\sqrt{10}$
3. (a) $(7 + \sqrt{3})(7 - \sqrt{3}) = 46$
   (b) $(2\sqrt{2} - 1)(2\sqrt{2} + 1) = 7$
   (c) $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5}) = 2$
4. (a) $\frac{5}{7}\sqrt{7}$ (b) $\frac{1}{3}(\sqrt{13} + 2)$
   (c) $4(\sqrt{3} + \sqrt{2})$ (d) $2 - \sqrt{3}$
5. (a) 1 (b) 1
6. (a) $\frac{1}{4}$ (b) $\frac{4}{7}$ (c) $\frac{16}{243}\sqrt{6}$
7. (a) 1 (b) $\frac{1}{5}\sqrt{15}$
8. (a) 7 (b) $\frac{1}{2}$ (c) 0
9. (a) $3 \log a - \log b - 2 \log c$
   (b) $n \log a - \log b$
   (c) $\log a + \log b - \log c$
10. (a) $\log \frac{a^3}{b}$ (b) $-\log a$

CHAPTER 4

Exercise 4a – p. 40

1. $x = -2$ or $x = -3$
2. $x = 2$ or $x = -3$
3. $x = 3$ or $x = -2$
4. $x = -2$ or $x = -4$
5. $x = 1$ or $x = 3$
6. $x = 1$ or $x = -3$
7. $x = -1$ or $x = -\frac{1}{2}$
8. $x = 2$ or $x = \frac{1}{4}$
9. $x = 1$ or $x = -5$
10. $x = 8$ or $x = -9$
11. $-1, 3$
12. $-1, -4$
13. $1, 5$
14. $2, -5$
15. $-2, 7$
16. $2, 7$

Exercise 4b – p. 42

1. $x = 2$ or $x = 5$
2. $x = 3$ or $x = -5$
3. $x = 4$ or $x = -1$
4. $x = 3$ or $x = 4$
5. $x = \frac{1}{3}$ or $x = -1$
6. $x = -1$ or $x = -6$
7. $x = 0$ or $x = 2$
8. $x = -1$ or $x = -\frac{1}{4}$
9. $x = \frac{2}{3}$ or $x = -1$
10. $x = 0$ or $x = -\frac{1}{2}$
11. $x = 0$ or $x = -6$
12. $x = 0$ or $x = 10$
13. $x = 0$ or $x = \frac{1}{2}$
14. $x = 5$ or $x = -4$
15. $x = 2$ or $x = -\frac{4}{3}$
16. $x = 2$ or $x = -1$
17. $x = 0$ or $x = 1$
18. $x = 0$ or $x = 2$
19. $x = 3$ or $x = -1$
20. $x = -1$ or $x = \frac{1}{2}$

Exercise 4c – p. 44

1. 4
2. 1
3. 9
4. 25
5. 2
6. $\frac{25}{4}$
7. 192
8. 81
9. 200
10. $\frac{1}{4}$
11. $\frac{1}{3}$
12. $\frac{9}{8}$
13. $x = -4 \pm \sqrt{17}$
14. $x = 1 \pm \sqrt{3}$
15. $x = -\frac{1}{2}(1 \pm \sqrt{5})$
16. $x = -\frac{1}{2}(1 \pm \sqrt{3})$
17. $x = -\frac{1}{2}(3 \pm \sqrt{5})$
18. $x = \frac{1}{4}(1 \pm \sqrt{17})$
19. $x = -2 \pm \sqrt{6}$
20. $x = -\frac{1}{6}(1 \pm \sqrt{13})$
21. $x = \frac{1}{2}(-2 \pm 3\sqrt{2})$
22. $x = \frac{1}{2}(1 \pm \sqrt{13})$
23. $x = -\frac{1}{8}(1 \pm \sqrt{17})$
24. $x = \frac{1}{4}(3 \pm \sqrt{41})$

Exercise 4d – p. 46

1. $x = -2 \pm \sqrt{2}$
2. $x = \frac{1}{4}(-1 \pm \sqrt{17})$
3. $x = \frac{1}{2}(-5 \pm \sqrt{21})$
4. $x = \frac{1}{4}(1 \pm \sqrt{33})$
5. $x = 2 \pm \sqrt{3}$
6. $x = \frac{1}{4}(1 \pm \sqrt{41})$
7. $x = \frac{1}{6}(1 \pm \sqrt{13})$
8. $x = -\frac{1}{6}(1 \pm \sqrt{13})$
9. $x = -0.260$ or -1.540
10. $x = 2.781$ or 0.719
11. $x = 1.883$ or -0.133
12. $x = 0.804$ or -1.554
13. $x = 0.804$ or -1.554
14. $x = 0.724$ or 0.276
15. $x = 3.303$ or -0.303
16. $x = 7.873$ or 0.127

Exercise 4e – p. 48

1. $x = 2, y = 1, z = 1$
2. $x = 3, y = 4, z = 1$
3. $x = 1, y = -1, z = 2$
4. $x = 3, y = 2, z = -1$
5. $x = 1, y = 2, z = 3$
6. $x = 4, y = -1, z = 2$

Exercise 4f – p. 50

1.

x	-2	1
y	-1	2

2. $x = -1, y = 3$
3. $x = 2, y = 3$

4.

x	-1	$\frac{1}{2}$
y	4	1

5.

x	2	$-\frac{1}{2}$
y	-3	2

6.

x	$\frac{7}{2}$	-2
y	$-\frac{1}{2}$	5

7.

x	1	2
y	2	1

8.

x	-1	3
y	-4	4

9. $x = 1, y = 5$

10.

x	6	-6
y	2	-4

11. $x = \frac{1}{2}, y = -1$

12.

x	1	0
y	$\frac{1}{3}$	$\frac{2}{3}$

13. $x = -1, y = -\frac{1}{2}$

14. $x = 1, y = -\frac{1}{3}$

15.

x	$-\frac{1}{3}$	$\frac{2}{3}$
y	$-\frac{1}{2}$	$\frac{1}{4}$

16. $x = 1, y = \frac{1}{2}$

17.

x	-3	6
y	-3	$\frac{3}{2}$

18.

x	1	$-\frac{1}{4}$
y	-2	3

19.

x	-1	2
y	$\frac{1}{3}$	$-\frac{1}{6}$

20.

x	$\frac{1}{2}$	0
y	$\frac{1}{2}$	1

21.

x	1	$3\frac{1}{2}$
y	1	-4

22.

x	-1	$7\frac{1}{2}$
y	2	$-\frac{7}{5}$

Exercise 4g – p. 54

1. 4
2. $-\frac{5}{3}$
3. 1
4. $\frac{4}{3}$
5. -3
6. $\frac{2}{5}$
7. real and different
8. not real
9. real and different
10. real and equal
11. real and different
12. real and equal
13. real and different
14. not real
15. real and different
16. real and equal
17. $k = \pm 12$
18. $a = 2\frac{1}{4}$
19. $p = 2$
22. $q^2 = 4p$

Mixed Exercise 4 – p. 56

1. (a) 5 (b) $-1, 6$ (c) 5
2. (a) 6 (b) $3 \pm \sqrt{14}$ (c) 6
3. (a) $-\frac{3}{2}$ (b) $\frac{1}{4}(-3 \pm \sqrt{17})$ (c) $-\frac{3}{2}$
4. (a) $-\frac{4}{3}$ (b) $\frac{1}{3}(-2 \pm \sqrt{19})$ (c) $-\frac{4}{3}$
5. (a) 2 (b) 1, 1 (c) 2
6. (a) $\frac{11}{4}$ (b) $-\frac{1}{4}, 3$ (c) $\frac{11}{4}$
7. (a) -1 (b) $\frac{1}{2}(-1 \pm \sqrt{13})$ (c) -1
8. (a) -4 (b) $-6, 2$ (c) -4
9. (a) -2 (b) $-1 \pm \sqrt{3}$ (c) -2
10. (a) -4 (b) $-2, -2$ (c) -4
11. $x = 0$ or 2
12. $x = -4$ or 1
13. $x = 2$ or $x \to \infty$
14. $x = 0$ or -1
15. $x = \frac{1}{2}(7 \pm \sqrt{89})$
16. $x = 1$ or $x \to \infty$
17. (a) not real (b) real and different
    (c) real and equal
    (d) real and different
18. 4, -1
20. 2
21.

x	4	-22
y	5	31

22.

x	2	4
y	5	9

23. $x = 5$ or $\frac{2}{3}$; (a) rational
    (b) it factorises

CONSOLIDATION A

Multiple Choice Exercise A - p. 58

1. A
2. E
3. D
4. E
5. B
6. A
7. E
8. E
9. A
10. E
11. C
12. C
13. B
14. A
15. B
16. B
17. A
18. A
19. B, C
20. A, B
21. T
22. F
23. T
24. T

Miscellaneous Exercise A – p. 61

1. $3[(x+1)^2 - \frac{7}{3}]$
2. -96
3. $A = \frac{5}{7}$ $B = -\frac{3}{7}$
4. (a) $\frac{3}{2}$ (b) $\frac{7}{4}$
5. 1
6. $\frac{81}{4}$
7. $p = 8, q = 2$
9. $x = \frac{1}{6}(y^3 + 12)$
10. $\dfrac{3}{x} - \dfrac{3}{x+1}$
11. $p = -4, q = \frac{1}{2}$
12. $a = 3, b = -36$
13. $x = \frac{1}{2}(\sqrt{21} - 3), y = \frac{1}{2}(5 - \sqrt{21})$
14. -4

CHAPTER 5

Exercise 5a – p. 66

1. 2 cm
2. 90 cm
3. 20 cm
4. (a) 84 cm (b) $7x$ cm
5. 5:7
6. 3:1
7. $y:(x-y)$
8. $\dfrac{ma}{n-m}$
9. $(a+b):b$

Exercise 5b – p. 68

1. 3.75 cm
2. TN = 2.5 cm, LN = 4.5 cm
3. BC = 2.25 cm
4. $\dfrac{yz}{x}$
5. (a) 1.25 cm (b) 2:7 externally
6. (a) Y (b) Y (c) Y
   (d) Y (e) N (f) Y

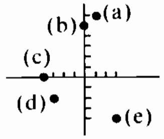
Exercise 5c – p. 73

2. $\frac{5}{26}$ cm
3. XZ = $\frac{7}{4}$ cm, QR = 16 cm
5. RT = 12 cm, AT = 8 cm
6. 9:4
7. 10°
8. BD = $\frac{24}{7}$, DC = $\frac{25}{7}$, AD = $\frac{120}{7}\sqrt{2}$
10. EC = 3 cm, AC = $\frac{25}{3}$ cm

CHAPTER 6

Exercise 6a – p. 77

1.



2. (9, 5) and (9, 1) or (−3, 5) and (−3, −1)
3. (−2, 2), (3, −3)

Exercise 6b – p. 82

1. (a) 5 (b) $\sqrt{2}$ (c) $\sqrt{13}$
2. (a) $(\frac{5}{2}, 4)$ (b) $(\frac{5}{2}, \frac{1}{2})$ (c) $(3, \frac{7}{2})$
3. (a) $\sqrt{109}, (\frac{1}{2}, 1)$ (b) $\sqrt{5}, (-\frac{1}{2}, -1)$
   (c) $2\sqrt{2}, (-2, -3)$
4. $\sqrt{65}$
5. $\sqrt{13}$
6. $(2, -4)$
8. (b) $(-3\frac{1}{2}, -\frac{1}{2})$ (c) $17\frac{1}{2}$ sq units
9. (a) $\sqrt{5}(2 + \sqrt{2})$ (b) $(0, 4\frac{1}{2})$
   (c) $2\frac{1}{2}$
11. $(-5, -3)$

Exercise 6c – p. 87

1. (a) 3 (b) $\frac{3}{2}$ (c) $\frac{1}{3}$ (d) $\frac{3}{4}$
   (e) -4 (f) 6 (g) $-\frac{7}{3}$
   (h) $-\frac{3}{2}$ (i) $\frac{k}{h}$
3. (a) yes (b) no (c) yes
   (d) yes
3. (a) parallel (b) perpendicular
   (c) perpendicular (d) neither
   (e) parallel

Exercise 6d – p. 89

1. $a = 0, b = 4$
2. (b) $22\frac{1}{2}$ square units
5. $\sqrt{(a^2 + 4b^2)}$
8. $\left(\frac{p+q}{2}, \frac{p+q}{2}\right)$
9. $(a-2)^2 + (b-1)^2 = 9$
10. 8
11. $b(d-b) = ac$
12. $b^2 = 8a - 16$

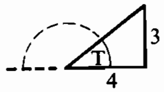
CHAPTER 7

Exercise 7a – p. 91

1. $\sin A = \frac{12}{13}, \cos A = \frac{5}{13}$
2. $\tan X = \frac{3}{4}, \sin X = \frac{3}{5}$
3. $\cos P = \frac{9}{41}, \tan P = \frac{40}{9}$
4. $\sin A = \frac{1}{\sqrt{2}} = \cos A$
5. $\sin Y = \frac{1}{3}\sqrt{5}, \tan Y = \frac{1}{2}\sqrt{5}$
6. $\cos A = \frac{1}{2}\sqrt{3}$; 30°
7. $\cos X = \frac{24}{25}$
8. $\cos X = \frac{4}{5}$;
   $\cos^2 X - \sin^2 X = 0.28 = \cos 2X$
9. 1
10. $2 \sin X \cos X = 0.96, \sin 2X = 0.96,$
    $\sin 2X = 2 \sin X \cos X$
11. $\frac{2 \tan X}{1 + \tan^2 X} = 0.96 = \sin 2X$
12. $\frac{1 - \tan^2 X}{1 + \tan^2 X} = 0.28 = \cos 2X$

Exercise 7b – p. 99

1. $\frac{3}{5}, -\frac{4}{5}, -\frac{3}{4}$
2. $\frac{5}{13}, -\frac{12}{13}, -\frac{5}{12}$
3. $\frac{3}{5}, -\frac{4}{5}, -\frac{3}{4}$
4. $\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}, -\frac{2}{3}$
5. 80° or 100°
6. 60°
7. 105°
8. 52° or 128°
9. 135°
10. 150°
11. 81° or 99°
12. 57°
13. 90°
14. 80°
15. 89°
16. 180°
17. $\pm\frac{4}{5}$
18. $\frac{5}{13}$
19. 53° or 127°
20. 150°
21. 135°
22. (a) yes, 90°
    (b) yes, 0
    (c) no
    (d) yes, 0 or 180°
23. A + B = 180°
24.



25. (a) Yes (b) No

Exercise 7c – p. 104

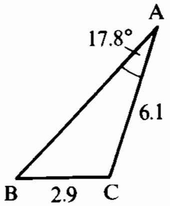
Answers are correct to 3 s.f.

1. 11.1
2. 13.7
3. 10.2
4. 8.83
5. 156
6. 113
7. 7.01
8. 89.1
9. 581
10. 141
11. 16.3
12. 28.1
13. 51.3
14. no; an angle and the side opposite to it are not known

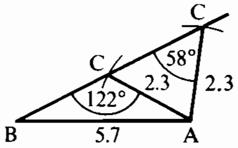
Exercise 7d – p. 108

Answers are correct to the nearest degree.

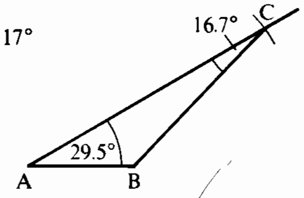
1. 18°



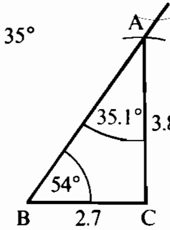
2. 58° or 122°



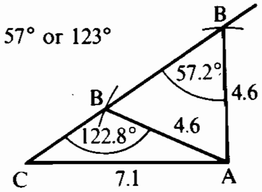
3. 17°



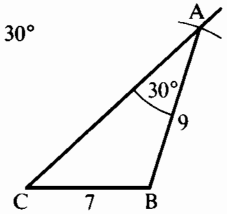
4. 35°



5. 57° or 123°



6. 30°



Exercise 7e – p. 111

1. 5.29
2. 12.9
3. 53.9
4. 4.04
5. 101
6. 12.0
7. 64.0
8. 31.8

Exercise 7f – p. 113

1. 38°
2. 55°
3. 45°
4. 94°
5. (a) 18° (b) 126°
6. 29°
7. 11.4 cm, 68°

Exercise 7g – p. 115

1. 87.4 cm
2. 23.8 cm
3. 17.5 cm
4. ∠B = 81°; a = 112 cm
5. a = 164 cm; c = 272 cm
6. ∠B = 34°; a = 37.0 cm
7. ∠C = 43°; b = 19.4 cm; c = 13.5 cm
8. ∠A = 52°; a = 33.2 cm; c = 41.5 cm
9. ∠B = 43°; ∠C = 60°; a = 27.1 cm
10. ∠A = 22°; ∠C = 33°; b = 30.3 cm
11. 14.1 m
12. 40°, 53°, 87°

Mixed Exercise 7 – p. 115

1. (a) 116° (b) 86°
2. $-\frac{24}{25}$
3. (a) $\frac{5}{\sqrt{39}}$ (b) $\frac{-5}{\sqrt{39}}$
4. (a) $\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$ (b) $\frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}$
5. $-\frac{5}{13}$
6. 9.05 cm
7. 4.82
8. 83°
9. 54° or 126°
10. 108°, 50°, 22°

CHAPTER 8

Exercise 8a – p. 118

1. 12 300 cm²
2. 2190 cm²
3. 1680 cm²
4. 453 square units
5. 42.9 square units
6. 51°, 21.0 cm²
7. 10.6 cm, 59.8 cm²
8. 52°, 151 cm (or 150 cm)
9. 5.25 cm

Exercise 8b – p. 121

1. 11.0 cm, 67.7 cm² (or 67.8 cm²)
2. 58.5 km
3. 477 m
4. ∠BAO = 74°, ∠CAO = 52°; 22 cm²

5. (a) 60.8 cm (b) 35° (c) 42.8 cm (d) 2140 cm^2 (e) 427 000 cm^3

Exercise 8c – p. 128

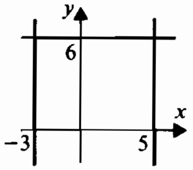
1. (a) $4\sqrt{2}$ cm (b) $2\sqrt{29}$ cm (c) $2\sqrt{29}$ cm (d) $2\sqrt{33}$ cm
2. (a) 6 cm, $6\sqrt{2}$ cm, $2\sqrt{34}$ cm (b) 53° (c) 43° (d) 53°
3. (a) $5\sqrt{2}$ cm (b) $5\sqrt{5}$ cm (c) $\sqrt{109}$ cm (d) 21° (e) 34°
4. $\frac{1}{3}$
5. P = 55.4°, Q = 31.2°, R = 93.4°
6. 71°
7. (a) 6 m (b) 9 m (c) 48° (d) 240 m^2
8. (a) 15.3 m (b) 2.29 cm (c) 15° (d) 30°
9. 420 m; 31°
10. 46°

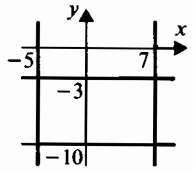
Mixed Exercise 8 – p. 133

1. 75.8 cm^2
2. $\frac{1}{2}$; yes, ∠A can be 30° or 150°
3. (a) 98° (b) 19.8 cm^2 (c) 3.96 cm
4. PQ = 8.29 cm, QR = 6 cm, RP = 3.46 cm; 9 cm^2
5. (a) 45° (b) 35°
6. 35°
7. $\dfrac{a\sqrt{3}}{49}; \dfrac{2a}{7\sqrt{7}}$
8. $a\sqrt{\dfrac{37}{39}} = 0.974a$

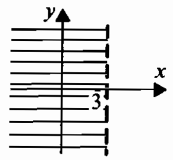
CHAPTER 9

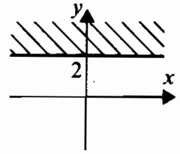
Exercise 9a – p. 139

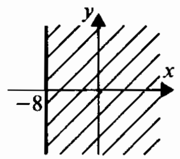
1.


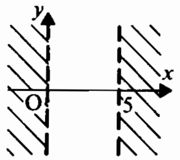
2.


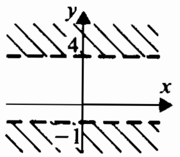
In questions 3–10, the unshaded region is the one required.

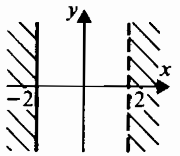
3.


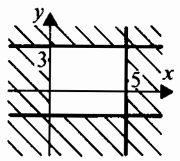
4.


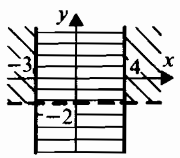
5.


6.


7.


8.


9.


10.


Exercise 9b – p. 142

1. (a) $y = 2x$ (b) $x + y = 0$ (c) $3y = x$ (d) $4y + x = 0$ (e) $y = 0$ (f) $x = 0$
2. (a) $2y = x + 2$ (b) $3y + 2x = 0$ (c) $y = 4x$
3. (a) $2y = x$ (b) $y + 2x - 6 = 0$ (c) $y = 2x + 5$
4. (a) $2y + x = 0$ (b) $2x - 3y = 0$ (c) $2x + y = 0$
5. (a) $x - 3y + 1 = 0$ (b) $2x + y - 5 = 0$
6. (a) $5x - y - 17 = 0$ (b) $x + 7y + 11 = 0$
7. $3x + 4y - 48 = 0$, 5

Exercise 9c, p. 146

1. (a) $y = 3x - 3$ (b) $5x + y - 6 = 0$ (c) $x - 4y - 4 = 0$ (d) $y = 5$ (e) $2x + 5y - 21 = 0$ (f) $15x + 40y + 34 = 0$
2. (a) $3x - 2y + 2 = 0$ (b) $3x - 2y + 7 = 0$ (c) $x = 3$
3. (a), (c) and (d)
4. $x + y - 7 = 0$
5. (a) $x + 2y - 5 = 0$ (b) $16x - 6y + 19 = 0$ (c) $10x - 16y + 23 = 0$
6. $2x + y = 0$
7. $4x + 5y = 0$
8. $5x + 4y = 0$
9. $x + 2y - 11 = 0$
10. $3x - 4y + 19 = 0$

Mixed Exercise 9 – p. 149

2. $\frac{1}{4}$ sq. units
3. $(-\frac{4}{3}, \frac{11}{3})$, $(-5, 0)$, $(6, 0)$
4. $10x - 26y - 1 = 0$
5. $x + 3y - 11 = 0$, $(\frac{13}{5}, \frac{14}{5})$
6. $[\frac{2}{5}(2 + 2a - b), \frac{1}{5}(4 - 2a + b)]$
7. $y = 2x - 3$
8. $y = x - 4$ (or $y + x = 10$ if line is inclined at −45° to Ox)
9. $8by - 2ax + 8b^2 + 3a^2 = 0$
10. (a) $\sqrt{20}$ (b) $x - 2y + 1 = 0$
11. $(\frac{9}{10}, \frac{17}{10})$ and $(-\frac{18}{10}, \frac{26}{10})$, $(-\frac{27}{10}, -\frac{1}{10})$ or $(\frac{36}{10}, \frac{8}{10})$, $(\frac{27}{10}, -\frac{19}{10})$
12. (a) (1, 2), (5, 2), (3, 6) (b) 8

CHAPTER 10

Exercise 10a – p. 152

1. (a) ∠ADE, ∠ACE (b) ∠AFE (c) ∠AOC (d) ∠ABC (e) ∠COE (f) ∠CAD, ∠CED (g) ∠BAC, . . .
3. (c) 106.3° (d) (5, 8), 53.1°

Exercise 10b – p. 156

1. (a) 50° (b) 40°, 40° (c) 60° (d) 54°, 108°
3. (4, 4)
4. 30°
7. $a^2 + b^2 - 9a - b + 14 = 0$
9. $(-\frac{115}{18}, -\frac{29}{6})$, 10.8 units to 3 s.f.
10. 60°, 60°

Exercise 10c – p. 158

1. 61.9°
2. 5 units
3. 67.4°
4. 7.22 cm
5. 60°, 30°, 90°
6. 8.43
7. 16 sq. units
8. 14.9 cm
9. 2.85
10. $\dfrac{2\sqrt{5}}{2}, -\dfrac{2\sqrt{5}}{5}$
12. $(-\frac{3}{4}, -\frac{9}{4})$
13. (a) $2x + y = 0$ (b) $(-\frac{4}{5}, \frac{8}{5})$
14. $x + y - 14 = 0$

CHAPTER 11

Exercise 11a – p. 165

1. $\frac{1}{4}\pi, \frac{5}{6}\pi, \frac{1}{6}\pi, \frac{3}{2}\pi, \frac{5}{4}\pi, \frac{1}{8}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi, \frac{7}{4}\pi$
2. 30°, 180°, 18°, 45°, 150°, 15°, 22.5°, 240°, 20°, 270°, 80°
3. (Angles in radians, correct to 3 s.f.) 0.611, 0.824, 1.62, 4.07, 0.246, 2.04, 6.46
4. (Angles to the nearest degree.) 97°, 190°, 57°, 120°, 286°, 360°

Exercise 11b – p. 168

1. 2.09 cm, 4.19 cm^2
2. 26.2 cm, 131 cm^2
3. 4.77 cm, 35.8 cm^2
4. 7.96 cm, 79.6 cm^2
5. 18.8 cm, 75.4 cm^2
6. 3.14 cm, 1.57 rad
7. 4.8 cm, 0.96 rad

8. π cm, 6 rad
9. $\frac{5}{8}\pi$, 8 cm
10. 146°
11. 0.283 rad
12. 0.52°
13. (a) 12 cm^2 (b) 23.2 cm^2
14. 14.5 mm^2, 139 mm^2
15. (a) 15.2 cm (b) 32.5 cm^2
16. 19.1°
17. 85.6 cm
18. 19.6 cm, 108 cm^2
20. 0.979a^2
21. (a) 135° (b) 9.41 cm
(c) 35.4 cm (d) 68.8 cm

CONSOLIDATION B

Multiple Choice Exercise B - p. 173

1. C
2. E
3. C
4. E
5. A
6. C
7. D
8. B
9. A
10. A
11. C
12. A
13. D
14. D
15. D
16. A, B
17. B
18. B, C
19. A, C
20. B
21. A, B
22. C
23. A, C
24. F
25. T
26. T
27. F
28. T
29. T
30. T
31. F

Miscellaneous Exercise B - p. 176

4. (2, 3), $\sqrt{13}$
6. 2.08 m, 44.3°
7. $(12 - 2x)/x$
8. (a) 65 cm (b) 45°; 21°
9. 0.464 cm^2
10. 3
11. (a) 43.3° (b) 53.1° (c) 111.1°
12. (a) 3 cm (b) 96°
13. (a) 79.9° (b) 13.4 cm (c) 36.7°
14. The points are $P_{-1}(0, -3)$, $P_0(1, -1)$, $P_1(2, 1)$ and $P_2(3, 3)$.
Grad $P_{-1}P_0$ = grad $P_{-1}P_1$
= grad $P_{-1}P_2 = 2$
so the four points are collinear,
$y = 2x - 3$
15. $(4\sqrt{2}, 8\sqrt{2} + 4)$, $(12 + 4\sqrt{2}, 8\sqrt{2})$, (0, 4)

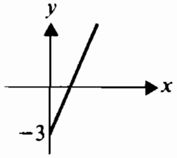
CHAPTER 12

Exercise 12a – p. 183

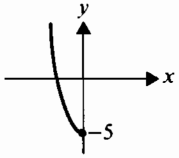
1. (a) yes (b) yes (c) yes, $x \neq 1$
(d) no (e) yes, $x \geqslant 0$ (f) yes
(g) yes (h) no
2. −4, −24
3. 25, 217
4. 1, not defined, 12
5. 1, $\frac{\sqrt{3}}{2}$

Exercise 12b – p. 186

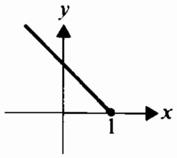
1. (a) $f(x) \geqslant -3$ (b) $f(x) \geqslant -5$
(c) $f(x) \geqslant 0$ (d) $0 < f(x) \leqslant \frac{1}{2}$
2. (a)



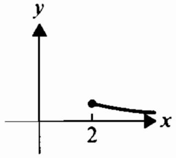
(b)



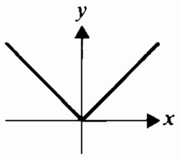
(c)



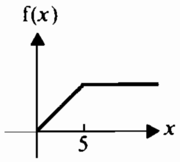
(d)



3. (a) 5, 4, 2, 0 (b)

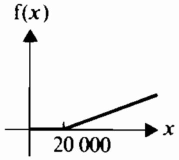


4. (a) 0, 2, 4, 5, 5 (b)



(c) $0 \leqslant f(x) \leqslant 5$

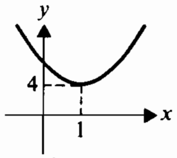
5. (a) 0, £5000
(b) $f(x) = 0$ for $0 \leqslant x \leqslant 20\,000$
$f(x) = \frac{1}{5}(x - 20\,000)$ for $x > 20\,000$



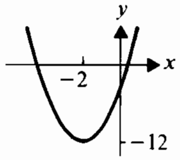
domain $x \geqslant 0$ (but $x <$ GNP!)
range $f(x) \geqslant 0$

Exercise 12c – p. 190

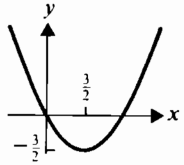
1. (a) $\frac{11}{4}$ (b) 3 (c) 4
2. (a) $f(x) \leqslant \frac{29}{4}$ (b) $f(x) \geqslant -2$
(c) $f(x) \leqslant 1$
3. (a)



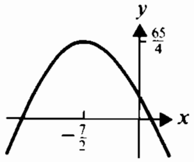
(b)



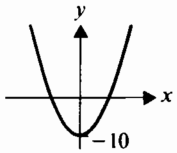
(c)



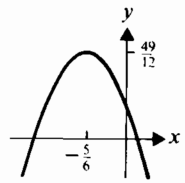
(d)



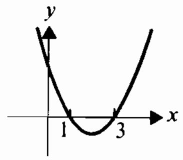
(e)



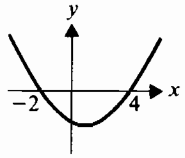
(f)



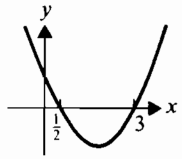
4. (a)



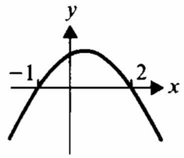
(b)



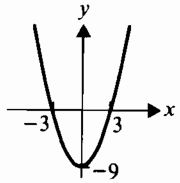
(c)



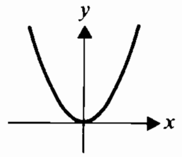
(d)



(e)

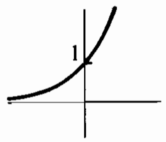


(f)

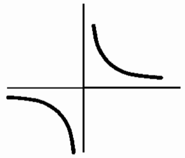


Exercise 12d – p. 196

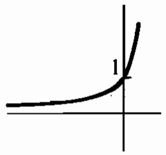
1. (a)



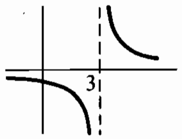
(b)



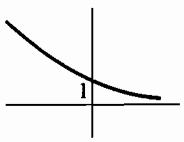
(c)



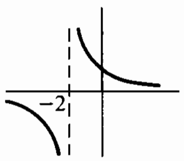
(d)



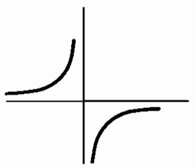
2. 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$
as $x \to \infty$, $f(x) \to 0$ and
as $x \to -\infty$, $f(x) \to \infty$



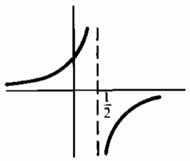
3. -2, as $x \to -2$ from below, $f(x) \to -\infty$
as $x \to -2$ from above, $f(x) \to \infty$



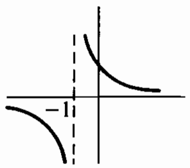
4. (a)



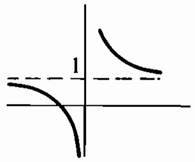
(b)



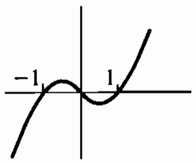
(c)



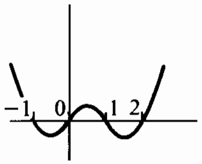
(d)



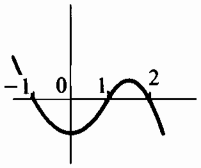
5. (a)



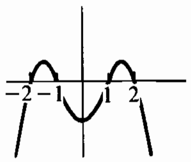
(b)



(c)



(d)



Exercise 12e – p. 197

1. (b) translation $\begin{pmatrix} 0 \\ c \end{pmatrix}$

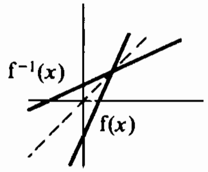
2. translation $\begin{pmatrix} -c \\ 0 \end{pmatrix}$
3. (a) reflection Ox
(b) reflection Oy

Exercise 12f – p. 200

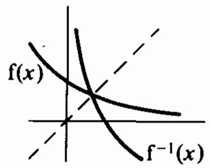
11. Reflection in line $y = x$
(there are several alternatives).
12. Reflection in line $y = x$
(there are several alternatives).
13. (5, 2)
14. (b, a)

Exercise 12g – p. 206

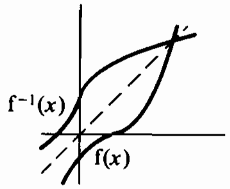
1. (a)



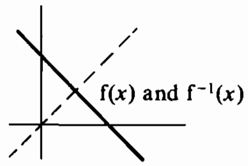
(b)



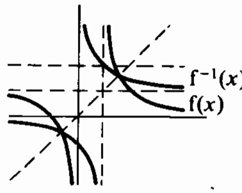
(c)



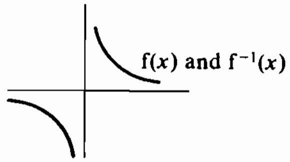
(d)



(e)



(f)



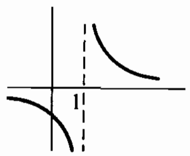
2. (d) and (f)
3. (a) $f^{-1} = (x - 1)$ (b) no
(c) $f^{-1}(x) = \sqrt[3]{(x-1)}$
(d) $f^{-1}(x) = \sqrt{(x+4)}, x \geqslant -4$
(e) $f^{-1}(x) = \sqrt[4]{x} - 1, x \geqslant 0$
4. (a) $-\frac{1}{3}$ (b) $\frac{1}{2}$ (c) there isn't one
5. (a) 9 (b) 2 (c) -1

Exercise 12h – p. 207

1. (a) $\frac{1}{x^2}$ (b) $(1-x)^2$ (c) $1 - \frac{1}{x}$
(d) $1 - x^2$ (e) $\frac{1}{x^2}$
2. (a) 125 (b) 15 (c) -1
(d) -1
3. (a) $(1+x)^2$ (b) $2(1+x)^2$
(c) $1 + 4x^2$
4. $g(x) = x^2$, $h(x) = 2 - x$
5. $g(x) = x^4$, $h(x) = (x + 1)$
6. (a) $f(x) = gh(x)$, $g(x) = 10^x$,
$h(x) = x + 1$
(b) $f(x) = gh(x)$, $g(x) = \frac{1}{x^2}$
$h(x) = 3x - 2$
(c) $f(x) = g(x) + h(x)$, $g(x) = 2^x$,
$h(x) = x^2$
(d) $f(x) = \frac{g(x)}{h(x)}$, $g(x) = 2x + 1$,
$h(x) = x$
(e) $f(x) = gh(x)$, $g(x) = x^4$,
$h(x) = 5x - 6$
(f) $f(x) = g(x)h(x)$, $g(x) = x - 1$,
$h(x) = x^2 - 2$

Mixed Exercise 12 – p. 208

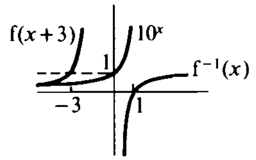
1. (a) $\frac{1}{0}$ is meaningless (b) $\frac{1}{4}$
(c)



(d) $f^{-1}(x) = 1 - \frac{1}{x}, x \neq 0$

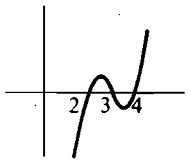
2. (a) $\frac{11}{4}$ when $x = \frac{3}{2}$
   (b) $-\frac{41}{8}$ when $x = \frac{7}{4}$
   (c) -9 when $x = -2$

3.

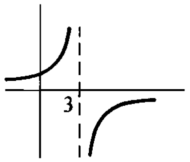


4. (a) $10000, \frac{1}{9}, \frac{1}{100}$
   (b) $10^{-x^2}, 10^{2/x}$
   (c) $\log x$, g^{-1} does not exist, $\frac{1}{x}$
   (d) $\pm\frac{1}{3}$
   (e) no for all x, yes for $x > 0$

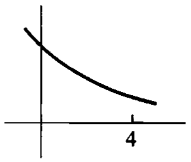
5. (a)



   (b)

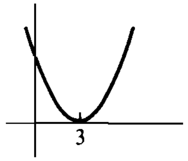


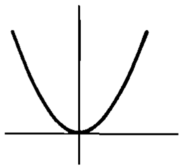
   (c)



6. (a) $g(x) = 2^x$, $h(x) = 3x - 2$
   (b) $2^{16}, 1$

7. (a)





   (b) $0, \frac{1}{6}$

CHAPTER 13

Exercise 13a – p. 210

1. $x < \frac{7}{2}$
2. $x > 4$
3. $x > -3$
4. $x > -2$
5. $x < \frac{1}{2}$
6. $x < -3$
7. $x < -\frac{1}{4}$
8. $x > \frac{8}{3}$
9. $x > \frac{3}{8}$

Exercise 13b – p. 212

1. $x > 2$ and $x < 1$
2. $x \geqslant 5$ and $x \leqslant -3$
3. $-4 < x < 2$
4. $x \geqslant \frac{1}{2}$ and $x \leqslant -1$
5. $x > 2 + \sqrt{7}$ and $x < 2 - \sqrt{7}$
6. $-\frac{1}{2} < x < \frac{1}{2}$
7. $-4 \leqslant x \leqslant 2$
8. $x > 1$ and $x < -\frac{2}{5}$
9. $x \geqslant \frac{3}{2}$ and $x \leqslant -5$
10. $x > 4$ and $x < -2$
11. $\frac{1}{2}(-3 - \sqrt{17}) \leqslant x \leqslant \frac{1}{2}(-3 + \sqrt{17})$
12. $x > 7$ and $x < -1$

Exercise 13c – p. 216

1. $x > -2$
2. $x < -2$
3. $1 < x < \frac{5}{2}$
4. $4 < x < \frac{24}{5}$
5. $-3 < x < 3$ and $x > 5$
6. $1 < x < 2$ and $x < 0$
7. $x > 3$ and $1 < x < 2$
8. $2 < x < 8$
9. $-1 < x < 1$ and $2 < x < 3$
10. $\frac{1}{6} < x < 1$
11. $x > -1$ and $-5 < x < -3$
12. $-4 < x < -\frac{2}{3}$
13. $-1 < x < 2$
14. $3 < x < 4$ and $-4 < x < -1$
15. $4 < x < 5$ and $-1 < x < 1$
16. $-2 < x < 3$

Exercise 13d – p. 219

1. (a) $p \geqslant 9$ and $p \leqslant 1$
   (b) $p \geqslant 5$ and $p \leqslant 1$
2. $-2 < a < 6$
3. $p < -1$

4. $-\frac{1}{7} \leqslant f(x) \leqslant 1$
5. $f(x) \geqslant 2$ and $f(x) \leqslant -2$
6. all values
7. $-\frac{1}{2} \leqslant f(x) \leqslant \frac{1}{2}$
8. $f(x) \geqslant 3 + \sqrt{8}$ and $f(x) \leqslant 3 - \sqrt{8}$
9. $f(x) \geqslant 2$ and $f(x) \leqslant -2$
10. $0 < k < \frac{4}{9}$
12. $-\frac{1}{2} \leqslant f(x) \leqslant 1$

Mixed Exercise 13 – p. 220

1. $x < 1$
2. $x > \frac{3}{2}$
3. $x > \frac{3}{2}$
4. $x > 3$, or $x < -2$
5. $-\frac{2}{3} < x < \frac{3}{2}$
6. $-\sqrt{13} < x < \sqrt{13}$
7. $x > 3 + \sqrt{2}$, or $x < 3 - \sqrt{2}$
8. $-2 < x < 7$
9. all values of x
10. $x > 6$, or $x < 1$
11. $1 < x < 4$
12. $-1 < x < 1$
13. $1 < x < 3$
14. $x > -1$
15. $x \geqslant 1$ and $x \leqslant -3$
16. $-2 < x \leqslant -1$ and $1 \leqslant x < 2$
    (Note: $x \neq 2$ or -2)
18. $k \leqslant 3$ and $k \geqslant 4$
19. $\frac{2}{3} \leqslant f(x) \leqslant 2$

CHAPTER 14

Exercise 14a – p. 229

1. 4; 4
2. 1; 1
3. $3x^2$; 3
4. $2x$; 4
5. $2x - 1$; 1
6. $4x^3$; 32

Exercise 14b – p. 231

1. $-\frac{2}{x^3}$
2. $-\frac{1}{x^2}$
3. $-\frac{2}{x^2}$

Exercise 14c – p. 233

1. $5x^4$
2. $-3x^{-4}$
3. $\frac{4}{3}x^{1/3}$
4. $-\frac{1}{x^2}$
5. $10x^9$
6. $-\frac{2}{x^3}$
7. $\frac{3}{2}\sqrt{x}$
8. $-\frac{1}{2}x^{-3/2}$
9. $-\frac{4}{x^5}$
10. $\frac{1}{3}x^{-2/3}$
11. $-\frac{1}{4}x^{-5/4}$
12. 1
13. $\frac{7}{2}\sqrt{x^5}$
14. $-\frac{7}{x^8}$
15. $\frac{1}{7}x^{-6/7}$
16. $3x^2$

Exercise 14d – p. 235

1. $3x^2 - 2x + 5$
2. $6x + \frac{4}{x^2}$
3. $\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$
4. $8x^3 - 8x$
5. $3x^2 - 4x - 8$
6. $2x + \frac{5}{2\sqrt{x}}$
7. $-\frac{3}{4}x^{-7/4} - \frac{3}{4}x^{-1/4} + 1$
8. $9x^2 - 8x + 9$
9. $\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$
10. $\frac{1}{2\sqrt{x}} + \frac{3\sqrt{x}}{2}$
11. $-\frac{2}{x^3} + \frac{3}{x^4}$
12. $\frac{-1}{2\sqrt{x^3}} + \frac{2}{x^2}$
13. $-\frac{1}{2}x^{-3/2} + \frac{9}{2}x^{1/2}$
14. $\frac{1}{4}x^{-3/4} - \frac{1}{5}x^{-4/5}$
15. $-\frac{12}{x^4} + \frac{3x^2}{4}$
16. $-\frac{4}{x^2} - \frac{10}{x^3} + \frac{18}{x^4}$
17. $\frac{3}{2\sqrt{x}} - 3$
18. $1 + 2x^{-2} + 9x^{-4}$
19. $\frac{3}{2}\sqrt{x} - \frac{5}{2}x\sqrt{x}$
20. $\frac{-3\sqrt{x}}{2x^3} + \frac{3x}{2\sqrt{x}}$

Exercise 14e – p. 236

1. $\dfrac{dy}{dx} = 2x + 2$
2. $\dfrac{dz}{dx} = -4x^{-3} + x^{-2}$
3. $\dfrac{dy}{dx} = 6x + 11$
4. $\dfrac{dy}{dz} = 2z - 8$
5. $\dfrac{ds}{dt} = -\dfrac{3}{2t^4}$
6. $\dfrac{ds}{dt} = \dfrac{1}{2}$
7. $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$
8. $\dfrac{dy}{dz} = \dfrac{5z^2 - 1}{2\sqrt{z}}$
9. $\dfrac{dy}{dx} = 18x^2 - 8$
10. $\dfrac{ds}{dt} = 2t$
11. $\dfrac{ds}{dt} = 1 - \dfrac{7}{t^2}$
12. $\dfrac{dy}{dx} = -\dfrac{3\sqrt{x} + 28}{2x^3}$

Exercise 14f – p. 238

1. $2; -\frac{1}{2}$
2. $-\frac{1}{3}; 3$
3. $\frac{1}{4}; -4$
4. $6; -\frac{1}{6}$
5. $1; -1$
6. $5; -\frac{1}{5}$
7. $11; -\frac{1}{11}$
8. $-11; \frac{1}{11}$
9. $4; -\frac{1}{4}$
10. $\frac{4}{27}; -\frac{27}{4}$
11. $\frac{5}{4}; -\frac{4}{5}$
12. $2; -\frac{1}{2}$
13. (2, 2) and (−2, 4)
14. (1, 0) and $(-\frac{1}{3}, \frac{4}{27})$
15. (3, 0) and (−3, 18)
16. (−1, −2) and (1, 2)
17. (1, −16)
18. $(-1, \frac{1}{4})$
19. (0, −5)
20. (1, −2) and (−1, 2)

Mixed Exercise 14 – p. 238

1. $6x + 1$
2. (a) $-3x^{-4} - 3x^2$
   (b) $\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$
   (c) $-\dfrac{2}{x^3} - \dfrac{6}{x^4}$
3. (a) $\dfrac{dy}{dx} = \frac{3}{2}x^{1/2} - \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-4/3}$
   (b) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} + \dfrac{1}{x^2} - \dfrac{3}{x^4}$
   (c) $-\dfrac{3}{4x^{7/4}} + \dfrac{1}{4x^{5/4}}$
4. (a) 5 (b) 5 (c) 17
5. (a) 5 (b) 6
6. (a) 1 (b) 7 and −7
7. (a) −3 (b) $(-\frac{1}{2}, 0)$ and (2, 0)
   (c) −5 and 5
8. (a) (1, 10) and (−1, 6)
   (b) $(\frac{1}{3}, 7\frac{7}{9})$ and $(-\frac{1}{3}, 8\frac{2}{9})$
9. (a) $4x^3 - 2x$ (b) $6(3x + 4)$
   (c) $\dfrac{x + 3}{2x\sqrt{x}}$
10. $1 - \sqrt{2}$
11. (−2, 4)
12. $\frac{1}{5}$ and $-\frac{1}{5}$
13. (1, 9) and (3, 11)
14. 1
15. (b), (d)

CHAPTER 15

Exercise 15a – p. 243

1. (a) $y = 2x - 5$
   (b) $2y + x + 5 = 0$
2. (a) $y = 4x - 2$
   (b) $4y + x + 8 = 0$
3. (a) $y + x + 2 = 0$ (b) $y = x$
4. (a) $y = 5$ (b) $x = 0$
5. (a) $y + x = 3$ (b) $y = x - 1$
6. (a) $y = 19x + 26$
   (b) $19y + x + 230 = 0$
7. $4y + x + 12 = 0$
8. $y = 7x - 29$
9. $y + x = 1, 2y = 2x - 3; (\frac{5}{4}, -\frac{1}{4})$
10. $4y - x + 1 = 0, 4y + x - 5 = 0$
11. $y = 5x - 1, 3y + 9x + 19 = 0$
12. $y = 7x - 4, y + 5x + 28 = 0$
13. $(2, 8), y = 8x - 8$
14. $(\frac{1}{2}, -\frac{1}{4})$
15. $y + x + 1 = 0$
16. $2y = x + 2$

17. $k = -\frac{7}{2}$
18. $8y + 121 = 0$
19. (1, −1)
20. $p = 12, q = 8; (-2, 24)$

Exercise 15b – p. 246

1. $x = 0$
2. $x = \frac{3}{4}$
3. $x = 0, x = \frac{8}{3}$
4. $x = \pm\frac{1}{2}$
5. $x = 0, x = \frac{4}{3}$
6. $x = \pm 1$
7. $x = 4$
8. $x = \pm 3$
9. $x = 1, x = -\frac{4}{3}$
10. $x = \pm\frac{5}{9}\sqrt{3}$
11. $x = 1, x = -4$
12. $x = \pm\frac{2}{3}\sqrt{3}$
13. (3, 3), (−3, −3)
14. $(1, -7), (\frac{1}{3}, -\frac{185}{27})$
15. $(\frac{1}{2}, -\frac{25}{4})$
16. $(\frac{1}{3}, \frac{-2}{9}\sqrt{3})$
17. (1, 2)
18. (4, 10), (−4, 6)

Exercise 15c – p. 252

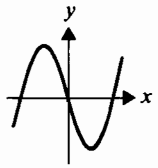
1. (1, 1)
2. (−1, −2) min; (1, 2) max
3. (3, 6) min; (−3, −6) max
4. (0, 0) max; $(\frac{10}{3}, -\frac{500}{27})$ min
5. (0, 0) min
6. $(1, \frac{3}{2})$ min
7. (−1, 1) max; (0, 0) min; (1, 1) max
8. (0, 0) min
9. $(\frac{5}{4}, -\frac{49}{8})$ min
10. (−1, 4) max; (1, −4) min
11. (−2, −16) min; (0, 0) max; (2, −16) min
12. (−2, 8) min; (2, 8) min
13. −2 max; 2 min
14. $2\frac{3}{4}$ min
15. 0 inflex; 27 max
16. 8 inflex
17. 7 inflex
18. $-\frac{5}{16}$ min, 0 max, −2 min

Exercise 15d – p. 256

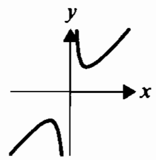
1. 800 m²; 20 m × 40 m
2. 20 cm × 20 cm × 10 cm
3. $r = \sqrt{(9 - h^2)}$; $12\pi\sqrt{3}$ cm³
4. 5 cm square
5. $\sqrt{35}$ cm square
6. $a = 1, b = -2, c = 3$
7. $p = q = 1, r = 2; (-1, 0)$
8. $5y = x^2 + 4x + 9$

Mixed Exercise 15 – p. 257

1. $11; y = 11x - 6; (\frac{2}{3}, \frac{4}{3})$
2. $2y = x - 1; (-\frac{3}{2}, -\frac{5}{4})$
3. (2, 14), (−2, −14)
4. $y = 3x + 6, y = 3x + 2$
5. $x + 2y + 9 = 0$
6. (−2, 16) max, (2, −16) min;



7. min 2, max −2;

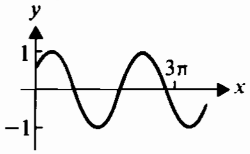


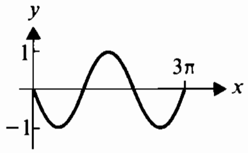
9. $h = \frac{1}{2}(7 - 2r - \pi r)$
10. 4 m × 4 m × 2 m
11. 12.5 cm²

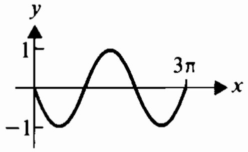
CHAPTER 16

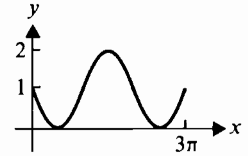
Exercise 16a – p. 264

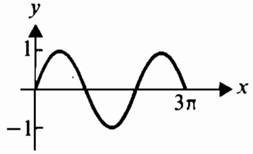
1. $\frac{1}{2}\sqrt{3}$
2. 0
3. $-\frac{1}{2}\sqrt{3}$
4. $\frac{1}{2}$
5. $\frac{1}{2}\pi, \frac{5}{2}\pi, \frac{9}{2}\pi$
6. $-\frac{1}{2}\pi, -\frac{5}{2}\pi$
7. sin 55°
8. −sin 70°
9. −sin 60°
10. $-\sin\frac{1}{6}\pi$
11.

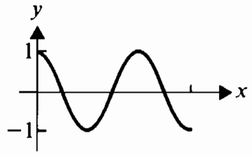


12.


13.


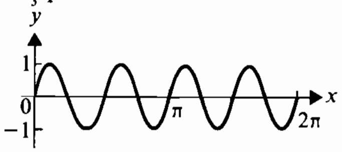
14.


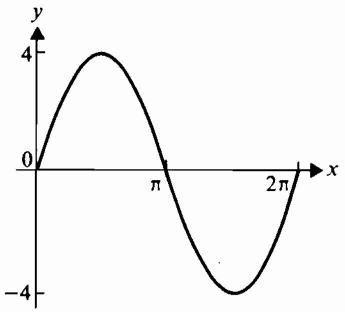
15.


16.


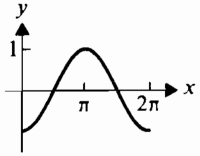
17. The curve $y = a\sin\theta$ is a one-way stretch of the curve $y = \sin\theta$ by a factor a parallel to the y-axis.

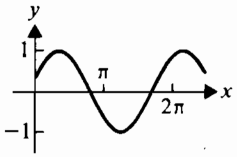
19. The curve $y = \sin 3\theta$ is a one-way shrinkage of the curve $y = \sin\theta$ by a factor $\frac{1}{3}$ parallel to the x-axis.

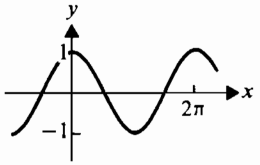
20. (a)


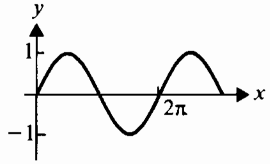
(b)


Exercise 16b – p. 268

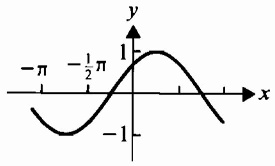
1. (a) $-\cos 57°$ (b) $-\cos 70°$ (c) $\cos 20°$ (d) $-\cos 26°$
2. (a) $-\frac{\sqrt{3}}{2}$ (b) 0 (c) $-\frac{1}{\sqrt{2}}$ (d) 1
3. (a)


(b)


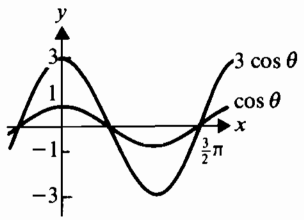
(c)


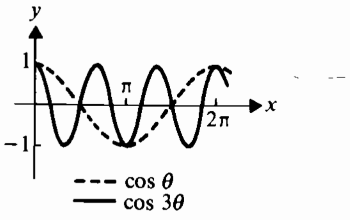
4.


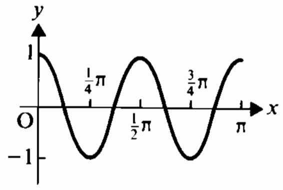
$\sin\theta = \cos(\theta - \frac{1}{2}\pi)$
$\cos\theta = -\sin(\theta - \frac{1}{2}\pi)$

5.


(a) $\theta = \frac{1}{4}\pi$ (b) $\theta = -\frac{3}{4}\pi$
(c) $\theta = -\frac{1}{4}\pi$ and $\frac{3}{4}\pi$

6.


7.


8.


$\frac{1}{8}\pi, \frac{3}{8}\pi, \frac{5}{8}\pi, \frac{7}{8}\pi$

Exercise 16c – p. 270

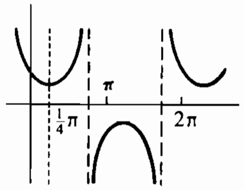
1. (a) 1 (b) $-\sqrt{3}$ (c) $\sqrt{3}$ (d) -1
2. (a) $\tan 40°$ (b) $-\tan \frac{2}{7}\pi$ (c) $-\tan 50°$ (d) $\tan \frac{2}{5}\pi$
3. (a) $\frac{1}{4}\pi, \frac{5}{4}\pi$
   (b) $\frac{3}{4}\pi, \frac{7}{4}\pi$
   (c) $0, \pi, 2\pi$
   (d) $\frac{1}{2}\pi, \frac{3}{2}\pi$

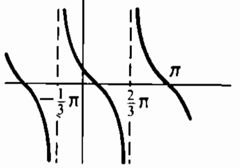
Exercise 16d – p. 273

1. (a) 0.412 rad, 2.73 rad, −5.87 rad, −3.55 rad
   (b) $-\frac{4}{3}\pi, -\frac{2}{3}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi$
   (c) 0.876 rad, 4.02 rad, −2.27 rad, −5.41 rad
2. (a) 141.3°, 321.3°, 501.3°, 681.3°
   (b) 191.5°, 348.5°, 551.5°, 708.5°
   (c) 84.3°, 275.7°, 444.3°, 635.7°
3. (a) 36.9° (b) −36.9° (c) 0.464 rad
4. $0, \pi, 2\pi$
5. 11.8°, 78.2°, 191.8°, 258.2°
6. $\frac{1}{3}\pi, \pi, \frac{5}{3}\pi$

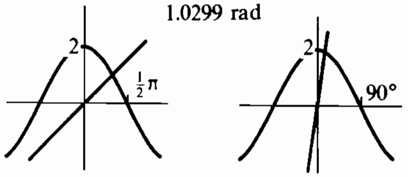
Exercise 16e – p. 276

1. (a) 60°, 300° (b) 59.0°, 239.0° (c) 41.8°, 138.2°
2. (a) −140.2°, 39.8°
   (b) −131.8°, 131.8°
   (c) −150°, −30°
3. $-\frac{1}{2}\pi, \frac{1}{2}\pi$
4. (a) 1 (b) $-\sqrt{2}$ (c) −2

5.

; $\frac{1}{4}\pi$

6.

; $-\frac{1}{12}\pi, \frac{11}{12}\pi$

Exercise 16f – p. 278

1. 1.0299 rad


2. (a) −1.89549, 0, 1.89549
   (b) 0, 0.8767 rad (c) 1.2834 rad

CHAPTER 17

Exercise 17a – p. 283

	$\sin\theta$	$\cos\theta$	$\tan\theta$
1. (a)	$-\frac{12}{13}$	$-\frac{5}{13}$	$\frac{12}{5}$
(b)	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$
(c)	$\frac{7}{25}$	$\frac{24}{25}$	$\frac{7}{24}$
(d)	0	± 1	0

2. $\tan^4 A$
3. 1
4. $\sec\theta \operatorname{cosec}\theta$
5. $\sec^2\theta$
6. $\tan\theta$
7. $\sin^3\theta$
8. $x^2 - y^2 = 16$
9. $b^2x^2 - a^2y^2 = a^2b^2$
10. $y^2(4 + x^2) = 36$
11. $(1 - x)^2 + (y - 1)^2 = 1$
12. $y^2(x^2 - 4x + 5) = 4$
13. $x^2(b^2 - y^2) = a^2b^2$

Exercise 17b – p. 286

1. 57.7°, 122.3°, 237.7°, 302.3°
2. 190.1°, 349.9°
3. 38.2°, 141.8°
4. 30°, 150°
5. 30°, 150°
6. 0°, 131.8°, 228.2°, 360°
7. ±0.723 rad
8. −0.314 rad, −2.83 rad
9. $-\frac{3}{4}\pi$, −0.245 rad, $\frac{1}{4}\pi$, 2.90 rad
10. $-\pi, -\frac{1}{3}\pi, \frac{1}{3}\pi, \pi$

Exercise 17c – p. 290

1. $\frac{1}{3}\pi + 2n\pi, \frac{2}{3}\pi + 2n\pi$
2. $\pm\frac{1}{2}\pi + 2n\pi \ (= \frac{1}{2}\pi + n\pi)$
3. $-\frac{1}{3}\pi + n\pi$
4. $-14.5° + 360n°, -165.5° + 360n°$
5. $\pm\frac{2}{3}\pi + 2n\pi$
6. $\frac{1}{4}\pi + n\pi$
7. $2n\pi$
8. $\frac{1}{6}\pi + 2n\pi, \frac{5}{6}\pi + 2n\pi$
9. $\pm\frac{1}{6}\pi + 2n\pi, \pm\frac{5}{6}\pi + 2n\pi$
10. $\pm 41.4° + 360n°$
11. $-18° + 360n°, -162° + 360n°$
12. $45° + 180n°, -14° + 180n°$
13. $\pm\frac{1}{3}\pi + 2n\pi, \pi + 2n\pi$
14. $n\pi, \pm\frac{1}{3}\pi + 2n\pi$
15. $\pm 90° + 360n°, 14.5° + 360n°, 165.5° + 360n°$
16. $n\pi, \pm\frac{1}{6}\pi + 2n\pi$
17. $\pm 51.8° + 360n°$

Exercise 17d – p. 292

1. $22.5° + 90n°$
2. $\pm 40° + 120n°$
3. $-270° + 720n°, 630° + 720n°$
4. $\pm 12° + 72n°$
5. $-42.1° + 540n°$
6. $\pm 25.5° + 180n°$
7. $67.5° + 90n°$
8. $-480° + 1440n°$
9. $60° + 180n°$
10. $\frac{1}{12}\pi + 2n\pi, -\frac{7}{12}\pi + 2n\pi$
11. $\frac{1}{24}\pi + \frac{1}{2}n\pi$
12. $n\pi, \frac{1}{3}\pi + n\pi$

Exercise 17e – p. 293

1. 149.5°, 59.5°, 30.5°, 120.5°
2. −105.2°, 14.8°, 134.8°, −74.8°, 45.2°, 165.2°
3. ±63.6°
4. $\frac{1}{6}\pi, \frac{5}{12}\pi, \frac{2}{3}\pi, \frac{11}{12}\pi, \frac{7}{6}\pi, \frac{17}{12}\pi, \frac{5}{3}\pi, \frac{23}{24}\pi$
5. $\frac{1}{15}\pi, \frac{1}{3}\pi, \frac{7}{15}\pi, \frac{11}{15}\pi, \frac{13}{15}\pi, \frac{17}{15}\pi, \frac{19}{15}\pi, \frac{23}{15}\pi, \frac{5}{3}\pi, \frac{29}{15}\pi$
6. $\frac{3}{2}\pi$

Mixed Exercise 17 – p. 293

1. $x^2 + \dfrac{1}{y^2} = 1$
2. $\sin\beta = \pm\dfrac{\sqrt{3}}{2}, \tan\beta = \pm\sqrt{3}$
3. $\dfrac{2}{\sin^2\theta}$; $\frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$
4. $\pm 60° + 360n°, \pm 109.5° + 360n°$
6. $-\frac{29}{36}\pi, -\frac{17}{36}\pi, -\frac{5}{36}\pi, \frac{7}{36}\pi, \frac{19}{36}\pi, \frac{31}{36}\pi$
7. (a) $(x-2)^2 + (y+1)^2 = 1$
   (b) $(x+3)^2 = 1 + (2-y)^2$
8. $\pm\frac{1}{8}\pi + n\pi, \pm\frac{3}{8}\pi + n\pi \ (= \frac{1}{8}\pi + \frac{1}{4}n\pi)$
10. $\sin^2 A$
11. $180n°, \pm 70.5° + 360n°$
12. $\sec^2\theta \tan^2\theta$

CHAPTER 18

Exercise 18b – p. 298

1. 0
2. $\frac{1}{2}$
3. $\frac{1}{4}(\sqrt{6} - \sqrt{2})$
4. $-(2 + \sqrt{3})$
5. $\frac{1}{4}(\sqrt{6} - \sqrt{2})$
6. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$
7. $\sin 3\theta$
8. 0
9. $\tan 3A$
10. $\tan\beta$
11. (a) $\frac{3}{5}$ (b) $-\frac{4}{5}$ (c) $-\frac{3}{4}$
12. (a) 1, 115° (b) 1, 30°
    (c) 1, 310° (d) 1, 330°
21. 67.5°, 247.5°
22. 7.4°, 187.4°
23. 37.9°, 217.9°
24. 15°, 195°
25. $\frac{1}{12}\pi + n\pi$
26. $\frac{7}{12}\pi + n\pi$

Exercise 18c – p. 302

1. $\frac{1}{2}$
2. $\dfrac{1}{\sqrt{2}}$

3. $\frac{1}{2}\sin 2\theta$
4. $\cos 8\theta$
5. $-\dfrac{1}{\sqrt{3}}$
6. $\tan 6\theta$
7. $\sqrt{2}\cos 3\theta$
8. $-\dfrac{1}{\sqrt{2}}$
9. $\tan(x + 45°)$
10. $\dfrac{1}{\sqrt{2}}$
11. (a) $\frac{24}{25}, -\frac{7}{25}$ (b) $\frac{336}{625}, \frac{527}{625}$
    (c) $\frac{120}{169}, -\frac{119}{169}$
12. (a) $-\frac{336}{527}$ (b) $\frac{527}{625}$ (c) $-\frac{336}{625}$
    (d) $\frac{164833}{390625}$
13. (a) $x(1 - y^2) = 2y$
    (b) $x = 2y^2 - 1$ (c) $x = 1 - \dfrac{2}{y^2}$
    (d) $2x^2y + 1 = y$
15. (a) $\frac{3}{2}\pi + 2n\pi, \frac{1}{6}\pi + 2n\pi, \frac{5}{6}\pi + 2n\pi$
    (b) $\pm\frac{1}{2}\pi + 2n\pi, \frac{7}{6}\pi + 2n\pi, \frac{11}{6}\pi + 2n\pi$
    (c) $\pm\frac{1}{3}\pi + 2n\pi$
    (d) $\pm 35.3° + 180n°$
    (e) $\frac{1}{2}\pi + n\pi, \frac{1}{4}\pi + n\pi$
    (f) $90° + 180n°, 23.6° + 360n°, 156.4° + 360n°$

Exercise 18d – p. 306

1. (a) $\frac{24}{25}$ (b) $-\frac{7}{24}$ (c) $\frac{1}{2}, -2$
   (d) $\pm\dfrac{2}{\sqrt{5}}, \pm\dfrac{1}{\sqrt{5}}$
2. t^2
3. $\dfrac{1}{t}$
4. $\dfrac{(1 - t^2)}{2t^2}$
5. $\dfrac{1}{(3 + 6t - 5t^2)}$
8. 0, 67.4°
9. −153°, 130.4°
10. 36.9°, 126.9°
11. −90°, 36.8°

Mixed Exercise 18 – p. 306

1. $y = 1 - 2x^2$
4. $\frac{56}{65}, -\frac{16}{65}$
5. $x = 2y - 1$
6. −155.7°, 24.3°, −114.3°, 65.7°
7. $n\pi$
9. $\cot^2 x$
10. 90°, 270°

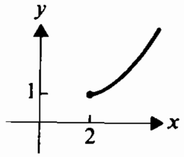
CONSOLIDATION C

Multiple Choice Exercise C – p. 313

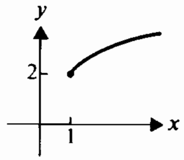
1.	C	19.	A, B
2.	B	20.	B, C
3.	D	21.	B, C
4.	C	22.	B
5.	E	23.	A, C
6.	B	24.	B
7.	B	25.	B, C
8.	B	26.	B, C
9.	D	27.	C
10.	A	28.	F
11.	C	29.	T
12.	D	30.	T
13.	B	31.	T
14.	B	32.	T
15.	B	33.	F
16.	B	34.	F
17.	C	35.	F
18.	B		

Miscellaneous Exercise C – p. 318

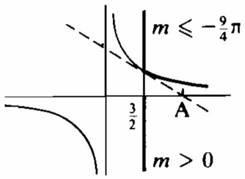
2. $x > 2$ and $x < -5$
3. (a) $(10x + 1)/(x + 5), x \neq -5$
   (b) $f^{-1}: x \to (2x + 1)/(x - 3), x \neq 3$
4. $8x^3 + 24x^2 + 32x + 14$
5. $\pm\frac{1}{4}\pi + n\pi, \pm\frac{1}{3}\pi + n\pi$
6. $b = -6, c = -8$; $f(x) \geqslant -17$;
   $g^{-1}(x) = 3 + \sqrt{(17 + x)}, x \geqslant -17$; 8
7. (a) 0 (b) $x < -3, -1 < x < 2$
8. (a) $h(x) > 1$
   (b) $x > 1, h^{-1}(x) > 2$
   (c)



   (d)



9. (a) $m > 0, m < -\frac{4}{9}; m = -\frac{4}{9};$
$9y + 4x - 12 = 0$
(b)



11. 30°, 60°, 120°, 150°
12. $(2, -9); (-\frac{1}{3}, 3\frac{19}{27});$
(a) (i) $-1 < x < \frac{1}{2}, x > 3$
(ii) $x < -\ln 2, x > \ln 3$
(b) $(2n + 1)\pi, (2n \pm \frac{1}{3})\pi$
13. Least value is -2, greatest value is $\frac{1}{4}$
14. 0, 60°, 120°, 180°
15. (a) $\frac{1}{1-x}$ (b) x (c) $\frac{1}{1-x}$
16. $2n\pi \pm \frac{1}{3}\pi, 2n\pi + \pi$
17. 40.9°, 220.9°
18. (a) $f(x) \leqslant 4$ (b) $\pm\sqrt{2}, \pm\sqrt{6}$
19. $\pm\frac{2}{3}\pi, 2n\pi \pm \frac{2}{3}\pi$
20. $x < 1, x > 2\frac{1}{2}$
21. The curve $y = x^2$ is translated 3 units in the direction Ox and translated $\frac{9}{2}$ units in the direction yO followed by a stretch of factor 2 parallel to Oy.
$f^{-1}(x) \geqslant 3, x \geqslant -\frac{9}{2};$
$f^{-1}(x) = 3 + \sqrt{[\frac{1}{2}(x + 9)]}$
22. $f \circ g: x \to (2x + 5)^2 - 3, -3 \leqslant f \circ g \leqslant 166$
23. (a) $\frac{(12 - 2x)}{x}$ (c) 9
24. $x - 9y + 73 = 0$
25. (a) $16.1° + 180n°$
(b) $27.2° + 360n°, 152.8° + 360n°$
26. $0, \pi, \frac{7}{6}\pi, \frac{11}{6}\pi, 2\pi$
27. $2n\pi, 2n\pi \pm \frac{1}{3}\pi$
28. $x < -4$ and $-1 < x < 2$
29. (a) $a = \frac{5}{4}$ (b) $a < -1$
30. (a) (1, 3) (b) (−1, 1) (c) $(\frac{1}{2}, 1)$
31. (b) $2n\pi \pm \frac{1}{3}\pi, n\pi$ (e) $a = b = \frac{1}{4}$
32. $a = 3, b = -4$

CHAPTER 19

Exercise 19a – p. 326

1. $6(3x + 1)$
2. $-4(3 - x)^3$
3. $20(4x - 5)^4$
4. $6x(x^2 + 1)^2$
5. $21(2 + 3x)^6$
6. $-18(2 - 6x)^2$
7. $4x^3(2x^4 - 5)^{-1/2}$
8. $-2x(x^2 + 3)^{-2}$
9. $\frac{9x^2}{2\sqrt{(3x^3 - 4)}}$
10. $-\left(\frac{1}{2\sqrt{x}} + 3\right)\left(\sqrt{x} + 3x\right)^{-2}$
11. $\frac{3x}{(4 - x^2)^{3/2}}$
12. $\frac{-7(x^2 + 1)}{(x^3 + 3x)^{4/3}}$
13. $-10(4 - 2x)^4$
14. $4x(x^2 + 3)$
15. $21(3x - 4)^6$
16. $4x(x^2 + 4)$
17. $-4x$
18. $-12x^2(2 - x^3)^3$
19. $\frac{3}{2}x(2 + x^2)^{-1/4}$
20. $\frac{1}{3}(2x - 1)(x^2 - x)^{-2/3}$
21. $6x(2 - 3x^2)^{-2}$
22. $4x(4 - x^2)^{-3}$
23. $-\frac{5}{2}x^4(x^5 - 3)^{-3/2}$
24. $\frac{-1}{8\sqrt{x}(6 - \sqrt{x})^{3/4}}$

Exercise 19b – p. 329

1. $2x(x - 3)^2 + 2x^2(x - 3)$
$= 2x(x - 3)(2x - 3)$
2. $\sqrt{(x - 6)} + \frac{x}{2\sqrt{(x - 6)}} = \frac{3(x - 4)}{2\sqrt{(x - 6)}}$
3. $(x - 2)^5 + 5(x + 2)(x - 2)^4$
$= (x - 2)^4(6x + 8)$
4. $(2x + 3)^3 + 6x(2x + 3)^2$
$= (2x + 3)^2(8x + 3)$
5. $4(x + 1)^2(x - 1)^3 + 2(x + 1)(x - 1)^4$
$= 2(x + 1)(x - 1)^3(3x + 1)$
6. $3\sqrt{x}(x - 3)^2 + \frac{1}{2\sqrt{x}}(x - 3)^3$
$= \frac{(x - 3)^2(7x - 3)}{2\sqrt{x}}$
7. $\frac{4(x - 3)(x + 5)^3 - (x + 5)^4}{(x - 3)^2}$
$= \frac{(x + 5)^3(3x - 17)}{(x - 3)^2}$
8. $\frac{(3x + 2)^2 - 6x(3x + 2)}{(3x + 2)^4}$
$= \frac{2 - 3x}{(3x + 2)^3}$

9. $\frac{4\sqrt{x}(2x - 7) - (2x - 7)^2\left(\frac{1}{2\sqrt{x}}\right)}{x}$
$= \frac{(2x - 7)(6x + 7)}{2x\sqrt{x}}$
10. $3x^2\sqrt{(x - 1)} + \frac{x^3}{2\sqrt{(x - 1)}}$
$= \frac{x^2(7x - 6)}{2\sqrt{(x - 1)}}$
11. $(x + 3)^{-1} - x(x + 3)^{-2} = 3(x + 3)^{-2}$
12. $2x(2x - 3)^2 + 4x^2(2x - 3)$
$= 2x(2x - 3)(4x - 3)$

Exercise 19c – p. 330

1. $\frac{2x(x - 3) - (x - 3)^2}{x^2} = \frac{(x - 3)(x + 3)}{x^2}$
2. $\frac{(x + 3)(2x) - x^2}{(x + 3)^2} = \frac{x(x + 6)}{(x + 3)^2}$
3. $\frac{-x^2 - 2x(4 - x)}{x^4} = \frac{x - 8}{x^3}$
4. $\frac{2x^3(x + 1) - (x + 1)^2(3x^2)}{x^6}$
$= \frac{-(x + 1)(x + 3)}{x^4}$
5. $\frac{4(1 - x)^3 + 12x(1 - x)^2}{(1 - x)^6} = \frac{4(1 + 2x)}{(1 - x)^4}$
6. $\frac{(x - 2)(4x) - 2x^2}{(x - 2)^2} = \frac{2x(x - 4)}{(x - 2)^2}$
7. $\frac{\frac{5}{3}x^{2/3}(3x - 2) - 3x^{5/3}}{(3x - 2)^2} = \frac{2x^{2/3}(3x - 5)}{3(3x - 2)^2}$
8. $\frac{-3(1 - 2x)^2}{x^4}$
9. $\frac{(3x - 2)(x + 1)^{3/2}}{2x^2}$

Mixed Exercise 19 – p. 331

1. $\frac{3x + 2}{2\sqrt{(x + 1)}}$
2. $6x(x^2 - 8)^2$
3. $\frac{1 - x^2}{(x^2 + 1)^2}$
4. $\frac{-4x^3}{3(2 - x^4)^{2/3}}$
5. $\frac{2x}{(x^2 + 2)^2}$
6. $\frac{1}{2}x(5\sqrt{x} - 8)$
7. $6x(x^2 - 2)^2$
8. $\frac{1 - 2x}{2\sqrt{(x - x^2)}}$
9. $\frac{\sqrt{x} + 2}{2(\sqrt{x} + 1)^2}$
10. $\frac{x(5x - 8)}{2\sqrt{(x - 2)}}$
11. $\frac{-(3x + 4)}{2x^3\sqrt{(x + 1)}}$
12. $6x^5(x^2 + 1)^2(2x^2 + 1)$
13. $\frac{x}{\sqrt{(x^2 - 8)}}$
14. $x^2(5x^2 - 18)$
15. $6x(x^2 - 6)^2$
16. $\frac{-(x^2 + 6)}{(x^2 - 6)^2}$
17. $-8x^3(x^4 + 3)^{-3}$
18. $\frac{(2 - x)^2(2 - 7x)}{2\sqrt{x}}$
19. $\frac{2 + 5x}{2\sqrt{x}(2 - x)^4}$
20. $(x - 2)(3x - 4)$
21. $30x^2(2x^3 + 4)^4$

CHAPTER 20

Exercise 20a – p. 336

1. (a) 7.39 (b) 0.368 (c) 4.48
(d) 0.741
2. (a) $2e^x$ (b) $2x - e^x$ (c) e^x
3. $e^2 - 2$
4. $2 + 2e$
5. 1
6. 0

Exercise 20b – p. 338

1. (a) 1.24 (b) 0.183 (c) 2.85
2. (a) $\ln x - \ln(x - 1)$
(b) $\ln 5 + 2\ln x$
(c) $\ln(x + 2) + \ln(x - 2)$
(d) $\ln \sin x - \ln \cos x$
(e) $2\ln \sin x$
(f) $\frac{1}{2}\ln(x + 1) - \frac{1}{2}\ln(x - 1)$
3. (a) $\ln\frac{x}{(1 - x)^2}$ (b) $\ln\frac{e}{x}$
(c) $\ln(\sin x \cos x) = \ln(\frac{1}{2}\sin 2x)$
(d) $\ln x^2\sqrt{(x - 1)}$
4. (a) $\frac{2}{3}\ln x$ (b) $5\ln x$

5. (a) 2.10 (b) 0 (c) 1.05
   (d) 0
6. (a) $x = 1$ (b) $x = 3$
   (c) $x = 0$

Exercise 20c – p. 341

1. (a) $\frac{3}{x}$ (b) $\frac{1}{x}$ (c) $-\frac{2}{x}$
   (d) $-\frac{1}{2x}$ (e) $-\frac{5}{x}$ (f) $\frac{1}{2x}$
   (g) $-\frac{3}{2x}$ (h) $\frac{5}{2x}$
2. (a) $(1, -1)$ (b) $(2^{1/3}, \{2 - 2\ln 2\})$
   (c) $(4, \{\ln 4 - 2\})$

Exercise 20d – p. 343

(pr ≡ product; f of f ≡ function of a function)

1. (a) pr (b) $u = e^x, v = x^2 + 1$
2. (a) f of f (b) $u = x^2 + 1, y = e^u$
3. (a) pr (b) $u = x, v = \ln x$
4. (a) f of f (b) $u = x + 1, y = e^{u/2}$
5. (a) pr (b) $u = e^x, v = \ln x$
6. (a) f of f (b) $u = 3 - x^2, y = \ln u$
7. (a) f of f (b) $u = \ln x, y = u^2$
8. (a) f of f (b) $u = -2x, y = e^u$
9. (a) f of f (b) $u = \ln x, y = \frac{1}{u}$
10. $fg(x) = e^{2x}$; $gf(x) = e^{x^2}$
11. (a) $\left(\frac{1}{x}\right)^2$ (b) $\ln x^2$
    (c) $\ln\left(\frac{1}{x}\right)$ (d) $(\ln x)^2$
    (e) $\ln\left(\frac{1}{x}\right)^2$ (f) $\left(\frac{1}{x}\right)^2$

Exercise 20e – p. 345

1. $e^x(x + 1)$
2. $x(2\ln x + 1)$
3. $e^x(x^3 + 3x^2 - 2)$
4. $12x\ln(x - 2) + \frac{6x^2}{(x - 2)}$
5. xe^x
6. $x\ln x + \frac{(x^2 + 4)}{2x}$
7. $\frac{4 + 3x}{2\sqrt{(2 + x)}}$
8. $\frac{1}{2}\ln(x - 5) + \frac{x}{2(x - 5)}$
9. $(x^2 + 2x - 2)e^x$
10. $\frac{1 - x}{e^x}$
11. $\frac{e^x(x - 2)}{x^3}$
12. $\frac{1 - 3\ln x}{x^4}$
13. $\frac{x\ln x - 2(x + 1)}{2x\sqrt{(x + 1)}(\ln x)^2}$
14. $\frac{e^x(x^2 - 2x - 1)}{(x^2 - 1)^2}$
15. $\frac{-2}{(e^x - e^{-x})^2}$
16. $4e^{4x}$
17. $\frac{2x}{x^2 - 1}$
18. $2xe^{x^2}$
19. $-6e^{(1-x)}$ or $-6e(e^{-x})$
20. $2xe^{(x^2+1)}$
21. $\frac{1}{2(x + 2)}$
22. $\frac{2\ln x}{x}$
23. $\frac{-1}{x(\ln x)^2}$
24. $\frac{1}{2}\sqrt{(e^x)}$

Mixed Exercise 20 – p. 345

1. $1 + \ln x$
2. $\frac{8}{3}(4x - 1)^{-1/3}$
3. $\frac{e^x(x - 2)}{(x - 1)^2}$
4. $\frac{-(x^3 + 4)}{2x^3\sqrt{(1 + x^3)}}$
5. $\frac{(x - 1)\ln(x - 1) - x\ln x}{x(x - 1)\{\ln(x - 1)\}^2}$
6. $3(\ln 10)10^{3x}$
7. $\frac{2x}{(1 + x^2)^2}$
8. $\frac{2}{x^2e^{2/x}}$
9. $\frac{-e^x}{1 - e^x}$ or $\frac{e^x}{e^x - 1}$
10. $3x^2e^{3x}(x + 1)$
11. $\frac{4}{5(2x - 1)^2} - \frac{6}{5(x - 3)^2}$
    $= \frac{2(3 - 2x^2)}{(2x - 1)^2(x - 3)^2}$

12. $\frac{e^{x/2}(x - 10)}{2x^6}$
13. $\frac{2}{x} - \frac{1}{x + 3} - \frac{2x}{x^2 - 1}$
14. $\frac{5x + 9}{x(x + 3)}$
15. $\frac{4}{x}(\ln x)^3$
16. $\frac{(x + 3)^2(x^2 - 6x + 6)}{(x^2 + 2)^2}$
17. $\frac{e^x - 1}{2\sqrt{(e^x - x)}}$
18. $\frac{8x}{x^2 + 1}$
19. $\frac{dy}{dx} = \frac{4}{(1 - 2x)^2}$; $\frac{d^2y}{dx^2} = \frac{16}{(1 - 2x)^3}$
20. $\frac{dy}{dx} = \frac{1}{x(x + 1)}$; $\frac{d^2y}{dx^2} = \frac{(2x + 1)}{x^2(x + 1)^2}$
21. $\frac{dy}{dx} = \frac{-4e^x}{(e^x - 4)^2}$; $\frac{d^2y}{dx^2} = \frac{4e^x(e^x + 4)}{(e^x - 4)^3}$

CHAPTER 21

Exercise 21a – p. 350

1. (a) $2\sin 2\theta\cos\theta$
   (b) $2\cos 4\theta\cos\theta$
   (c) $2\cos 3\theta\sin\theta$
   (d) $-2\sin 4\theta\sin 3\theta$
   (e) $2\cos 4A\sin A$
   (f) $+2\sin 3A\sin 2A$
   (g) $2\sin(A + 10°)\cos(A - 10°)$
   (h) $2\sin(\theta + 45°)\cos(\theta - 45°)$
2. (a) $\sin 3\theta + \sin\theta$
   (b) $\cos 5\theta + \cos\theta$
   (c) $\sin 5\theta + \sin 3\theta$
   (d) $\cos 4\theta - \cos 2\theta$
   (e) $\cos 2\theta - \cos 6\theta$
   (f) $\frac{1}{2}\cos 5\theta + \frac{1}{2}\cos 3\theta$
3. (a) $\cos\theta$ (b) $-\sqrt{2}\cos x$
   (c) 0
4. (a) $\frac{1}{2}\sqrt{6}$ (b) $\frac{1}{2}\sqrt{6}$
5. (a) $\pm\frac{1}{2}\pi + 2n\pi, \pm\frac{1}{6}\pi + \frac{2}{3}n\pi$
   (b) $\pm\frac{1}{4}\pi + n\pi, n\pi$
   (c) $\frac{1}{2}n\pi$ (d) $\frac{1}{3}n\pi, \frac{1}{2}\pi + n\pi$
6. $\cos 2\theta(2\cos\theta + 1), \pm\frac{1}{4}\pi + n\pi,$
   $\pm\frac{2}{3}\pi + 2n\pi$
7. $4\cos\theta\sin 4\theta\cos 2\theta, \pm\frac{1}{8}\pi + \frac{1}{2}n\pi,$
   $\pm\frac{1}{4}\pi + n\pi$
8. (a) 20°, 90°, 100°, 140°,
   (b) $22\frac{1}{2}°$, 90°, $112\frac{1}{2}°$,
   (c) 0°, 30°, 40°, 60°, 80°, 90°, 120°, 150°, 160°, 180°,
   (d) 30°, 45°, 135°, 150°,
   (e) 0°, 60°, 105°, 120°, 165°, 180°,
   (f) 10°, 130°

Mixed Exercise 21 – p. 352

1. $\frac{3\sqrt{3} - 4\sqrt{2}}{2(3 + \sqrt{6})}$ (b) $\frac{1}{6}(1 - 2\sqrt{6})$
2. $\pm 70.5°$
3. $\pm\frac{1}{4}\pi + n\pi, \frac{1}{6}\pi + 2n\pi, \frac{5}{6}\pi + 2n\pi$
5. $n\pi, \pm\frac{1}{3}\pi + 2n\pi,$
7. $\pm\frac{1}{6}\pi + \frac{2}{3}n\pi, \frac{1}{12}\pi + n\pi, \frac{5}{12}\pi + n\pi$
8. $\sqrt{2} - 1$
9. $1 - 2y^2 = x$
10. $x = 60°, 300°, y = 120°, 240°$
12. $\frac{1}{4}\pi + 2n\pi, \frac{3}{4}\pi + 2n\pi$
13. 257.6°, 349.8°
14. $y = \frac{1}{2}x, |x| \leqslant 2$
15. 0, 36°, 108°, 180°, 252°, 324°, 360°
16. $ax = a^2 - 2y^2$
17. 49.1°, 229.1°
19. $\pm\frac{1}{6}\pi + \frac{2}{3}n\pi, n\pi$
20. $\alpha = 90° - \theta$; 135°, 315°, $22\frac{1}{2}°$, $112\frac{1}{2}°$, $202\frac{1}{2}°$, $292\frac{1}{2}°$

CHAPTER 22

Exercise 22a – p. 358

1. (a) 2, 30° (b) $\sqrt{10}$, 71.6°
   (c) 5, 36.9°
2. $\sqrt{2}\cos(2\theta + \frac{1}{4}\pi)$
3. $\sqrt{29}\sin(3\theta + 21.8°)$
4. $-2\sin(\theta - \frac{1}{6}\pi)$; max 2 at $\theta = 300°$, min -2 at $\theta = 120°$
5. $25\cos(\theta + 73.7°)$; max 28 at $\theta = 286.3°$, min -22 at $\theta = 106.3°$
6. $\sqrt{2}, -\sqrt{2}$; $-\frac{1}{\sqrt{2}}$ max, $\frac{1}{\sqrt{2}}$ min
7. $-\sqrt{\frac{2}{3}}$ max, $\sqrt{\frac{2}{3}}$ min
8. (a) $45° + 360n°$
   (b) $118.1° + 360n°, -36.9° + 360n°$
   (c) $360n°, 360n° - 143.2°$
   (d) $360n°, 360n° - 53.1°$

Exercise 22b – p. 361

1. $\frac{2}{3}n\pi$
2. $\frac{1}{5}n\pi$

3. $\frac{1}{2}(4n-1)\pi, \frac{1}{10}(4n+1)\pi$
4. $\frac{1}{18}(2n+1)\pi$
5. $2n\pi, \frac{1}{7}(2n+1)\pi$
6. $\frac{1}{2}n\pi, \frac{1}{3}n\pi$
7. $\frac{2}{3}n\pi - \frac{1}{6}\pi, \frac{2}{5}n\pi$
8. $\frac{1}{6}(2n+1)\pi$
9. $0, \frac{1}{5}\pi, \frac{2}{5}\pi, \frac{1}{2}\pi, \frac{3}{5}\pi, \frac{4}{5}\pi, \pi$
10. $\frac{1}{10}\pi, \frac{3}{10}\pi, \frac{1}{2}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi$
11. $0, \frac{1}{9}\pi, \frac{2}{5}\pi, \frac{1}{3}\pi, \frac{5}{9}\pi, \frac{7}{9}\pi, \frac{4}{5}\pi, \pi$
12. $0, \frac{2}{11}\pi, \frac{4}{11}\pi, \frac{6}{11}\pi, \frac{8}{11}\pi, \frac{10}{11}\pi$
13. $-140°, -60°, -20°, 100°$
14. $-115°, -25°, 65°, 155°$
15. $-150°, -110°, 10°, 130°$
16. $-110°, -30°, 10°, 130°$
17. $-150°, -60°, 30°, 120°$
18. $-155°, -140°, -65°, 25°, 40°, 115°$

Exercise 22c – p. 366

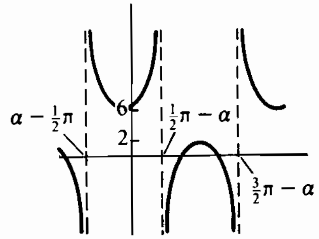
1. (a) $\frac{1}{3}\pi$ (b) $-\frac{1}{2}\pi$ (c) $\frac{1}{2}\pi$
   (d) $-\frac{1}{3}\pi$ (e) $\frac{2}{3}\pi$ (f) $-\frac{1}{4}\pi$
2. (a) $\frac{1}{4}\pi$ (b) $\frac{1}{2}\pi$
3. (a) $\frac{1}{2}\pi$ (b) $\dfrac{2x}{x^2+1}$ (c) $\pm\frac{1}{2}\pi$
   (d) $\frac{7}{11}$
5. (a) $\frac{2}{9}$ (b) 0 (c) ± 1

Exercise 22d – p. 368

1. (a) $\frac{1}{2}$ (b) $\dfrac{\theta^2}{1-2\theta^2}$ (c) $\dfrac{\theta}{2-\theta^2}$
   (d) 2 (e) 1 (f) 1
3. (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\sqrt{\frac{3}{2}}$ (d) 1

Mixed Exercise 22 – p. 368

1. $5\sin(\theta-\alpha)$ where $\tan\alpha = \frac{3}{4}$;
   1 min, $-\frac{7}{3}$ max, $\pm\infty$
2. (a) $\frac{2}{9}$ (b) $\dfrac{9}{\sqrt{85}}$ (c) $\dfrac{2}{\sqrt{85}}$
3. $\sqrt{2}\sin(2\theta - \frac{1}{4}\pi)$; $\frac{3}{8}\pi$
5. $-\frac{1}{4}$
6. $\sqrt{2}\cos(x - \frac{1}{4}\pi)$; $\frac{1}{4}\pi$
7. $0, \frac{2}{5}\pi, \frac{4}{5}\pi, \frac{6}{5}\pi, \frac{8}{5}\pi, 2\pi$
8. $24°, 96°, 168°, 80°$
10. $\frac{1}{14}(1+2n)\pi$
11. $5\cos(x+\alpha)$ where $\tan\alpha = \frac{4}{3}$;
    $4 + 2\sec(x+\alpha)$



CHAPTER 23

Exercise 23a – p. 372

2. (a) $\cos x + \sin x$ (b) $\cos\theta$
   (c) $-3\sin\theta$ (d) $5\cos\theta$
   (e) $3\cos\theta - 2\sin\theta$
   (f) $4\cos x + 6\sin x$
3. (a) -1 (b) 1 (c) -1
   (d) 1 (e) $2(\pi-1)$ (f) 4
4. (a) $\frac{1}{6}\pi$ (b) $\frac{1}{6}\pi$ (c) $\frac{1}{4}\pi$
   (d) π
5. (a) $(\frac{1}{3}\pi, \sqrt{3} - \frac{1}{3}\pi)$, max;
   $(\frac{5}{3}\pi, -\sqrt{3} - \frac{5}{3}\pi)$, min
   (b) $(\frac{1}{6}\pi, \frac{1}{6}\pi + \sqrt{3})$, max;
   $(\frac{5}{6}\pi, \frac{5}{6}\pi - \sqrt{3})$, min
6. $y + \theta = 3 + \frac{1}{2}\pi$
7. $2\pi y + x = 2\pi^3 - \pi$
8. (0, 1)

Exercise 23b – p. 375

1. $4\cos 4x$
2. $2\sin(\pi - 2x)$ or $2\sin 2x$
3. $\frac{1}{2}\cos(\frac{1}{2}x + \pi)$ or $-\frac{1}{2}\cos\frac{1}{2}x$
4. $\dfrac{x\cos x - \sin x}{x^2}$
5. $-\dfrac{(\cos x + \sin x)}{e^x}$
6. $\dfrac{\cos x}{2\sqrt{(\sin x)}}$
7. $2\sin x\cos x$ or $\sin 2x$
8. $\cos^2 x - \sin^2 x$ or $\cos 2x$
9. $\cos x\, e^{\sin x}$
10. $-\tan x$
11. $e^x(\cos x - \sin x)$
12. $x^2\cos x + 2x\sin x$
13. $2x\cos x^2$
14. $-\sin x\, e^{\cos x}$
15. $3\cot x$

16. $\dfrac{\sin x}{\cos^2 x} = \sec x\tan x$
17. $\sec^2 x$
18. $\dfrac{-\cos x}{\sin^2 x} = -\operatorname{cosec} x\cot x$
19. $\dfrac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$

Exercise 23c – p. 377

3. $2x\arcsin x + \dfrac{x^2}{\sqrt{(1-x^2)}}$
4. $\dfrac{2\arctan x}{1+x^2}$
5. $\dfrac{3}{\sqrt{(1-9x^2)}}$
6. $\dfrac{e^{\arctan x}}{1+x^2}$
7. $\dfrac{1}{(\arcsin x)\sqrt{(1-x^2)}}$
8. $\dfrac{-3x^2}{\sqrt{(1-x^6)}}$

Exercise 23d – p. 378

1. $\dfrac{1}{2x\sqrt{(1+\ln x)}}$
2. $-2x\sin(x^2+3)$
3. $(3x^2-1)\,e^{(x^3-x)}$
4. $\dfrac{1}{x}\cos(\ln x)$
5. $-2x\tan(x^2)$
6. $4x\,e^{x^2}(1+e^{x^2})$
7. $\dfrac{-\sin x\cos x}{\sqrt{(3-\sin^2 x)}}$
8. $\dfrac{2\sec^2 x\ln(\tan x)}{\tan x}$
9. $\dfrac{-x\,e^{\sqrt{(2-x^2)}}}{\sqrt{(2-x^2)}}$
10. $-4x\cos(x^2+1)\sin(x^2+1)$ or
    $-2x\sin\{2(x^2+1)\}$

Exercise 23e – p. 379

1. $e^{x^2}(\cos x + 2x\sin x)$
2. $-\left(\dfrac{x\sin 2x + \cos^2 x}{x^2}\right)$
3. $\ln\sin x + x\cot x$
4. $\dfrac{2x^2\ln x - x^2 + 1}{x(\ln x)^2}$
5. $\dfrac{e^x(x^2+x+2)}{\sqrt{(x^2+2)}}$
6. $\dfrac{2\cos x + x\sin x\ln x}{2x\sqrt{(\cos^3 x)}}$
7. $e^{\sin x}[1 + (x+1)\cos x]$
8. $\dfrac{2x(\sin x - x\cos x)}{\sin^3 x}$
9. $\dfrac{e^{x^2}}{x}(1 + 2x^2\ln x)$

Mixed Exercise 23 – p. 380

1. (a) $-4\cos 4\theta$ (b) $1 + \sin\theta$
   (c) $3\sin^2\theta\cos\theta + 3\cos 3\theta$
2. (a) $3x^2 + e^x$ (b) $2e^{(2x+3)}$
   (c) $e^x(\sin x + \cos x)$
3. $-\dfrac{3}{x}$ (b) $-\dfrac{2}{x}$ (c) $\dfrac{1}{2x}$
4. (a) $3\cos x + e^{-x}$ (b) $\dfrac{1}{2x} + \frac{1}{2}\sin x$
   (c) $4x^3 + 4e^x - \dfrac{1}{x}$
   (d) $-\frac{1}{2}(e^{-x} + x^{-3/2}) - \dfrac{1}{x}$
5. $1 + \dfrac{1}{x} + \ln x$
6. $3\sin 6x$
7. $\frac{8}{3}(4x-1)^{-1/3}$
8. $9 - 18\sqrt{x} + 8x$
9. $(x^4 + 4x^3 + 3)/(x+1)^4$
10. $\dfrac{(x-1)\ln(x-1) - x\ln x}{x(x-1)\{\ln(x-1)\}^2}$
11. $-1/\sin x\cos x$ or $-2\operatorname{cosec} 2x$
12. $2x\sin x + x^2\cos x$
13. $e^x(x-2)/(x-1)^2$
14. $2\cos x/(1-\sin x)^2$
15. $x(5x-4)/2\sqrt{(x-1)}$
16. $-2(1-x)^2(2x+1)$
17. $\dfrac{3}{2(x+3)} - \dfrac{x}{x^2+2}$
18. $e^x\{(x-5)\cos x - x\sin x\}/x^6$
19. $\cos^2 x(4\cos^2 x - 3)$
20. $-\sin 2x\, e^{\cos^2 x}$
21. $e^{\arcsin x}/\sqrt{(1-x^2)}$
22. $-3(\arccos x)^2/\sqrt{(1-x^2)}$
23. $e^x/(1+e^{2x})$
24. $a^x(1 + x\ln a)$
25. $2x/(x^2+1)\ln a$
26. (a) $x = \ln 3$ (b) $x = 1$ (*not* -1)
    (c) $x = \frac{1}{4}$

27. (a) 1 (b) $y - x = 1 - \frac{1}{2}\pi$
(c) $y + x = 1 + \frac{1}{2}\pi$
28. (a) $1 + e$ (b) $y = x(1 + e)$
(c) $y(1 + e) + x = (1 + e)^2 + 1$
29. (a) 2 (b) $y = 2x + 1$
(c) $2y + x = 2$
30. (a) -1 (b) $x + y = 3$
(c) $x - y + 1 = 0$
31. (a) $(\frac{1}{2}\pi, 0)$, min; $(\frac{3}{2}\pi, 2)$, max
(b) $(\frac{1}{6}\pi, \{\frac{1}{12}\pi + \frac{1}{2}\sqrt{3}\})$, max;
$(\frac{5}{6}\pi, \{\frac{5}{12}\pi - \frac{1}{2}\sqrt{3}\})$, min
(c) $(\ln 3, \{3 - 3\ln 3\})$, min; only one turning point
32. (a) $(\frac{1}{6}\pi, \{\frac{1}{2}\pi - \sqrt{3}\})$ (b) $(1, -1)$

CHAPTER 24

Exercise 24a – p. 386

1. $2x + 2y\frac{dy}{dx} = 0$
2. $2x + y + (x + 2y)\frac{dy}{dx} = 0$
3. $2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx}$
4. $-\frac{1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} = e^y\frac{dy}{dx}$
5. $-\frac{2}{x^3} - \frac{2}{y^3}\frac{dy}{dx} = 0$
6. $\frac{x}{2} - \frac{2y}{9}\frac{dy}{dx} = 0$
7. $\cos x + \cos y\frac{dy}{dx} = 0$
8. $\cos x\cos y - \sin x\sin y\frac{dy}{dx} = 0$
9. $e^y + xe^y\frac{dy}{dx} = 1$
10. $(1 + x)\frac{dy}{dx} = 2x - 1 - y$
11. $\frac{dy}{dx} = \pm\frac{1}{\sqrt{(2x + 1)}}$
12. $\frac{d^2y}{dx^2} = \pm\frac{x}{\sqrt{(2 - x^2)^3}}$
13. $\pm\frac{1}{4}\sqrt{2}$
15. (a) $xx_1 - 3yy_1 = 2(y + y_1)$
(b) $x(2x_1 + y_1) + y(2y_1 + x_1) = 6$
17. $3x + 12y - 7 = 0$

Exercise 24b – p. 389

1. $\frac{dy}{dx} = \frac{x - y}{x\ln x}$
2. $\frac{dy}{dx}\left(\frac{1}{y + 1} - \ln x\right) = \frac{y}{x}$
3. $\frac{1}{y}\frac{dy}{dx} = \ln(x + x^2) + \frac{1 + 2x}{1 + x}$
4. $\frac{-(x^2 + 8)}{(x + 2)^2(x - 4)^2}$
5. $\frac{2x(3 - x)}{(x - 1)^2(x - 3)^2}$
6. $-(1 - x)^4(7x^2 - 2x + 10)$
7. $2x\sqrt{\frac{(x - 3)}{(x + 1)}}$
8. $\frac{x}{(x + 2)^2(x^2 - 1)}\left\{\frac{1}{x} - \frac{2}{x + 2} - \frac{2x}{x^2 - 1}\right\}$
$= \frac{-(3x^3 + 2x^2 - x + 2)}{(x + 2)^3(x^2 - 1)^2}$
9. $-\frac{y}{2}\left\{\frac{9x^2 - 4x + 12}{(x^2 + 4)(3x - 2)}\right\}$
$= \frac{-(9x^2 - 4x + 12)}{2\sqrt{(x^2 + 4)^3(3x - 2)^3}}$

Exercise 24c – p. 396

1. (a) $\frac{1}{4t}$ (b) $-\tan\theta$ (c) $-\frac{4}{t^2}$
2. $\frac{dy}{dx} = 2t - t^2$; $\frac{3}{4}$
3. (a) $\frac{3}{2}t$ (b) $\frac{3}{2}\sqrt{x}$
(c) $x = t^2 \Rightarrow t = \sqrt{x}$
4. (a) $x = 2y^2$; $\frac{dy}{dx} = \frac{1}{4y} = \frac{1}{4t}$
(b) $x^2 + y^2 = 1$; $\frac{dy}{dx} = -\frac{x}{y} = -\tan\theta$
(c) $xy = 4$; $\frac{dy}{dx} = -\frac{4}{x^2} = -\frac{4}{t^2}$
5. $(-\frac{1}{3}\sqrt{3}, \frac{2}{9}\sqrt{3})$, max; $(\frac{1}{3}\sqrt{3}, -\frac{2}{9}\sqrt{3})$, min
6. $(n\pi + \frac{1}{2}\pi, 1)$
7. $2x + y + 2 = 0$
8. $ty + t^4 = t^3x + 1$
9. $ty = 2x + 2t^2$
10. $y = x$; $(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$
11. (a) $\frac{\cos t}{e^t}$ (b) $y = \sin(\ln x)$
(c) $\frac{\cos(\ln x)}{x}$

12. $t^2y + x = 2t$; $(2t, 0)$, $\left(0, \frac{2}{t}\right)$; $A = 2$
13. $2y + tx = 8t + t^3$; $(8 + t^2, 0)$,
$(0, \frac{1}{2}t\{8 + t^2\})$; $A = |\frac{1}{4}t(8 + t^2)^2|$

Mixed Exercise 24 – p. 397

1. (a) $4y^3\frac{dy}{dx}$ (b) $y^2 + 2xy\frac{dy}{dx}$
(c) $-\frac{1}{y^2}\frac{dy}{dx}$ (d) $\ln y + \frac{x}{y}\frac{dy}{dx}$
(e) $\cos y\frac{dy}{dx}$ (f) $e^y\frac{dy}{dx}$
(g) $\frac{dy}{dx}\cos x - y\sin x$
(h) $(\cos y - y\sin y)\frac{dy}{dx}$
2. $\frac{x}{2y}$
3. $-\frac{y^2}{x^2}$
4. $-\frac{2y}{3x}$
5. $-\frac{x^3(3x^3 + 6x^2 - 2x - 8)}{(1 + x)^4}$
6. $-\frac{(x - 1)^3(2x^2 - 3x - 15)}{x^6(x + 3)^2}$
7. $-\frac{y(y + 1)(3x + 2)}{x(x + 1)(y + 2)}$
8. $3t/2$
9. $\frac{t}{t + 1}$
10. $-\frac{3}{2}\cos\theta$
11. $-\frac{1}{t^2}$
12. $\frac{2y + 7}{2y - 2x + 3}$
13. $2t - t^2$

CHAPTER 25

Exercise 25a – p. 402

1. (a) 10.003 333; 10.003 332
(b) 2.083 333; 2.080 084
(c) 3.979 167; 3.979 057
2. (a) 0.857 (b) 0.515 (c) 0.719
3. $\left\{\frac{x}{1 + x} + \ln(1 + x)\right\}\delta x$; 0.75
4. $(\sec^2 x)\delta x$; $1 + \frac{1}{16}\pi$
5. 2.0125
6. $\frac{1}{8}\sqrt{2}$; $\frac{1}{8}a\sqrt{2}$

Exercise 25b – p. 405

1. 0.099 cm/s
2. 30 cm^3/s
3. 8 m^2/s
4. Decreasing at 0.126 cm/s
5. -2
6. $4a$ cm/minute

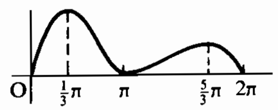
CONSOLIDATION D

Multiple Choice Exercise D – p. 411

1.	C	11.	A, D
2.	C	12.	B, C
3.	A	13.	C
4.	A	14.	T
5.	D	15.	F
6.	D	16.	F
7.	A	17.	F
8.	C	18.	T
9.	B, C	19.	T
10.	A, B	20.	T

Miscellaneous Exercise D – p. 413

1. (a) $\frac{1}{(2x + 1)^2}$ (b) $\frac{2x}{\sqrt{(1 - x^4)}}$
2. 45°, 90°, 135°
3. (a) $4x\cos(2x^2)$ (b) $2^x\ln 2$
4. $0, \frac{1}{3}\pi, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, \frac{5}{3}\pi$
5. (a) $\frac{1 + \cos x + \sin x}{(1 + \cos x)^2}$
(b) $\frac{1}{1 - x^2}$
8. (a) $-\frac{1}{4}$ (b) 4
9. $2n\pi + \frac{1}{2}\pi$, $2n\pi - \frac{1}{6}\pi$
10. (b) 5.25 cm/s
11. (a) $80(4x - 1)^{19}$ (b) $\frac{1}{2\sqrt{x}(1 + x)}$
12. $(\frac{25}{4}a, -\frac{9}{4}a)$
13. $e^x\left\{\frac{2\cos 2x}{1 + \sin 2x} + \ln(1 + \sin 2x)\right\}$
14. $x = 0, \pi, 2\pi$; $(\frac{1}{3}\pi, \frac{3}{4}\sqrt{3})$, $(\pi, 0)$,
$(\frac{5}{3}\pi, -\frac{3}{4}\sqrt{3})$



15. $\theta = 0, \frac{1}{10}\pi, \frac{3}{10}\pi, \frac{1}{3}\pi$
16. $x + y + 3 = 0$
17. $R = 5, \tan\alpha = \frac{4}{3}$
 (a) 2.1 rad, 2.9 rad
 (b) 0.3 rad, 0.9 rad, 2.4 rad, 4.5 rad
 A is (0, 1); B is (−2.2, 5);
 C is $(0.9, \frac{5}{6})$
18. 2
19. $2\sin 2x\cos x$;
 $x = -\frac{1}{2}\pi, -\frac{1}{3}\pi, 0, \frac{1}{3}\pi, \frac{1}{2}\pi, \pi$;
 $x = \frac{1}{2}n\pi, 2n\pi \pm \frac{1}{3}\pi$
20. $\theta = -\frac{1}{4}\pi, \frac{5}{4}\pi$ (or $\frac{7}{4}\pi$ etc);
 $x + y + 2 = 0, x - y = 4$
21. $3x - y + (y - x)\dfrac{dy}{dx} = 4$; (1, 3), (3, 1)
23. $\dfrac{1}{48\sqrt{3}}; \frac{1}{48}a\sqrt{3}$
24. (a) $1 + \theta - \frac{1}{2}\theta^2$ (b) $\frac{1}{2}$
 (c) $\frac{1}{2}\theta$ (d) $\sqrt{2} - \theta$
25. 1600π cm^3/s
26. 0.002 cm/s; 2.16 cm^3/s
27. $A = -1, B = 11, C = 3$

CHAPTER 26

Exercise 26a – p. 424

1. (a) (1, 5) (b) $(1.6, \frac{19}{2})$
 (c) (5, 1) (d) (11, −18)
3. (a) $\dfrac{\sqrt{5}}{5}$ (b) $\frac{1}{5}$
 (c) $\dfrac{|ma - b + c|}{\sqrt{(m^2 + 1)}}$ (d) $\dfrac{3\sqrt{2}}{2}$
 (e) $\dfrac{|ax + by + c|}{\sqrt{(a^2 + b^2)}}$ (f) $\dfrac{|c|}{\sqrt{(a^2 + b^2)}}$
4. (a) opposite (b) same
 (c) same
5. (a) $\frac{15}{16}$ (b) $\frac{5}{31}$
 (c) $\left|\dfrac{(a_1b_2 - a_2b_1)}{(a_1a_2 + b_2b_1)}\right|$
6. $\frac{3}{14}, \frac{9}{83}, \frac{3}{29}; \dfrac{9}{\sqrt{170}}, \dfrac{9}{\sqrt{41}}, \dfrac{3}{\sqrt{5}}$
7. $5y = x, 5x + y = 0$
8. (a) (−1, 8) (b) (−11, 10)
 (c) (−3, −14)
9. $4X^2 + Y^2 - 14X + 12Y - 4XY + 11 = 0$
10. no

11. $\dfrac{|5X - 12Y + 3|}{13}, \dfrac{|3X + 4Y - 6|}{5}$;
 $14X + 112Y = 93$ and $64X - 8Y = 63$
12. $x + 2y = 7$

Exercise 26b – p. 434

1. (a) $\left(\dfrac{1}{y}\right) = a(x) + b$
 (b) $(y^2 - x) = b(y) - a$
 (c) $(y) = a(e^x/y) + b$
 (d) $(x) = (y^2 + y) - k; m = 1$
 (e) $(\ln y) = (n + 2)(\ln x) + \ln a$
 (f) $(x) = a(\ln y) - k$
 There are other alternatives.
2. $a = 2, b = -4$ (exactly)
3. $a = 6, b = 4$
4. $a = 0.5, b = -2$
5. $a = 30, b = 2$
6. $a = 2, b = \frac{1}{2}$
7. $a = 3, b = -2$ (or $a = -2, b = 3$)
8. $k = 4, n = 0.07$
9. $k = 5500, n = 1.5$
10. The 'incorrect' values are −0.26 and −0.40; the estimated correct values are −0.50 and −1.00

CHAPTER 27

Exercise 27a – p. 438

1. $y^2 = 2y + 12x - 13$
2. $x + y = 4$
3. $8x^2 + 9y^2 - 20x = 28$
4. $11y = 3x$ and $99x + 27y + 130 = 0$
5. $x^2 + y^2 - 6x - 10y + 30 = 0$
6. $4x - 3y = 26$ and $4x - 3y + 24 = 0$
7. $x^2 + y^2 = 1$

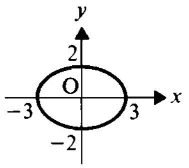
Exercise 27b – p. 441

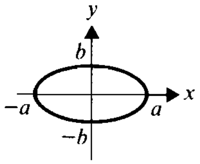
1. (a) $x^2 + y^2 - 2x - 4y = 4$
 (b) $x^2 + y^2 - 8y + 15 = 0$
 (c) $x^2 + y^2 + 6x + 14y + 54 = 0$
 (d) $x^2 + y^2 - 8x - 10y + 32 = 0$
2. (a) (−4, 1); 5
 (b) $(-\frac{1}{2}, -\frac{3}{2}); 3\sqrt{2}/2$
 (c) $(-3, 0); \sqrt{14}$
 (d) $(\frac{3}{4}, -\frac{1}{2}); \sqrt{5}/4$
 (e) (0, 0); 2
 (f) (2, −3); 3
 (g) (1, 3); 3
 (h) $(-1, \frac{1}{2}); \sqrt{69}/6$
3. (a) and (f)

Exercise 27c – p. 448

1. (a) yes (b) yes (c) no
 (d) yes
2. (a) $3y + 4x = 23$
 (b) $4y = x + 22$
 (c) $3y = 2x + 25$
3. $2x^2 + 2y^2 - 15x - 24y + 77 = 0$
4. $x^2 + y^2 - 8x + 6y = 0$
5. $x^2 + y^2 - 8x - 6y + 17 = 0$
6. $x^2 + y^2 - 4x - 14y - 116 = 0$
7. $x^2 + y^2 - 6y = 0$
8. $x^2 + y^2 + 4x + 4y + 4 = 0$
 or $196(x^2 + y^2) + 84(x + y) + 9 = 0$
9. $y = 0, 8y = 15x$
10. $x + 2y = 0$
11. $2x + y - 11 = 0, 2x + y + 9 = 0$
12. $(-\frac{4}{3}, \frac{2}{3}); 4\sqrt{5/3}$
13. (a) and (c)
14. $9 \pm 6\sqrt{5}$
15. $2mac = a^2 - c^2$

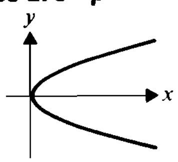
Exercise 27d – p. 453

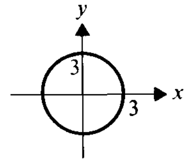
1. $3x^2 - y^2 - 60x + 252 = 0$; hyperbola
2. $16y = x^2$; parabola
3. $3x^2 + 4y^2 = 48$; ellipse
4. (a) parabola (b) circle
 (c) rectangular hyperbola
 (d) hyperbola (e) parabola
 (f) ellipse
5. 

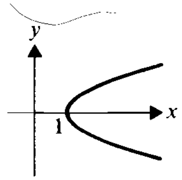
 (3, 0), (−3, 0) and (0, 2), (0, −2)
6. 

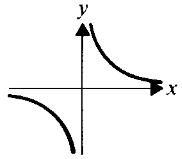
7. (a) (±4, 0), (0, ±2); (±4, 0)
 (b) they touch at (±4, 0)

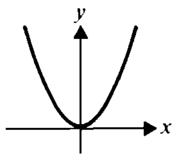
Exercise 27e – p. 458

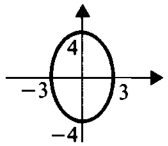
1. (a) 

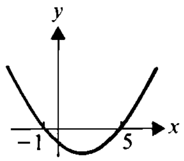
 (b) 

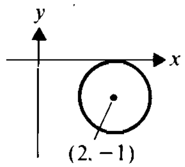
 (c) 

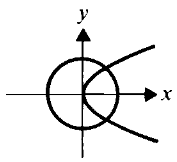
 (d) 

 (e) 

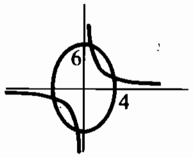
 (f) 

 (g) 

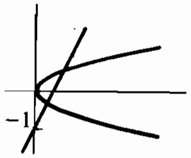
 (h) 

2. (a) 

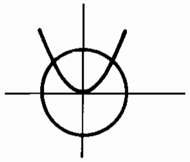
(b)



(c)



3.



$(2\sqrt{(2\sqrt{5}-2)}, 2\sqrt{5}-2)$,
$(-2\sqrt{(2\sqrt{5}-2)}, 2\sqrt{5}-2)$

4. (a) cuts (b) misses
 (c) touches (d) cuts
5. (a) $y^2 = 8x$ (b) $x - 2y + 8 = 0$
 (c) $2x - y(p+q) + 4pq = 0$
 (d) (18, 12) and (2, −4) (e) 2
 (f) (2, 4) and (2, −4) (g) (8, 8)

CHAPTER 28

Exercise 28a – p. 462

1. $\frac{1}{6}x^6 + K$
2. $-\frac{1}{4}x^{-4} + K$
3. $\frac{4}{5}x^{5/4}$
4. $-\frac{1}{2}x^{-2}$
5. $-\frac{2}{3}x^{-3/2}$
6. $2x^{1/2}$
7. $\frac{1}{2}x^2$
8. $\frac{3}{2}x^{2/3}$

Exercise 28b – p. 466

1. 4
2. $\frac{2}{7}(8\sqrt{2}, -1)$
3. $26\frac{2}{3}$
4. $12\frac{2}{3}$
5. 15
6. −2
7. $\frac{1}{2}$
8. 6
9. $6\frac{1}{7}$
10. 15

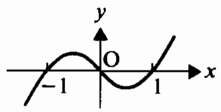
Exercise 28c – p. 468

Answers in square units.

1. $5\frac{1}{3}$
2. $12\frac{2}{3}$
3. $2\frac{2}{3}$
4. $13\frac{1}{2}$
5. $5\frac{1}{3}$
6. 60
7. $5\frac{1}{3}$
8. $4\frac{7}{8}$
9. 24

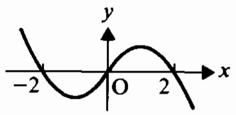
Exercise 28d – p. 470

1. $\frac{4}{3}$
2. $15\frac{1}{4}$
3. $\frac{1}{3}$
4.



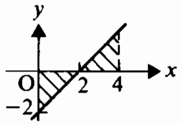
(a) $\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$

5.



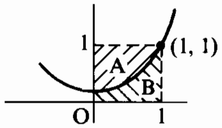
(a) 4 (b) 4 (c) 8

6. (a) −2 (b) 2 (c) 0



Exercise 28e – p. 471

1. (a) $49\frac{1}{2}$ (b) 2 (c) 3
2. 36
3. $2\frac{1}{3}$
4. $5\frac{1}{3}$
5. $4\frac{1}{2}$
6. (a) $42\frac{2}{3}$ (v) $42\frac{2}{3}$
7. $A = \int_0^1 x\,dy$ $B = \int_0^1 y\,dx$
 $A + B = 1$



Exercise 28f – p. 475

1. (a) (i) 24 (ii) 22
 (b) (i) $21\frac{1}{3}$ (ii) $21\frac{1}{3}$
2. (a) (i) 0.806 (ii) 0.705
 (b) (i) 0.704 (ii) 0.671
3. (a) (i) 1.462 (ii) 1.625
 (b) (i) 1.625 (ii) 1.679
4. (a) (i) 1.243 (ii) 1.282
 (b) (i) 1.290 (ii) 1.295
5.

22	0.705
21.33	0.671
21.33	0.667

Simpson's rule is more accurate each time and is exact for x^2 because this is a parabola.

Mixed Exercise 28 – p. 476

1. $\frac{1}{3}x^3 + \frac{1}{x} + K$
2. $\frac{2}{9}(3x+7)^{3/2} + K$
3. $\frac{2}{3}x^{3/2} + 2x^{1/2} + K = \frac{2}{3}\sqrt{x}(x+3) + K$
4. $-\frac{1}{4(4x-3)} + K$
5. $\frac{1}{4}x^4 - \frac{1}{2(1-x)^2} + K$
6. $\frac{2}{5}\sqrt{x}(x^2-5) + K$
7. $\frac{1}{5}$
8. 18
9. $33\frac{3}{4}$
10. $\frac{3}{4}$
11. $10\frac{2}{3}$
12. $\frac{1}{2}$
13. (a) 4 square units
 (b) $-4\frac{1}{2}$
14. (a) $27\frac{1}{2}$
 (b) $27\frac{1}{2}$
 (c) (a) is exact because the area is that of a trapezium.

CHAPTER 29

All indefinite integrals in this chapter require the term $+K$.

Exercise 29a – p. 478

1. $\frac{1}{4}e^{4x}$
2. $-4e^{-x}$
3. $\frac{1}{3}e^{(3x-2)}$
4. $-\frac{2}{5}e^{(1-5x)}$
5. $-3e^{-2x}$
6. $5e^{(x-3)}$
7. $2e^{(x/2+2)}$
8. $\frac{2^x}{\ln 2}$
9. $\frac{4^{(2+x)}}{\ln 4}$
10. $\frac{1}{2}e^{2x} - \frac{1}{2e^{2x}}$
11. $\frac{a^{(1-2x)}}{-2\ln a}$
12. $\frac{2^x}{\ln 2} + \frac{1}{3}x^3$
13. $\frac{1}{2}\{e^4 - 1\}$
14. $2\{e^2 - 1\}$
15. $1 - \frac{1}{e}$
16. $1 - e^2$

Exercise 29b – p. 481

Answers can be expressed in other forms also.

1. (a) $\frac{1}{2}\ln|x| + K$
 (b) $\frac{1}{2}\ln\{A|x|\}$
2. (a) $4\ln|x| + K$
 (b) $4\ln\{A|x|\}$ or $\ln(Ax^4)$
3. (a) $\frac{1}{3}\ln|3x+1| + K$
 (b) $\frac{1}{3}\ln A|3x+1|$ or $\ln A|\sqrt[3]{(3x+1)}|$
4. (a) $-\frac{3}{2}\ln|1-2x| + K$
 (b) $-\frac{3}{2}\ln A|1-2x|$
 or $\ln A|(1-2x)^{-3/2}|$
5. (a) $2\ln|2+3x| + K$
 (b) $\ln A(2+3x)^2$
6. (a) $-\frac{3}{2}\ln|4-2x| + K$
 (b) $\ln A|(4-2x)^{-3/2}|$
 or $\ln\frac{A}{|(4x-2x)^{3/2}|}$
 (or each of these results using $\ln|2-x|$)
7. (a) $-4\ln|1-x| + K$
 (b) $\ln\frac{A}{(1-x)^4}$
8. (a) $-\frac{5}{7}\ln|6-7x| + K$
 (b) $\ln\frac{A}{|(6-7x)^{5/7}|}$
9. $3\ln 2$
10. $\frac{1}{2}\ln 2 = \ln\sqrt{2}$
11. $2\ln 2 = \ln 4$
12. $\ln 2$

Exercise 29c – p. 482

1. $-\frac{1}{2}\cos 2x$
2. $\frac{1}{7}\sin 7x$
3. $\frac{1}{4}\tan 4x$
4. $-\cos(\frac{1}{4}\pi + x)$
5. $\frac{3}{4}\sin(4x - \frac{1}{2}\pi)$
6. $\frac{1}{2}\tan(\frac{1}{3}\pi + 2x)$
7. $-\frac{1}{4}\cot 4x$
8. $-\frac{2}{3}\cos(3x - \alpha)$
9. $-10\sin(\alpha - \frac{1}{2}x)$
10. $\frac{5}{4}\sec 4x$
11. $\frac{1}{3}\sin 3x - \sin x$
12. $\frac{1}{2}\tan 2x + \frac{1}{4}\cot 4x$
13. $\frac{1}{3}$
14. $-\frac{1}{4}$
15. 0
16. $\frac{1}{2}$

Exercise 29d – p. 484

1. $\arcsin x$
2. $\arctan(\frac{1}{2}x)$
3. $\arcsin(\frac{1}{2}x)$
4. $3\arctan x$
5. $\frac{5}{3}\arctan(\frac{1}{3}x)$
6. $\arcsin(\frac{1}{3}x)$
7. $\frac{1}{4}\arctan(\frac{1}{4}x)$
8. $5\,\text{arc sin}\,(x/\sqrt{2})$

Exercise 29e – p. 484

1. $\frac{1}{2}\cos(\frac{1}{2}\pi - 2x)$
2. $\frac{1}{4}e^{(4x-1)}$
3. $\frac{1}{7}\tan 7x$
4. $\frac{1}{2}\ln|2x - 3|$
5. $\sqrt{(2x-3)}$
6. $-1/\{3(3x-2)\}$
7. $5^x/\ln 5$
8. $-2\,\text{cosec}\,\frac{1}{2}x$
9. $\frac{1}{9}(3x-5)^3$
10. $\frac{1}{4}e^{(4x-5)}$
11. $\frac{1}{6}(4x-5)^{3/2}$
12. $-\frac{1}{3}\cot 3x$
13. $-\frac{3}{2}\ln|1-x|$
14. $10^{(x+1)}/\ln 10$
15. $\frac{1}{3}\sin(3x - \frac{1}{3}\pi)$
16. $\frac{2}{3}\sqrt{2}$
17. $2e(e-1)$
18. $-\frac{1}{2}$

Exercise 29f – p. 488

1. e^{x^4}
2. $-e^{\cos x}$
3. $e^{\tan x}$
4. e^{x^2+x}
5. $e^{(1-\cot x)}$
6. $e^{(x+\sin x)}$
7. $e^{(1+x^2)}$
8. $e^{(x^3-2)}$
9. $\frac{1}{10}(x^2-3)^5$
10. $-\frac{1}{3}(1-x^2)^{3/2}$
11. $\frac{1}{6}(\sin 2x + 3)^3$
12. $-\frac{1}{6}(1-x^3)^2$
13. $\frac{2}{3}(1+e^x)^{3/2}$
14. $\frac{1}{5}\sin^5 x$
15. $\frac{1}{4}\tan^4 x$
16. $\dfrac{1}{3(n+1)}(1+x^{n+1})^3$
17. $-\frac{1}{3}\cot^3 x$
18. $\frac{4}{9}(1+x^{3/2})^{3/2}$
19. $\frac{1}{12}(x^4+4)^3$
20. $-\frac{1}{4}(1-e^x)^4$
21. $\frac{2}{3}(1-\cos\theta)^{3/2}$
22. $\frac{1}{3}(x^2+2x+3)^{3/2}$
23. $\frac{1}{2}e^{(x^2+1)}$
24. $\frac{1}{2}(1+\tan x)^2$

Exercise 29g – p. 490

1. $\frac{1}{2}(e-1)$
2. $\frac{1}{5}$
3. $\frac{1}{2}(\ln 2)^2$
4. $\dfrac{7^5}{15}$
5. $e-1$
6. 9
7. $\frac{1}{2}(e^3-1)$
8. $\frac{13}{24}$
9. $\frac{1}{3}(\ln 3)^3$
10. $\frac{7}{3}$

Exercise 29h – p. 495

1. $x\sin x + \cos x$
2. $e^x(x^2-2x+2)$
3. $\frac{1}{16}x^4(4\ln|3x| - 1)$
4. $-e^{-x}(x+1)$
5. $3(\sin x - x\cos x)$
6. $\frac{1}{5}e^x(\sin 2x - 2\cos 2x)$
7. $\frac{1}{5}e^{2x}(\sin x + 2\cos x)$

8. $\frac{1}{32}e^{4x}(8x^2-4x+1)$
9. $-\frac{1}{2}e^{-x}(\cos x + \sin x)$
10. $x(\ln|2x| - 1)$
11. $x\,e^x$
12. $\frac{1}{72}(8x-1)(x+1)^8$
13. $\sin(x + \frac{1}{6}\pi) - x\cos(x + \frac{1}{6}\pi)$
14. $\dfrac{1}{n^2}(\cos nx + nx\sin nx)$
15. $\dfrac{x^{n+1}}{(n+1)^2}[(n+1)\ln|x| - 1]$
16. $\frac{3}{4}(2x\sin 2x + \cos 2x)$
17. $\frac{1}{5}e^x(\sin 2x - 2\cos 2x)$
18. $(2-x^2)\cos x + 2x\sin x$
19. $\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
20. $\frac{1}{3}\sin\theta(3\cos^2\theta + 2\sin^2\theta)$
21. $\frac{1}{2}e^{x^2-2x+4}$
22. $(x^2+1)e^x$
23. $-\frac{1}{4}(4+\cos x)^4$
24. $e^{\sin x}$
25. $\frac{2}{15}\sqrt{(1+x^5)^3}$
26. $\frac{1}{5}(e^x+2)^5$
27. $\frac{1}{4}e^{2x-1}(2x-1)$
28. $-\frac{1}{20}(1-x^2)^{10}$
29. $\frac{1}{6}\sin^6 x$

Exercise 29i – p. 497

1. 1
2. $\frac{32}{3}\ln 2 - \frac{7}{4}$
3. e
4. $-\frac{1}{2}(e^\pi + 1)$
5. $\frac{16}{15}$
6. $\frac{1}{4}\pi^2 - 2$
7. $\frac{1}{2}\pi - 1$
8. $e - 2$
9. $\frac{1}{12}\pi(4\sqrt{3} - 3) - \ln\sqrt{2}$

Mixed Exercise 29 – p. 498

1. $\frac{1}{4}e^{2x}(2x^2-2x+1)$
2. e^{x^2}
3. $\frac{1}{6}(3\tan x - 4)^2$
4. $\frac{1}{4}(x+1)^2\{2\ln(x+1) - 1\}$
5. $\frac{1}{4}\tan^4 x$
6. $(x^2-2)\sin x + 2x\cos x$
7. $-e^{\cos x}$
8. $\frac{1}{288}(2x+3)^8(16x-3)$
9. $-\frac{1}{2}e^{(1-x)^2}$
10. $\frac{1}{4}e^{(2x-1)}(2x-1)$
11. $\frac{1}{6}\sin^6 x$
12. $-\frac{1}{4}(4+\cos x)^4$
13. $\frac{1}{2}e^{(x^2-2x+3)}$
14. $-\frac{1}{30}(1-x^3)^{10}$
15. $\frac{1}{3}(e^9 - e^3)$
16. $\frac{1}{4}$
17. $\ln\frac{1}{2}$
18. 1
19. $\frac{1}{9}(e^3-1)$
20. $\frac{1}{8}(\pi - 2)$
21. $\frac{1}{2}(\ln 2)^2$
22. $\frac{1}{3}\ln 2^8 - \frac{7}{9}$

CHAPTER 30

Exercise 30a – p. 502

1. (a) $3, 2$ (b) $-\frac{7}{4}, -\frac{3}{4}$
   (c) $4, -4$ (d) $-1, -8$
   (e) k, k^2 (f) $(a+2)/a, -1$
2. (a) $x^2 - 3x + 4 = 0$
   (b) $2x^2 + 4x + 1 = 0$
   (c) $15x^2 - 5x - 6 = 0$
   (d) $4x^2 + x = 0$
   (e) $x^2 - ax + a^2 = 0$
   (f) $x^2 + (k+1)x + k^2 - 3 = 0$
   (g) $abx^2 - b^2x + ac^2 = 0$
3. (a) $\frac{4}{5}$ (b) $5\frac{1}{2}$ (c) -1
   (d) 5 (e) -6 (f) $-\frac{2}{5}$
4. (a) $x^2 - 6x + 11 = 0$
   (b) $3x^2 - 2x + 1 = 0$
   (c) $x^2 + 2x + 9 = 0$
   (d) $3x^2 + 2x + 3 = 0$

Exercise 30b – p. 504

1. (a) Q: $x + 1$, R: $-6x + 3$
   (b) Q: $x^3 - x^2 - 4x + 4$, R: -2
   (c) Q: $2x - 4$, R: $5x - 5$
   (d) Q: $3x^2 + 6x + 12$, R: 19
   (e) Q: x^2, R: $-6x^2 + 1$
2. (a) $1 + \dfrac{3}{x+1}$
   (b) $2 + \dfrac{4}{x-2}$
   (c) $1 + \dfrac{4}{x^2-1}$
   (d) $x + 2 + \dfrac{4}{x-2}$

(e) $x + 7 + \dfrac{28}{x - 4}$

(f) $1 - \dfrac{x + 4}{x(x + 1)}$

Exercise 30c – p. 508

1. (a) $\dfrac{1}{x-1} - \dfrac{1}{x+1}$

(b) $\dfrac{1}{x-2} - \dfrac{1}{x+1}$

(c) $\dfrac{1}{3(x-3)} - \dfrac{1}{3x}$

(d) $\dfrac{1}{x-1} - \dfrac{1}{x+3}$

(e) $\dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

(f) $\dfrac{1}{2x-1} - \dfrac{1}{2x+1}$

2. (a) $\dfrac{1}{x-1} - \dfrac{x+1}{x^2+1}$

(b) $\dfrac{1}{x} - \dfrac{x}{2x^2+1}$

(c) $\dfrac{3}{2x} - \dfrac{x}{2(x^2+2)}$

(d) $\dfrac{22}{19(x-3)} + \dfrac{1-6x}{19(2x^2+1)}$

(e) $\dfrac{3}{5(x+2)} - \dfrac{3}{5(2x+1)} + \dfrac{x-1}{5(x^2+1)}$

(f) $\dfrac{1}{x} - \dfrac{6x+3}{2x^2-1} + \dfrac{2}{x-1}$

(g) $\dfrac{1}{x-1} - \dfrac{1}{x-2} + \dfrac{2}{(x-2)^2}$

(h) $\dfrac{2}{x} - \dfrac{1}{x^2} - \dfrac{3}{2x+1}$

(i) $\dfrac{3}{x} - \dfrac{9}{3x-1} + \dfrac{9}{(3x-1)^2}$

(j) $1 + \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

(k) $1 - \dfrac{7}{4(x+3)} - \dfrac{1}{4(x-1)}$

(l) $x + \dfrac{2}{x-1} - \dfrac{1}{x+1}$

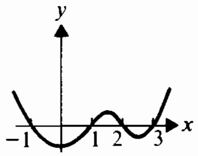
Exercise 30d – p. 511

1. (a) 3 (b) 18 (c) 47
(d) $\frac{35}{16}$ (e) $-\frac{16}{27}$
(f) $a^3 - 2a^2 + 6$ (g) $c^2 - ac + b$
(h) $\dfrac{1}{a^4} - \dfrac{2}{a} + 1$

2. (a) yes (b) no (c) no
(d) yes (e) no (f) yes
3. (a) $(x-1)(x+2)(x+1)$
(b) $(x-2)(x^2+x+1)$
(c) $(x-1)(x+1)(x^2+1)$
(d) $(x-2)(x+2)(x^2+x+1)$
(e) $(2x-1)(x^2+1)$
(f) $(3x-1)(9x^2+3x+1)$
(g) $(x+a)(x^2-ax+a^2)$
(h) $(x-y)(x^2+xy+y^2)$
4. -7
5. 5
6. -3
7. 1

Exercise 30e – p. 513

1. $(x-1)(2x-1)(x+1)$;
$-1 < x < \frac{1}{2}, x > 1$
2. $f(x) = (x-2)(x^2+x+1) \Rightarrow f(x) = 0$ only when $x = 2$; $x < 2$
3. 8
4. $x = \pm 1, \pm 2$
5. $(x-1)(x+1)(x-2)(x-3)$

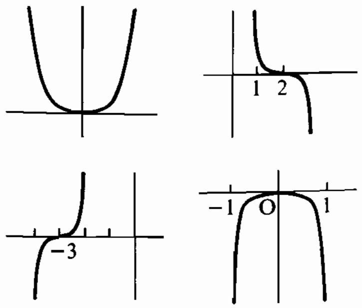


6. (a) $p = 11, q = -5$ (b) 3

Exercise 30f – p. 517

1. touch at (1, 2)
2. (1, 1) and (−5, 4)
3. no intersection
4. (0, 2) and $(\frac{8}{5}, \frac{14}{5})$
5. (1, 0) and (2, 0)
6. (0, 0), (−3, 0) and (−2, 0)
7. touch at (1, 0) and (2, 0)
8. touch and cross at (−3, 0); (−2, 0)
9. touch at (0, 0)
10. $k = 8$
11. $k = \frac{1}{4}$
12. $k = \pm 2\sqrt{19}$
13. $(\frac{1}{2}, 0)$
14. (1, 2) and (1, −2)
15. touch at (−1, −2); cut at (1, 2) and $(4, \frac{1}{2})$
16. (0, 0) and $(\frac{2}{3}, 2)$
17. (a) $\pm 2\sqrt{3}$ (b) $-2\sqrt{3} < \lambda < 2\sqrt{3}$
(c) $\lambda < -2\sqrt{3}, \lambda > 2\sqrt{3}$

18.



19. (a) (i) $y = \pm(x-1)$
(ii) $2y + x = 4$
(b) (i) $y = 0, y = 36(1-x)$
(ii) $2y + x = \pm 6\sqrt{2}$
(c) (i) $3y = 2(x-1), y = 0$
(ii) $8y + 4x + 3 = 0$

CHAPTER 31

All indefinite integrals in this chapter require the addition of a constant of integration.

Exercise 31a – p. 522

1. $\ln(4 + \sin x)$
2. $\frac{1}{3}\ln|3e^x - 1|$
3. $\dfrac{1}{4(1-x^2)^2}$
4. $\dfrac{1}{2\cos^2 x}$
5. $\frac{1}{4}\ln(1+x^4)$
6. $\ln|x^2 + 3x - 4|$
7. $\frac{2}{3}\sqrt{(2+x^3)}$
8. $\dfrac{-1}{\sin x - 2}$
9. $\ln|\ln x|$
10. $\dfrac{-1}{5\sin^5 x}$
11. $\dfrac{1}{3\cot^3 x}$
12. $-2\sqrt{(1-e^x)}$
13. $\frac{1}{6}\ln|3x^2 - 6x + 1|$
14. $\dfrac{-1}{(n-1)\sin^{n-1}x}$ $(n \neq 1)$
15. $\dfrac{1}{(n-1)\cos^{(n-1)}x}$ $(n \neq 1)$
16. $\ln|4 + \sec x|$
17. $\dfrac{1}{2(1-\tan x)^2}$
18. $\dfrac{1}{(3+\cos x)}$
19. $\ln|\sec x + \tan x|$
20. $-\ln|\operatorname{cosec} x + \cot x|$
21. $\ln 3$
22. $\ln\sqrt{2}$
23. $\frac{1}{18} - \frac{1}{128}$
24. $\dfrac{e-1}{2(e+1)}$
25. 0
26. $\dfrac{1}{\ln 4}$

Exercise 31b – p. 525

1. $2\ln\left|\dfrac{x}{x+1}\right|$
2. $\ln\left|\dfrac{x-2}{x+2}\right|$
3. $\frac{1}{2}\ln|x^2 - 1|$
4. $\frac{1}{2}\ln\left|\dfrac{(x+2)^3}{x}\right|$
5. $\ln\dfrac{(x-3)^2}{|x-2|}$
6. $\frac{1}{2}\ln\dfrac{|x^2-1|}{x^2}$
7. $x - \ln|x+1|$
8. $x + 4\ln|x|$
9. $x - 4\ln|x+4|$
10. $\ln\dfrac{|1-x|}{x^4}$
11. $x - \frac{1}{2}\ln\left|\dfrac{x+1}{x-1}\right|$
12. $x + \ln\dfrac{|x+1|}{(x+2)^4}$
13. $\frac{1}{2}\ln|x^2 - 1|$
14. $\dfrac{-1}{x^2-1}$
15. $\ln\left|\dfrac{x-1}{x+1}\right|$
16. $\ln|x^2 - 5x + 6|$
17. $\ln\dfrac{(x-3)^6}{(x-2)^4}$
18. $\ln\left|\dfrac{(x-3)^3}{x-2}\right|$
19. $4 + \ln 5$

20. $\ln \frac{1}{6}$
21. $\frac{1}{2}\ln \frac{12}{5}$
22. $\ln \frac{5}{3}$
23. $\frac{5}{36}$
24. $1 - \frac{3}{2}\ln \frac{7}{5}$

Exercise 31c – p. 526

1. $\frac{2}{x-2} - \frac{2}{x-1}$; $\frac{-2}{(x-2)^2} + \frac{2}{(x-1)^2}$; $\frac{4}{(x-2)^3} - \frac{4}{(x-1)^3}$;
2. $\frac{\frac{9}{5}}{x-3} - \frac{\frac{3}{5}}{2x-1}$; $\frac{6}{5(2x-1)^2} - \frac{9}{5(x-3)^2}$; $\frac{-24}{5(2x-1)^3} + \frac{18}{5(x-3)^3}$
3. $\frac{\frac{2}{3}}{x-4} + \frac{\frac{1}{3}}{x+2}$; $\frac{-1}{3(x+2)^2} - \frac{2}{3(x-4)^2}$; $\frac{2}{3(x+2)^3} + \frac{4}{3(x-4)^3}$
4. $\frac{1}{x-3} - \frac{1}{x+2}$; $\frac{1}{(x+2)^2} - \frac{1}{(x-3)^2}$; $\frac{-2}{(x+2)^3} + \frac{2}{(x-3)^3}$;
5. $\frac{3}{2x+3} - \frac{1}{x+1}$; $\frac{-6}{(2x+3)^2} + \frac{1}{(x+1)^2}$; $\frac{24}{(2x+3)^3} - \frac{2}{(x+1)^3}$
6. $\frac{\frac{3}{2}}{x-1} - \frac{\frac{9}{2}}{3x-1}$; $\frac{27}{2(3x-1)^2} - \frac{3}{2(x-1)^2}$; $\frac{-81}{(3x-1)^3} + \frac{3}{(x-1)^3}$

Exercise 31d – p. 530

1. $\frac{1}{4}(2x + \sin 2x)$
2. $\sin x - \frac{1}{3}\sin^3 x$
3. $-\frac{1}{15}\cos x(15 - 10\cos^2 x + 3\cos^4 x)$
4. $\tan x - x$
5. $\frac{1}{32}\{12x - 8\sin 2x + \sin 4x\}$
6. $\frac{1}{2}\tan^2 x - \ln|\sec x|$
7. $\frac{1}{32}\{12x + 8\sin 2x + \sin 4x\}$
8. $\frac{1}{3}\cos x(\cos^2 x - 3)$
9. $\frac{1}{15}\sin^3\theta(5 - 3\sin^2\theta)$
10. $(\sin^{11}\theta)(\frac{1}{11} - \frac{1}{13}\sin^2\theta)$
11. $(\sin^{(n+1)}\theta)\left(\frac{1}{n+1} - \frac{1}{n+3}\sin^2\theta\right)$ $(n \neq -1 \text{ or } -3)$
12. $\frac{1}{32}(4\theta - \sin 4\theta)$
13. $\frac{1}{15}(10\sin 3\theta - 3\sin 5\theta - 15\sin\theta)$
14. $\frac{1}{15}\tan^3\theta(3\tan^2\theta + 5)$
15. $-\frac{1}{7}(\cos 7t + 7\cos t)$
16. $\frac{1}{21}(3\sin 7t + 7\sin 3t)$
17. $\frac{1}{16}(2\cos 4t - \cos 8t)$
18. $\frac{1}{8}(2\sin 2t - \sin 4t)$
19. $\left(\frac{-1}{n+m}\right)\cos(n+m)t - \left(\frac{1}{n-m}\right)\cos(n-m)t$ $(n^2 \neq m^2)$
20. $\frac{1}{2(n+m)}\sin(n+m)t + \frac{1}{2(n-m)}\sin(n-m)t$ $(n^2 \neq m^2)$
21. $\frac{1}{12}$
22. $\frac{13}{15} - \frac{1}{4}\pi$
23. $\frac{1}{5}(2\sqrt{3} - 3)$
24. $\frac{1}{16}\sqrt{3}$

CHAPTER 32

All indefinite integrals require the term $+K$.

Exercise 32a – p. 522

1. $\frac{1}{21}(x+3)^6(3x+2)$
2. $\frac{1}{2}\arctan\frac{x}{2}$
3. $-\frac{2}{3}(x+6)\sqrt{(3-x)}$
4. $\frac{2}{15}(3x-2)(x+1)^{3/2}$
5. $\frac{1-5x}{10(x-3)^5}$
6. $\ln|x + \sqrt{(1+x^2)}|$
7. $\frac{4}{135}(9x+8)(3x-4)^{3/2}$
8. $\frac{3}{10}\arctan\frac{2}{5}x$
9. $\ln|x + 2 + \sqrt{(x^2+4x+3)}|$

10. $-\frac{1}{36}(8x+1)(1-x)^8$
11. $\frac{1}{2}\arcsin 2x$
12. $\frac{5+4x}{12(4-x)^4}$

Exercise 32b – p. 537

1. $\frac{1}{2}e^{2x+3}$
2. $\frac{1}{6}(2x^2-5)^{3/2}$
3. $\frac{1}{12}(6x - \sin 6x)$
4. $-\frac{1}{2}e^{-x^2}$
5. $-\frac{1}{8}(\cos 4\theta + 2\cos 2\theta)$
6. $\frac{1}{110}(10u-7)(u+7)^{10}$
7. $\frac{1}{12(x^3+9)^4}$
8. $\frac{1}{2}\ln|1 - \cos 2y|$ $(y \neq n\pi)$
9. $\frac{1}{2}\ln|2x+7|$
10. $\arcsin u$
11. $-\frac{2}{9}(1+\cos 3x)^{3/2}$
12. $\frac{1}{16}(\sin 4x - 4x\cos 4x)$
13. $\frac{1}{2}\ln|x^2+4x-5|$
14. $\frac{1}{2}\ln|x^2+4x+5|$
15. $-\frac{1}{4}(x^2+4x-5)^{-2}$
16. $-(9-y^2)^{3/2}$
17. $\frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x)$
18. $x(\ln|5x| - 1)$
19. $\frac{1}{6}\sin 2x(3 - \sin^2 2x)$
20. $-e^{\cot x}$
21. $-2\sqrt{(7+\cos y)}$
22. $e^x(x^2 - 2x + 2)$
23. $\frac{1}{2}\ln|x^2-4|$
24. $x + \ln\left|\frac{x-2}{x+2}\right|$
25. $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|$
26. $\frac{1}{30}(3\sin 5x + 5\sin 3x)$
27. $-\frac{1}{30}\cos 2\theta(15 - 10\cos^2 2\theta + 3\cos^4 2\theta$
28. $\frac{1}{15}\cos^3 u(3\cos^2 u - 5)$
29. $\tan\theta - \theta$
30. $\arcsin x + 2\sqrt{(1-x^2)}$
31. $\frac{1}{27}(9y^2\sin 3y + 6y\cos 3y - 2\sin 3y)$
32. $-\ln|1 - \tan x|$
33. $\frac{1}{3}(7+x^2)^{3/2}$
34. $-\frac{1}{5}\cos(5\theta - \frac{1}{4}\pi)$
35. $\sin\theta\{\ln|\sin\theta| - 1\}$
36. $e^{\tan u}$
37. $\frac{2x-1}{10(3-x)^6}$
38. $\frac{1}{3}\tan^3 x$

Exercise 32c – p. 540

1. $y^2 = A - 2\cos x$
2. $\frac{1}{y} - \frac{1}{x} = A$
3. $2y^3 = 3(x^2 + 4y + A)$
4. $x = A\sec y$
5. $(A - x)y = 1$
6. $y = \ln\frac{A}{\sqrt{(1-x^2)}}$
7. $y = A(x-3)$
8. $x + A = 4\ln|\sin y|$
9. $u^2 = v^2 + 4v + A$
10. $16y^3 = 12x^4\ln|x| - 3x^4 + A$
11. $y^2 + 2(x+1)e^{-x} = A$
12. $\sin x = A - e^{-y}$
13. $2r^2 = 2\theta - \sin 2\theta + A$
14. $u + 2 = A(v+1)$
15. $y^2 = A + (\ln|x|)^2$
16. $y^2 = Ax(x+2)$
17. $4v^3 = 3(2+t)^4 + A$
18. $1 + y^2 = Ax^2$
19. $Ar = e^{\tan\theta}$
20. $y^2 = A - \operatorname{cosec}^2 x$
21. $v^2 + A = 2u - 2\ln|u|$
22. $e^{-x} = e^{1-y} + A$
23. $A - \frac{1}{y} = 2\ln|\tan x|$
24. $y - 1 = A(y+1)(x^2+1)$

Exercise 32d – p. 544

1. $y^3 = x^3 + 3x - 13$
2. $e^t(5 - 2\sqrt{s}) = 1$
3. $3(y^2 - 1) = 8(x^2 - 1)$
4. $y = x^2 - x$
5. $y = e^x - 2$
6. $y = 5 - \frac{3}{x}$
7. $y = \pm\sqrt{(4 + e^{-3} - e)}$ $= \pm 1.154$
8. $(y+1)^2(x+1) = 2(x-1)$
9. $2y = x^2 + 6x$
10. $4y^2 = (y+1)^2(x^2+1)$

Mixed Exercise 32 – p. 545

1. $\frac{1}{10}(1+x^2)^5$
2. $-\frac{1}{9}e^{-3x}(3x+1)$

3. $\frac{1}{10}(\sin 5x + 5\sin x)$
4. $x + \ln|x+2|$
5. $\dfrac{-1}{3(x^3+1)}$
6. $\ln\left|\dfrac{x-4}{x-1}\right|$
7. $\ln\left|\dfrac{x}{2x+1}\right|$
8. $\ln\sqrt{(x^2+1)} - \arctan x$
9. $-2\sqrt{(\cos x)}$
10. $\sqrt{2}$
11. $\frac{256}{15}$
12. $\frac{1}{2} - \ln\sqrt{2}$
13. $9 - \frac{16}{3}\sqrt{2}$
14. $\ln\frac{9}{4}$
15. $\ln\left\{\dfrac{\ln 3}{\ln 2}\right\}$
16. $\ln\frac{1}{6}$
17. $3\frac{1}{2} + 6\ln\frac{2}{3}$
18. $x^3y = y - 1$
19. $y = \tan\{\frac{1}{2}(x^2-4)\}$
20. $y^2 = 2x$

CHAPTER 33

Exercise 33a – p. 549

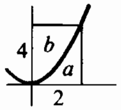
1. $v = \frac{1}{2}t^2 - 6t$; $s = \frac{1}{6}t^3 - 3t^3$
   $v_6 = -18$ m/s; $s_6 = -72$ m
2. $v = \frac{3}{2}t^2 - t - 1$; $31\frac{1}{2}$ m/s
3. 129 m
4. (a) $s = 2t - \dfrac{1}{t+1} + 1$
   (b) $v \to 2$ as $t \to \infty$
5. $6t$ m/s; 504 m ($t = 2$ at A)

Exercise 33b – p. 550

1. $s\dfrac{ds}{dt} = k$
2. $\dfrac{dh}{dt} = k\ln|H - h|$
3. (a) $\dfrac{dn}{dt} = 0$ (b) $\dfrac{dn}{dt} = k\sqrt{n}$
4. (a) $\dfrac{dn}{dt} = k_1 n$ (b) $\dfrac{dn}{dt} = \dfrac{k_2}{n}$
   (c) $\dfrac{dn}{dt} = -k_3$
5. (a) $\dfrac{dp}{dt} = kp(s-p)$

Exercise 33c – p. 555

1. 9
2. (a) $\frac{8}{3}$ (b) $\frac{16}{3}$



3. 18
4. $\frac{4}{3}$
5. 1
6. $e - 2$
7. $\frac{1}{6}$
8. $\frac{1}{2}\sqrt{3} - \frac{1}{6}\pi$
9. (a) 1 (b) $2 - \ln 3$
10. $8(\ln 2)$
11. (a) $\frac{4}{3}$ (b) $\frac{1}{3}$
12. $4\sqrt{3}$

Exercise 33d – p. 562

1. $\frac{512}{15}\pi$
2. $\frac{1}{2}\pi(e^6 - 1)$
3. $\frac{1}{2}\pi$
4. $\frac{64}{5}\pi$
5. 2π
6. 8π
7. 8π
8. $\frac{3}{5}\pi(\sqrt[3]{32} - 1)$
9. $\frac{1}{2}\pi(e^2 - 1)$
10. $\frac{16}{15}\pi$
11. $\frac{16}{15}\pi$
12. $\frac{1}{2}\pi^2$
13. $\frac{5}{2}\pi$
14. $\frac{3}{10}\pi$

Exercise 33e – p. 568

1. (a) 18 (b) $\frac{348}{10}$ (c) $\frac{58}{15}$
2. 8; (a) $\frac{64}{3}$ (b) $\frac{32}{3}$; $(\frac{8}{3}, \frac{4}{3})$
3. (a) 2 (b) $\frac{1}{4}\pi$ (c) $\frac{1}{2}\pi, \frac{1}{8}\pi)$
   (d) $\frac{1}{2}\pi^2$ (e) $\frac{1}{4}\pi^2$ (f) $(\frac{1}{2}\pi, 0)$
4. (a) $\frac{1}{6}$ (b) $\frac{1}{60}$ (c) $(\frac{1}{2}, -\frac{1}{10})$
   (d) $\frac{1}{30}\pi$ (e) $\frac{1}{60}\pi$ (f) $(\frac{1}{2}, 0)$
5. (a) $\frac{32}{3}$ $\frac{128}{5}$ $\frac{12}{5}$
   (d) 8π (e) $\frac{64}{3}\pi$ (f) $(0, \frac{8}{3})$

6. (a) $e - 1$ (b) 1 (c) $\dfrac{1}{e-1}$
   (d) $\frac{1}{2}\pi(e^2 - 1)$ (e) $\frac{1}{4}\pi(e^2 + 1)$
   (f) $\left(0, \frac{1}{2}\left\{\dfrac{e^2+1}{e^2-1}\right\}\right)$
7. $\dfrac{e^2+1}{e^2-1}$

Mixed Exercise 33 – p. 572

1. $t = \frac{1}{2}$ or 1; 7.5 m
2. $v = \frac{1}{2} - \dfrac{1}{2t^2}$, $v_2 = \frac{3}{8}$ m/s
   $s = \dfrac{1}{2t^2} + \frac{1}{2}t - 1$, $s_2 = \frac{1}{4}$ m
3. -2 m/s^2; $8\frac{1}{3}$ m
4. $\frac{1}{3}$
5. (a) $e^4 - 1$ (b) $\frac{1}{4}(e^8 - 1)$
   (c) $\frac{1}{4}(e^4 + 1)$
6. $\frac{1}{2}(e^4 - 1)$
7. $\frac{16}{3}$ (a) 8π (b) $(0, \frac{14}{3})$
8. (a) $\frac{32}{3}$ (b) $\frac{192}{5}$ (c) $(0, \frac{18}{5})$
9. $\frac{256}{15}\pi$; $\frac{32}{3}\pi$; $\bar{y} = 2$, $\bar{x} = \frac{5}{8}$
10. $\frac{1}{2}\left(\sqrt{3} - \dfrac{3}{\pi}\right)$

CONSOLIDATION E

Multiple Choice Exercise E – p. 578

1. C
2. C
3. D
4. D
5. B
6. B
7. C
8. D
9. D
10. B
11. A
12. A
13. D
14. E
15. A, C
16. B, C
17. A, C
18. B, C
19. B
20. A, C
21. B
22. B, C
23. A, D
24. A, D
25. F
26. T
27. F
28. T
29. F
30. F
31. F
32. T

Miscellaneous Exercise E – p. 583

1. $-\frac{9}{4}$
2. $\dfrac{5}{x-3} - \dfrac{2}{3x+2}$; $-\dfrac{157}{32}$
3. 2
4. $p = 6, q = 10$
5. (a) (3, 3) (c) $(\frac{14}{3}, \frac{1}{2})$
6. $\dfrac{-1}{x-2} + \dfrac{4}{x+6}$; $\dfrac{1}{(x-2)^2} - \dfrac{4}{(x+6)^2}$;
   $\dfrac{-2}{(x-2)^3} + \dfrac{8}{(x+6)^3}$; $x = -\frac{2}{3}$;
   $\frac{1}{8}$ max, $\frac{9}{8}$ min
7. $a = -11, b = 3$;
   $(x-3)(x-4)(2x+3)$
8. Centre $(2, -3)$, radius 5;
   $\frac{1}{5}|6+k|$; $k = 19$ or -31
9. $\frac{4}{3}$; $y - 3 = -\frac{3}{4}(x-2)$; $2y = x + 9$;
   63.4°
10. (b) 45°
    (c) $(6, -1)$; 10;
    $(x-6)^2 + (y+1)^2 = 100$
11. 33 000, -1.4
12. 3.3 to 3.5
13. 1.8, 2.5
14. (a) $\frac{1}{4}e^{2x}(6x+5)$ (b) $\frac{1}{64}(\pi + 2)$
15. $p^2 - 4$; $2x^2 - 3(p^2-4)x + 18 = 0$
16. (a) $\frac{1}{12}(6x + \sin 6x) + K$
    (b) $\frac{1}{2}\arctan(\frac{1}{2}e^x) + K$
    (c) $\frac{1}{4}e^{2x}(2x-1) + K$
17. $A = -1, B = 1$; $\ln 2 - \frac{1}{2}$
18. $\arctan e - \frac{1}{4}\pi = 0.43$ to 2 d.p.
19. $\frac{1}{20}$
20. 2, 3, -1; $y - 3 = -\frac{2}{5}(x-1)$
21. $5\frac{1}{3}$ sq. units
22. (a) $\dfrac{\pi(e^{4a}-1)}{2e^{2a}}$
    (b) (i) $\frac{1}{110}$
    (ii) $(x^2+1)\arctan x - x + K$
23. $2\ln A\left|\dfrac{8-x}{4-x}\right|$; $t = 2\ln\left|\dfrac{8-x}{3(4-x)}\right|$;
    $2\ln\frac{5}{3}$ minutes, i.e. 61 minutes
24. $\dfrac{\frac{1}{2}}{x+1} - \dfrac{\frac{1}{2}}{x+3}$; $y^2(x+3) = 8(x+1)$
25. $y = x^2 - 5$
26. $\frac{9}{2} - 6\ln 2$
27. (a) -2, 3
    (b) $(k+2)x^2 - 4k^2x + 4k^2 = 0$
28. $\dfrac{dV}{dt} = kV$ (a) £4495
    (b) 57 months
29. $y = \exp(2x^2 - 2)$
30. 10.6

31. $\frac{1}{12}\pi$; $\frac{1}{12}\pi\sqrt{2}$
32. $5\ln 5 - 4$; 15.4
33. (c) $\frac{1}{2}\pi - 1$ (d) $\frac{1}{8}\pi$; $\frac{1}{4}\pi^2$
34. (a) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + K$
(b) $K + \arctan(\tan x)$
(c) 1.3
35. (a) $4p^2 - 2q$

CHAPTER 34

Exercise 34a – p. 593

1. (a) $\sum_{r=1}^{5} r^3$ (b) $\sum_{r=1}^{10} 2r$
(c) $\sum_{r=2}^{50} \frac{1}{r}$ (d) $\sum_{r=0}^{\infty} \frac{1}{3^r}$
(e) $\sum_{r=0}^{7} (-4 + 3r)$
(f) $\sum_{r=0}^{\infty} \left(\frac{8}{2^r}\right) = \sum_{r=0}^{\infty} \frac{1}{2^{r-3}}$
2. (a) $1 + \frac{1}{2} + \frac{1}{3} + \ldots$
(b) $0 + 2 + 6 + 12 + \ldots + 30$
(c) $2 + \frac{1}{2} + \frac{4}{15} + \ldots + \frac{22}{861}$
(d) $1 + \frac{1}{2} + \frac{1}{5} + \ldots$
(e) $0 + 0 + 6 + \ldots + 720$
(f) $-1 + a - a^2 + \ldots$
3. (a) 8; 9 (b) 9; 10 (c) -1; 6
(d) $\frac{1}{420}$; ∞ (e) $(\frac{1}{2})^n$; ∞
(f) -48; 23 (g) 4; 10

Exercise 34b – p. 599

1. (a) 9, $2n - 1$ (b) 16, $4(n - 1)$
(c) 15, $3n$ (d) 17, $3n + 2$
(e) -2, $8 - 2n$
(f) $p + 4q$, $p + (n - 1)q$
(g) 18, $8 + 2n$ (h) 17, $4n - 3$
(i) 0, $\frac{1}{2}(5 - n)$ (j) 8, $3n - 7$
2. (a) 100 (b) 180 (c) 165
(d) 185 (e) -30
(f) $5(2p + 9q)$ (g) 190
(h) 190 (i) $-\frac{5}{2}$ (j) 95
3. $a = 27.2$, $d = -2.4$
4. $d = 3$; 30
5. 1, $\frac{1}{2}$, 0; $-8\frac{1}{2}$
6. (a) $28\frac{1}{2}$ (b) 80 (c) 400
(d) 80 (e) 108 (f) $3n(1 - 6n)$
(g) 40 (h) $2m(m + 3)$
7. 4, $2n - 4$
9. 2, 364

10. 39
11. 64
12. (a) 1, 5 (b) 270
13. (a) $a = 21, d = -3$
(b) less than 4 or more than 11

Exercise 34c – p. 604

1. (a) 32, 2^n (b) $\frac{1}{8}$, $\frac{1}{2^{n-2}}$
(c) 48, $3(-2)^{n-1}$
(d) $\frac{1}{2}$, $(-1)^{n-1}(\frac{1}{2})^{n-4}$ (e) $\frac{1}{27}$, $(\frac{1}{3})^{n-2}$
2. (a) 189 (b) -255
(c) $2 - (\frac{1}{2})^{19}$ (d) 781/125
(e) 341/1024 (f) 1
3. $\frac{1}{2}$, 2
4. $-\frac{1}{2}$
5. $-\frac{1}{2}$, 1/1024
6. 13.21 to 4 s.f.
7. (a) $(x - x^{n+1})/(1 - x)$
(b) $(x^n - 1)/x^{n-2}(x - 1)$
(c) $(1 + (-1)^{n+1}y^n)/(1 + y)$
(d) $x(2^n - x^n)/2^{n-1}(2 - x)$
(e) $[1 - (-2)^n x^n]/(1 + 2x)$
8. $\frac{8}{3}(1 - (\frac{1}{4})^n)$, 4
9. 62 or 122
10. 8.493 to 4 s.f.
11. 8 (last repayment is less than £2000)
12. £23.31

Exercise 34d – p. 610

1. (a) yes (b) no (c) yes
(d) yes (e) no (f) yes
2. $-1 < x < 1$ (b) $x < -1, x > 1$
(c) $-\frac{1}{2} < x < \frac{1}{2}$ (d) $0 < x < 2$
(e) $-1 - a < x < 1 - a$
(f) $x < -(1 + a), x > 1 - a$
3. (a) 6 (c) $13\frac{1}{3}$ (d) $\frac{5}{9}$ (f) $\frac{9}{4}$
4. (a) 161/990 (b) 34/99 (c) 7/330
5. $\frac{1}{2}$
7. 8, 4, 2, 1

Exercise 34e – p. 613

1. $\frac{1}{2r} - \frac{1}{2(r + 2)}$; $\frac{n(3n + 5)}{4(n + 1)(n + 2)}$
2. $\frac{1}{r + 1} - \frac{1}{r + 2}$; $\frac{n - 2}{4(n + 2)}$
3. $\frac{1}{r} - \frac{1}{r + 1}$; $\frac{n + 1}{n(2n + 1)}$

4. $\frac{1}{16(2r - 1)} + \frac{1}{8(2r + 1)} - \frac{3}{16(2r + 3)}$;
$\frac{n(n + 1)}{2(2n + 1)(2n + 3)}$
5. $\frac{1}{2}n(n + 1)(2n^2 + 2n + 1)$
6. $\frac{n(n + 3)}{4(n + 1)(n + 2)}$
7. $\frac{n(n + 2)}{(n + 1)^2}$
8. $\frac{1}{4}n^2(n + 1)^2$
9. $\frac{\cos 2\theta - \cos (2n + 2)\theta}{2 \sin \theta}$

Exercise 34f – p. 615

1. $\frac{1}{3}n(n + 1)(n + 2)$
2. $\frac{n}{4}(n + 1)(n + 2)(n + 3)$
3. $\frac{1}{12}n(n + 1)(45n^2 + 37n + 2)$
4. 42 075
5. $\frac{1}{6}n(2n + 5)(n - 1)$
6. $-n(2n + 1)$
7. $\frac{1}{3}n(4n^2 + 6n - 1)$

Mixed Exercise 34 – p. 617

1. $\frac{2}{3}$
2. $\frac{1}{2}(1 + 3^{11}) = 88\,574$
3. $\frac{ab^4(1 - b^{2(n-1)})}{1 - b^2}$
4. 650
5. $2(n + 5)(n - 4)$
6. $\frac{1}{4}n(n + 1)(n^2 + n + 2)$
7. $\frac{(n + 4)(3n - 1)}{4n(n + 1)}$
8. $210 \ln 3$
9. $\frac{e(1 - e^n)}{1 - e}$
10. $\frac{1}{6}n(n + 1)(3n^2 + 13n + 11)$
11. $\frac{1}{2(n + 1)}$
12. 1
13. $\frac{2}{3}n(n + 1)(2n + 1)$
14. $-\ln n$
15. 1, 7, 19, 37; $3n^2 - 3n + 1$
16. 16, -8
17. 2
18. 1
19. $a = 2 \pm \sqrt{2}$, $r = \frac{1}{4}(2 \pm \sqrt{2})$

CHAPTER 35

Exercise 35a – p. 620

1, **4** and **5** are permutations
2, **3** and **6** are combinations

Exercise 35b – p. 625

1. 7!
2. (a) $\frac{5!}{2!}$ (b) 5
3. 6!
4. 3!
5. $9(10^5)$
6. (a) 4(5!) (b) 2(5!) (c) 2(5!)
7. (a) 3! (b) 3
8. 9!
9. (a) $\frac{9!}{4!}$ (b) 9 (c) $4\left(\frac{8!}{4!}\right)$
10. 11!
11. (a) 2(4!) (b) 5! − 2(4!)
12. $\frac{1}{2}(4!)$
13. 12
14. (a) 2(11!) (b) 12! − 2(11!)
15. (a) $\frac{4!}{1!}$ (b) 4^3
16. $\frac{3!}{1!}$

Exercise 35c – p. 630

Answers are given in factorial form to facilitate method checks.

1. (a) $\frac{11!}{5!}$ (b) $\frac{11!}{6!5!}$ (c) $\frac{20!}{3!17!}$
2. (a) Arranging 4 out of 8 different objects.
(b) Choosing a group of 5 (or a group of 9) from 14 different objects.
(c) Choosing a group of 5 (or 2) from 7 different objects.
3. $\frac{8!}{6!2!}$
4. $\frac{8!}{6!2!}$
5. (a) $\frac{19!}{3!16!}$ (b) $\frac{20!}{4!16!}$
6. (a) $\frac{9!}{3!6!}$ (b) $\frac{9!}{2!7!} \times 2$
7. $\frac{9!}{5!4!}$; (a) $\frac{9!}{4!5!} - 1$
(b) $\frac{9!}{5!4!} - \frac{7!}{3!4!}$ (c) 1

8. $\dfrac{13!}{5!8!}$

9. $\dfrac{12!}{4!8!}$

10. (a) $\dfrac{3!}{2!1!}$ (b) $2\left(\dfrac{3!}{2!1!}\right)$

11. (a) $\dfrac{9!}{4!5!}$ (b) $\dfrac{9!}{2!7!}$

(c) $2\left(\dfrac{9!}{2!7!}\right)$ (d) 9

Exercise 35d – p. 633

1. 6
2. 24
3. 120
4. 720
5. 30
6. 132
7. 840
8. 60
9. 28
10. 1260
11. 70
12. $\frac{3}{4}$
13. $\dfrac{5!}{2!}$
14. $\dfrac{11!}{9!}$
15. $\dfrac{(n+1)!}{(n-2)!}$
16. $\dfrac{(n+2)!}{n!}$
17. $\dfrac{20!}{3!17!}$
18. $\dfrac{8!3!}{6!5!}$
19. $\dfrac{5!}{3!2!}$
20. $\dfrac{40!}{38!2!}$
21. $\dfrac{n!}{3!(n-3)!}$
22. $\dfrac{(n-1)!}{4!(n-5)!}$
23. $8!(1+9)$
24. $5!(42-2)$
25. $(n-1)!(n+1)$
26. $n!(n+1-1)$

28. (a) $\dfrac{2n(2n-1)(2n-2)\dots(2n-r+1)}{r!}$

(b) $\dfrac{2n(2n-1)(2n-2)\dots(n+1)}{n!}$

Exercise 35e – p. 639

1. (a) 10 (b) 27
2. 13
3. $\dfrac{18!}{13(5!)^2(7!)}$
4. $\dfrac{11!}{(2!)^3}$
5. 2454
6. 18
7. 75 600
8. 120
9. (a) $5! \times 3! \times 2$ (b) $5! \times \dfrac{6!}{3!}$
10. 16
11. 28
12. (a) 56 (b) 35
13. 27

Mixed Exercise 35 – p. 641

1. (a) 225 (b) 480 (c) 240
2. 81; 1, 8, 24, 32, 16
3. $9 \times 8 \times 10!$
4. 72
5. 11 760
6. $\dfrac{12!}{7!5!}$ (a) $\dfrac{10!}{5!5!} + \dfrac{10!}{3!7!}$

(b) $2 \times \dfrac{10!}{6!4!}$

7. (a) $3 \times 4!$ (b) 3×5^4
8. $\dfrac{9!}{(2!)^3}$

(a) $\dfrac{8!}{(2!)^2}$ (b) $2 \times \dfrac{8!}{(2!)^2}$

(c) $\dfrac{(5!)^2}{(2!)^3}$

CHAPTER 36

Exercise 36a – p. 646

1. (a) $1 + 36x + 594x^2 + 5940x^3$

(b) $1 - 18x + 144x^2 - 672x^3$

(c) $1024 + 5120x + 11\,520x^2 + 15\,360x^3$

(d) $1 - \frac{20}{3}x + \frac{190}{9}x^2 - \frac{380}{9}x^3$

(e) $128 - 672x + 1512x^2 - 1890x^3$

(f) $(\frac{3}{2})^9 + \dfrac{3^{10}}{2^7}x + \dfrac{3^9}{8}x^2 + \frac{7}{2}(3^7)x^3$

2. (a) $336x^2$ (b) $-10x$

(c) $-21\,840x^{11}$ (d) $3360p^6q^4$

(e) $16(3a)^7b$ (f) $7920x^4$

(g) $63x^5$ (h) $56a^3b^5$

3. (a) $1 - 8x + 27x^2$

(b) $1 + 19x + 160x^2$

(c) $2 - 19x + 85x^2$

(d) $1 - 68x + 2136x^2$

Exercise 36b – p. 649

1. 0.8171
2. 1.062
6. (a) $1 - 50x$ (b) $256 - 1024x$

(c) $1 - 19x$

Exercise 36c – p. 655

1. $1 - x - \dfrac{x^2}{2} - \dfrac{x^3}{3}, -\frac{1}{2} < x < \frac{1}{2}$
2. $\frac{1}{3} - \dfrac{x}{9} + \dfrac{x^2}{27} - \dfrac{x^3}{81}, -3 < x < 3$
3. $1 - \dfrac{x}{4} + \dfrac{3x^2}{32} - \dfrac{5x^3}{128}, -2 < x < 2$
4. $1 + 2x + 3x^2 + 4x^3, -1 < x < 1$
5. $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3, -1 < x < 1$
6. $1 + \frac{1}{2}x - \frac{5}{8}x^2 - \frac{3}{16}x^3, -1 < x < 1$
7. $-2 - 3x - 3x^2 - 3x^3, -1 < x < 1$
8. $2 + 2x + \frac{21}{4}x^2 + \frac{27}{8}x^3, -\frac{1}{3} < x < \frac{1}{3}$
9. $\frac{1}{2} - \frac{3}{4}x + \frac{13}{8}x^2 - \frac{51}{16}x^3, -\frac{1}{2} < x < \frac{1}{2}$
10. $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3, -1 < x < 1$
11. $1 - \frac{1}{9}x^2, -3 < x < 3$
12. $x - x^2 + x^3, -1 < x < 1$
13. $1 - 3p^{-1} + 6p^{-2} - 10p^{-3} + 15p^{-4}, |p| < 1$
14. 1.732
15. 3.162 28
16. $1 + 2x + 2x^2$
18. $1 - 3x + \frac{7}{2}x^2$
19. $1 + 2x + 5x^2$
20. $\frac{3}{2} + \frac{15}{4}x$
23. $\frac{1}{729}[27 + 27x + 18x^2 + 10x^3]$

Exercise 36d – p. 658

1. $1 + 2x + 2x^2$
2. $1 - 2x + 2x^2$
3. $1 + \frac{1}{2}x + \frac{1}{8}x^2$
4. $1 + \frac{3}{2}x + \frac{9}{8}x^2$
5. $-x - \frac{1}{2}x^2 - \frac{1}{6}x^3$
6. $2 + x + \frac{1}{2}x^2$
7. $-1 + x - x^2$
8. $1 + \frac{1}{2}x^2 - \frac{1}{3}x^2$
9. $1 + 0.5 + 0.125 + 0.020\,833\,33$
$+ 0.002\,606\,16 + 0.000\,260\,411$
$+ 0.000\,021\,70 + 0.000\,001\,55$
$+ 0.000\,000\,096 + \dots$
$= 1.648\,72$ correct to 5 d.p.

Exercise 36e – p. 659

1. $3x - \dfrac{9x^2}{2} + 9x^3, -\frac{1}{3} < x \leqslant \frac{1}{3}$
2. $2x - x^2 + \frac{2}{3}x^3, -1 < x \leqslant 1$
3. $-\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3, -2 \leqslant x < 2$
4. $-x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6, -1 < x < 1$
5. $-x - x^2 - \frac{4}{3}x^3, -\frac{1}{2} \leqslant x < \frac{1}{2}$
6. $x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4, -1 < x \leqslant 1$
7. $x + \frac{1}{2}x^2 - \frac{1}{6}x^3, -1 < x \leqslant 1$
8. $-x - \dfrac{x^3}{3} - \dfrac{x^5}{5}, -1 < x < 1$
9. $2x - \frac{1}{2}x^2 + \frac{1}{3}x^3$
(note that $\ln e^x = x$), $- < x \leqslant 1$

Mixed Exercise 36 – p. 660

1. $1 + 18x + 144x^2 + \dots + 2^9x^9$; all x
2. $1 - 2x + 4x^2 - \dots; -\frac{1}{2} < x < \frac{1}{2}$
3. $x + x^2 + x^3 + \dots; -1 < x < 1$
4. $1 + 3x + 6x^2 + \dots; -\frac{1}{2} < x < \frac{1}{2}$
5. $\dfrac{1}{3(1+x)} + \dfrac{2}{3(1-2x)}; 1 + x + 3x^2$
6. 8
8. $1 - \frac{1}{6}x^3$
9. $1 + x\ln 2 + \dfrac{x^2}{2}(\ln 2)^2; 1.2577\dots$
10. (a) $1 + \frac{3}{2}x + \frac{27}{8}x^2; |x| < \frac{1}{3}$

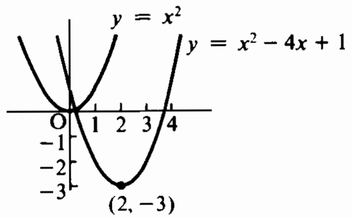
(b) 1.343 75 with $x = \frac{1}{6}$

11. (a) 1 (b) $-16x^3$

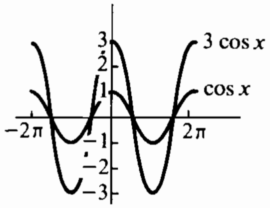
CHAPTER 37

Exercise 37a – p. 665

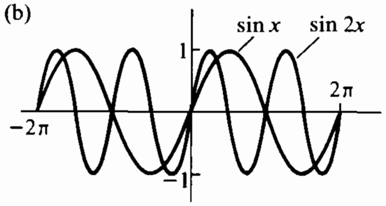
1. $a = 2, b = -3$



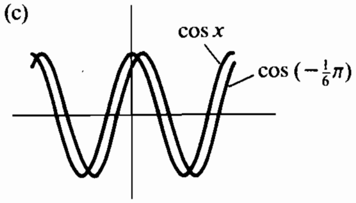
2. (a)



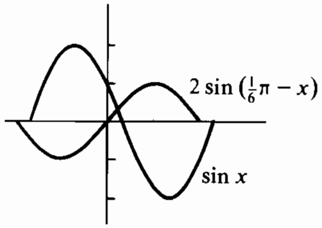
(b)



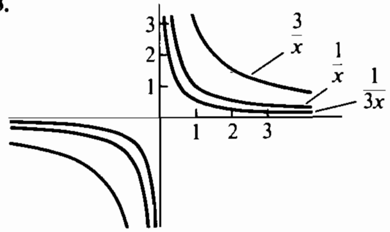
(c)



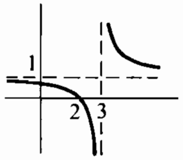
(d)



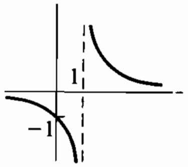
3.



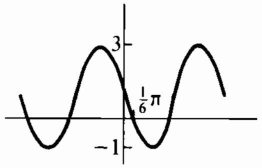
4.



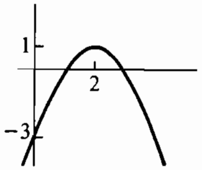
5.



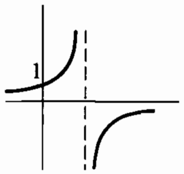
6.



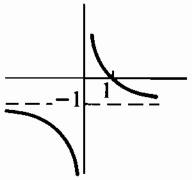
7.



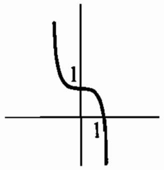
8.



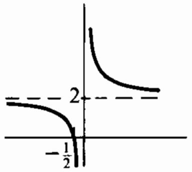
9.



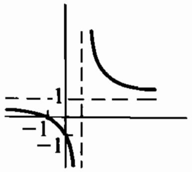
10.



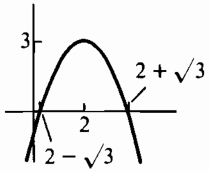
11.



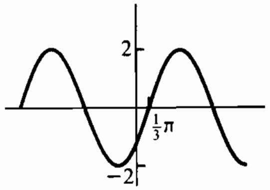
12.



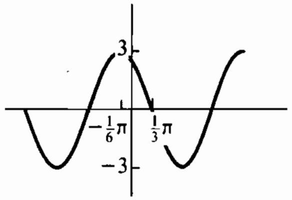
13.



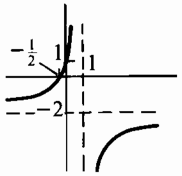
14.



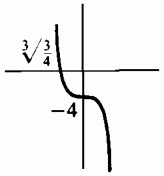
15.



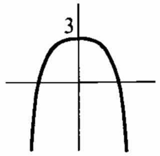
16.



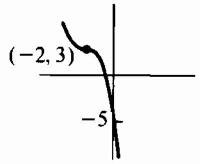
17.



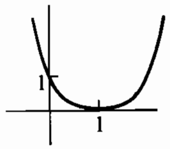
18.



19.

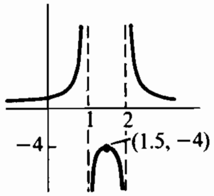


20.

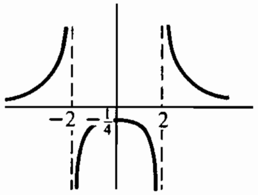


Exercise 37b – p. 669

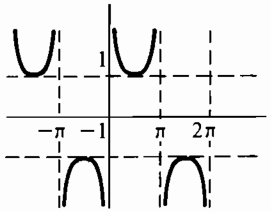
1.



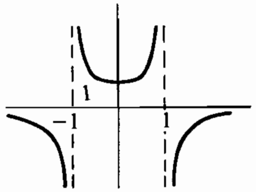
2.



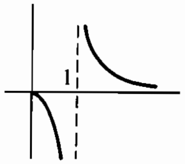
3.



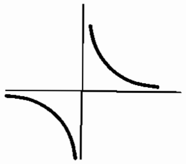
4.



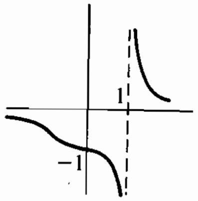
5.



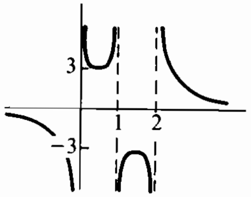
6.



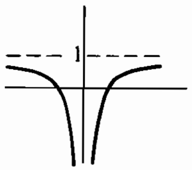
7.



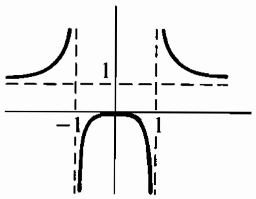
8.



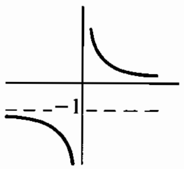
9.



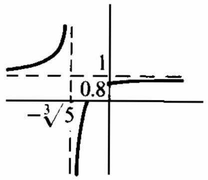
10.



11.



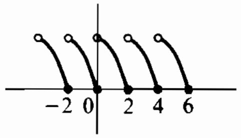
12.



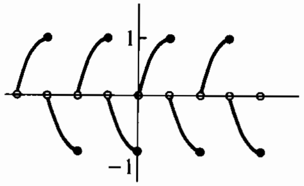
Exercise 37c – p. 673

1. (a) even, periodic (b) odd, periodic
   (e) odd, periodic (h) odd

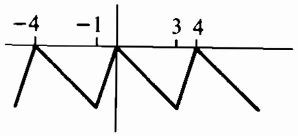
2.



3.

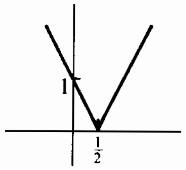


4.

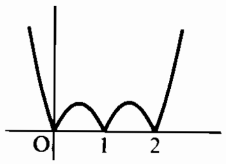


Exercise 37d – p. 676

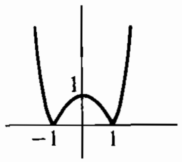
1.



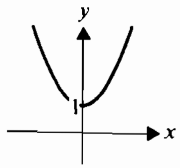
2.



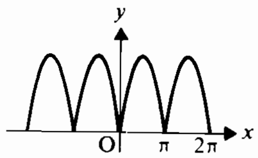
3.



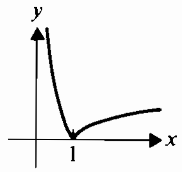
4.



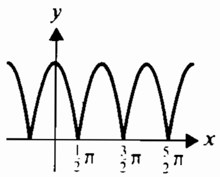
5.



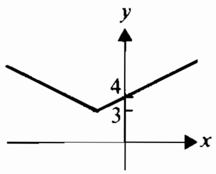
6.



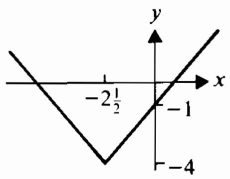
7.



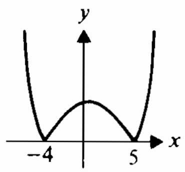
8.



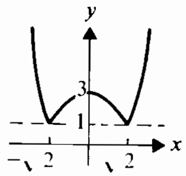
9.



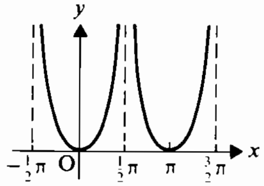
10.



11.



12.

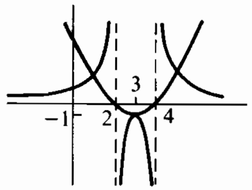


Exercise 37e – p. 679

1. $(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})$
2. $(0, 0), (1, 1)$
3. $(\sqrt{3}, 2\sqrt{3}), (2 - \sqrt{7}, 2\sqrt{7} - 4)$
4. $(1, 1), (-1, 1)$
5. $(1, 3), (1 + \sqrt{6}, 3 + 2\sqrt{6})$
6. 3 and $\frac{1}{2}(\sqrt{17} - 3)$
7. $\frac{4}{5}$ and $\frac{2}{3}$
8. $\frac{2}{3}$ and 2
9. $\sqrt{2} - 1$ and 3

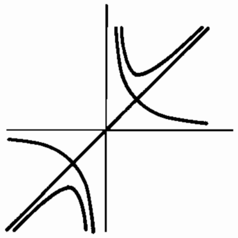
Exercise 37f – p. 681

1. $\frac{1}{2}(1 - \sqrt{5}) < x < 0, x > \frac{1}{2}(1 + \sqrt{5})$
2. $0 < x < 1$
3. $0 < x < 1$
4. $-1 < x < 0$
5. $-2 < x < -1$
6. $x > 0$
7. $-\frac{1}{2} < x < \frac{1}{2}$
8. $x > -\frac{1}{2}$
9. $-1 < x < \frac{1}{3}$
10. $x > 1$
11. $x < 1$
12. $x < -\frac{5}{4}, x > -\frac{1}{4}$
13. $x < 0, x > 2$
14. $x < 0, x > 2$
15. $0 < x < 1 + \sqrt{3}$
16. $-1 < x < 1$
17. $0 < x < \frac{1}{4}\pi, \frac{3}{4}\pi < x < \frac{5}{4}\pi, \frac{7}{4}\pi < x < 2\pi$
18.



$f(x) \leqslant -1$,
$f(x) > 0$

19.

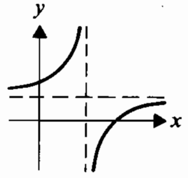


$f(x) \leqslant -2$, $f(x) \geqslant 2$

20. (a) $f(x) < 0$, $f(x) > 0$
(b) $f(x) \leqslant -\frac{1}{4}$, $f(x) > 0$
(c) $f(x) \geqslant -\frac{1}{4}$
(d) $f(x) \leqslant -4$, $f(x) > 0$

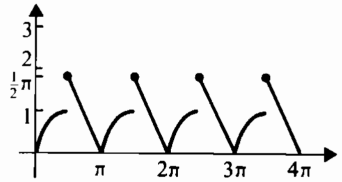
Mixed Exercise 37, p. 683

1.

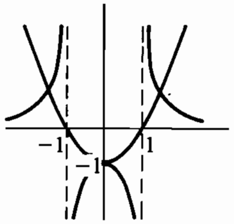


$y = 1$ and $x = 3$; $(4, 0)$ and $(0, \frac{4}{3})$

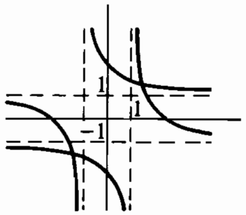
2.



3.



4.

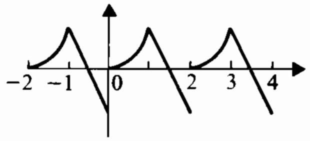


$g^{-1}(x) > 1$, $g^{-1}(x) < 1$

5. $x < -\frac{1}{2}$

6. $-1 < x < 0$ and $x > 2$

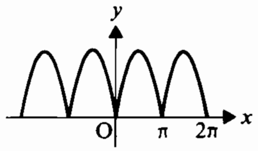
7.



$x = -2, 0, 2, 4$

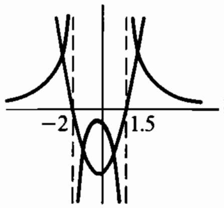
8. (a) odd with period π (b) odd
(c) even (d) even with period 2π

9.



(a) and (c)

10. $-\frac{49}{8}$; $x = -2$ and $\frac{3}{2}$



; 4

CONSOLIDATION F

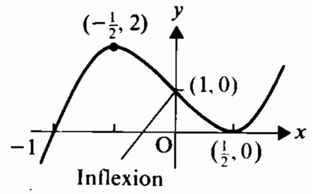
Multiple Choice Exercise F - p. 688

1. B
2. C
3. A
4. E
5. E
6. C
7. A
8. C
9. B
10. A
11. E
12. B
13. C
14. C
15. D
16. E
17. A, B
18. A, C
19. A, C
20. A, C
21. B, C
22. A
23. A, B
24. A
25. F
26. T
27. F
28. T
29. F
30. F
31. T

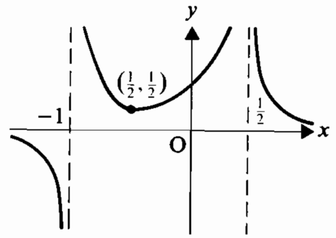
Miscellaneous Exercise F – p. 692

1. $\frac{1}{2}$
2. 999×26^4; $26 \times 25 \times 24 \times 23 \times 738$
3. $a = 2n - \frac{1}{2}$
4. $1 - 2x - 2x^2 - 4x^3$
5. 110

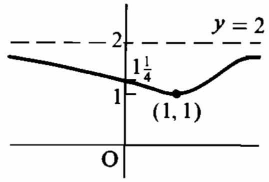
6. 1.5, 6.25
7. $2 < x < 4$
8. 3
9. $1 - x + 2x^2 - \frac{14}{3}x^3$, $-\frac{1}{3} < x < \frac{1}{3}$
10. $x < -1$
11. (a) 182 (b) 41
12. (b) 1.72
13. 8
14. 22
17. $1 - \frac{2}{3}x - \frac{4}{9}x^2$, $-\frac{1}{2} < x\ \frac{1}{2}$
18. $x < 1$
19. 33, $3\frac{1}{2}$; 1450
20. 4, $2x^2$, $-12x^3$
21. 10 080, 7560
22. $a = 2, b = -3$; 1365
23. (a) $-\frac{2}{3} < x < 2$
(b) $-7 \leqslant y \leqslant -1$, $y \geqslant 1$
24. $-3 \leqslant x \leqslant 1$
25. (a)



(b)



26.



27. (a) 5050 ln 2 (b) $\dfrac{\ln 2}{1 - \ln 2}$
28. $1 + \frac{1}{2}y - \frac{1}{8}y^2$; $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$; 0.0103
29. $x = 0, y = 0, x = -2, y = 1$; $(0, 0)$; $(2, \frac{1}{2})$, $(-1, -1)$; $0 < x < 2$, $-2 < x < -1$
30. $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$; $\frac{22}{7} - \pi$
31. (a) $1 + 11x + 64x^2$
(b) $11x + \frac{7}{2}x^2 + \frac{59}{3}x^3$
32. $1 - px + \dfrac{p^2x^2}{2}$, $1 - 2qx + 2q(q + 1)x^2$, $p = 4, q = 2$
33. $1 - \frac{1}{6}x + \frac{1}{24}x^2$, $|x| < 3$

CHAPTER 38

Exercise 38a – p. 700

1. $\frac{1}{6}\pi(1 + 4n)$
2. $\frac{1}{8}\pi(1 + 4n)$, $\frac{1}{4}\pi(n - 1)$
3. $2n\pi$, $\pm\frac{2}{3}\pi + 2n\pi$
4. $2n\pi + \tan^{-1}\frac{12}{5}$
5. $n\pi$
6. $\pm\frac{1}{4}\pi + n\pi$, $\pm\frac{2}{3}\pi + 2n\pi$
7. $\pm\frac{1}{2}\pi + 2n\pi$
8. $\tan^{-1}\frac{3}{4} + \frac{1}{2}\pi + 2n\pi$
9. $\pm\frac{1}{2}\pi + 2n\pi$
10. $2n\pi + \sin^{-1}(\sqrt{2} - 1)$, $(2n + 1)\pi + \sin^{-1}(\sqrt{2} - 1)$
11. 0, 30°, 150°, 180°
12. 53.1°, 180°
13. 0, 180°
14. 201.5°, 338.5°
15. 26.6°, 135°, 206.6°, 315°
16. 34.6°, 325.4°
17. 120°, 300°
18. 0, 72°, 90°, 144°, 180°, 216°, 270°, 288°
19. 20°, 100°, 140°, 220°, 260°, 340°

Exercise 38b – p. 702

1. 1.63
2. 0.861
3. 1.16
4. 2.77
5. −0.104
6. 1.22
7. −1, 1
8. 2
9. 3
10. 2
11. 3
12. $\sqrt{3}$ or $1/\sqrt{3}$
13. $x = \frac{1}{2}, y = 1$
14. $x = 2, y = 4$
15. $x = 1, y = 0$
16. $x = 1, y = 0$

Exercise 38c – p. 705

1. $-\frac{3}{4}, -\frac{3}{4}, \frac{2}{5}$
2. $-1, -1, \frac{1}{5}, 5$
3. $\frac{3}{2}, \frac{3}{2}, -\frac{2}{5}$
4. 1, 2, 3
5. 4
6. $6 + 2\sqrt{3}$
7. $\frac{1}{2}(3 \pm \sqrt{41})$
8. 4
9. 5
10. $\pm 3, \pm\sqrt{3}$
11. $1, 8, -8$
12. $-1, 2$
13. -2
14. $x = \sqrt{2}, y = \frac{1}{2}\sqrt{2}$ and $x = -\sqrt{2}$, $y = -\frac{1}{2}\sqrt{2}$
15. $x = 2, y = 3$ and $x = -2, y = -1$
16. $x = -3, y = 4, x = -6, y = 3$
17. $-2, -2, 1, 4$

Exercise 38d – p. 709

1. (a) ∞ (b) 2 (c) ∞ (d) ∞ and all +ve (e) 3 (f) 1 (g) 1 (h) 2 (i) 1
2. (b) $1 < x < 1.5$ (e) $2 < x < 2.5$ (f) $0.5 < x < 1$ (g) $1.5 < x < 2$ (h) $0.5 < x < 1$ (i) $0 < x < 0.5$
3. (0, 1) max, (2, −3) min, 3
4. (a) 2 (b) 3 (c) 3
5. −1 (exact)
6. $-3 < x < -2$

Exercise 38e – p. 715

1. 0.20
2. 0.22
3. 0.16
4. 0.90
5. 5.2
6. −1.5

In the remaining questions, the degree of accuracy depends upon the accuracy of the first approximation: these answers are correct to 5 significant figures.

7. 1.1656
8. −2.6691
9. 0 (exact), 1.2564
10. 0.51097
11. 0 (exact), 0.74685
12. 0.37482
13. 0 (exact), 1.9038
14. 1 (exact)
15. −0.91735, 0.86366
16. 3.1038
17. −1.4537, 0.53979

Mixed Exercise 38 – p. 716

1. 1/2 lg 3
2. 1
3. $2^{\pm\sqrt{1/2}}$
4. $\pm\cos^{-1}\frac{2}{5} - \tan^{-1}\frac{4}{3} + 2n\pi$
5. $x = \frac{1}{2}(1 + \sqrt{7}), y = \frac{1}{2}(1 - \sqrt{7})$; $x = \frac{1}{2}(1 - \sqrt{7}), y = \frac{1}{2}(1 + \sqrt{7})$
6. $\pm\frac{1}{4}\pi + n\pi, \frac{7}{6}\pi + 2n\pi, \frac{11}{6}\pi + 2n\pi$
7. 1
8. $-\frac{1}{3}, \frac{1}{3}$
9. $2n\pi$
10. 3
11. 2.20707
12. 1.3
13. 0.7

CHAPTER 39

Exercise 39a – p. 724

1. (a) $\overrightarrow{AC}$ (b) $\overrightarrow{BD}$ (c) $\overrightarrow{AD}$ (d) $\overrightarrow{DB}$
2. $\mathbf{a}, \mathbf{b} - \mathbf{a}, 2\mathbf{b} - 2\mathbf{a}$
5. $\mathbf{b} - \mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{a} - 3\mathbf{b}, 2\mathbf{b}, 2\mathbf{a} - 2\mathbf{b}$
6. $\overrightarrow{DE} = \overrightarrow{CH} = \overrightarrow{BG} = \mathbf{c}$, $\overrightarrow{DC} = \overrightarrow{EH} = \overrightarrow{FG} = \mathbf{a}$, $\overrightarrow{FE} = \overrightarrow{GH} = \overrightarrow{BC} = \mathbf{b}$
8. $\mathbf{c} - \mathbf{a}, \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{c}$
9. $\mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{b} - \mathbf{c} - \mathbf{a}, \mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b} - \mathbf{c}$
10. (a) 90° (b) 19.1° (c) 40.9°

Exercise 39b – p. 728

1. (a) $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ (b) $\frac{1}{2}(\mathbf{b} + \mathbf{d})$
2. $\frac{1}{4}(3\mathbf{a} + \mathbf{d})$
3. $\frac{1}{7}(3\mathbf{b} + 4\mathbf{c})$
4. (a) $\frac{1}{4}(\mathbf{a} + 2\mathbf{b} + \mathbf{c})$ (b) $\frac{1}{4}(\mathbf{b} + 2\mathbf{c} + \mathbf{d})$
5. $\mathrm{PQ} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$, $\mathrm{AC} = (\mathbf{c} - \mathbf{a}) = 2\mathrm{PQ}$
6. $\frac{3}{4}(\mathbf{d} - \mathbf{a})$
7. $\frac{1}{2}(\mathbf{c} + \mathbf{d} - 2\mathbf{a})$
8. (a) $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (b) $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

Exercise 39c – p. 734

1. (a) $3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$ (b) $\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$ (c) $\mathbf{i} - 3\mathbf{k}$

2. (a) (5, −7, 2) (b) (1, 4, 0) (c) (0, 1, −1)
3. (a) $\sqrt{21}$ (b) 5 (c) 3
4. (a) 6 (b) 7 (c) $\sqrt{206}$
5. (a) $3\mathbf{i} + 4\mathbf{k}$ (b) $2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ (c) $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ (d) $-6\mathbf{i} + 12\mathbf{j} - 8\mathbf{k}$
6. (b), (d) and (e)
7. (c)
8. $\lambda = \frac{1}{2}$
9. (b), (e) and (f)
10. (a) neither (b) parallel (c) equal
11. (a) $-2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ (b) $-\mathbf{j} + 3\mathbf{k}$ (c) $-2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
12. $\sqrt{62}, \sqrt{10}, 2\sqrt{6}$
13. $\sqrt{5}$
14. $\overrightarrow{AB} = -\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{BD} = -\mathbf{j} - 4\mathbf{k}$, $\overrightarrow{CD} = -2\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, $\overrightarrow{AD} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
15. $y^2 = 4x$; parabola

Exercise 39d – p. 738

1. (a) 2 : 2 : −1 (b) 6 : −2 : −3 (c) 3 : 0 : 4 (d) 1 : 8 : 4
2. (a) $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (b) $\frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$ (c) $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$ (d) $\frac{1}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}$
3. (a) $(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$ $(\frac{3}{7}, \frac{2}{7}, \frac{6}{7})$ (c) $(\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9})$ (d) $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$
4. (a) $(\frac{12}{7}, -\frac{18}{7}, \frac{36}{7})$ (b) $(\frac{16}{9}, \frac{8}{9}, -\frac{2}{9})$
5. (a) $4\mathbf{j} + 5\mathbf{k}$ (b) $16\mathbf{i} - 16\mathbf{j} - 8\mathbf{k}$ (c) $\pm(\frac{8}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k})$
6. (a) $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ (b) $\frac{5}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$ (c) $\mathbf{i}$
7. $\sqrt{22}$, 3 : 2 : 3; $3\sqrt{2}$, 1 : −4 : −1

Exercise 39e – p. 742

1. $\sqrt{13}$; 3
2. 2 : 0 : −1; −1 : 0 : 2
3. $\frac{1}{3}(4\mathbf{i} - \mathbf{j} + 4\mathbf{k})$
4. (a) $\frac{2}{5}\mathbf{i} + \frac{1}{5}\mathbf{j} + \frac{7}{5}\mathbf{k}$ (b) $-2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$
5. (a) no (b) no (c) no

Exercise 39f – p. 747

1. (a) $x - 2 = y - 3 = z + 1$ (b) $\frac{x}{3} = \frac{z}{5}$ and $y = 4$ (c) $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$
2. (a) $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ (b) $\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ (c) $\mathbf{r} = \mathbf{i} + \lambda(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$
3. (a) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$; $\frac{x - 1}{5} = \frac{y + 3}{4} = \frac{z - 2}{-1}$
   (b) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(3\mathbf{j} - \mathbf{k})$; $x = 2, \frac{y - 1}{3} = \frac{z}{-1}$
   (c) $\mathbf{r} = \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$; $x = -y = -z$
4. (a) no (b) yes (c) yes (d) yes (e) no
5. $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, $4 - x = 5 - y = \frac{10 - z}{3}$
   $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 5\mathbf{k})$, $2 - x = 3 - y = \frac{4 - z}{5}$
6. (a) $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} - 6\mathbf{k})$; $(\frac{7}{3}, 3, 0)$; $(0, 10, 14)$; $(\frac{10}{3}, 0, -6)$
   (b) $\mathbf{r} = \mathbf{i} + \mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$; $(\frac{10}{3}, \frac{9}{2}, 0)$; $(0, -\frac{1}{2}, 10)$; $(\frac{1}{3}, 0, 9)$
7. $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$; $(\frac{13}{2}, 0, -\frac{3}{2})$
8. 6 : 3 : 2; $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$
9. (a) $x = 1 - \lambda, y = 2 - 2\lambda, z = 4 + 2\lambda$

Exercise 39g – p. 751

1. parallel
2. intersecting; $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$
3. skew
4. $a = -3$; $\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$
5. $t = 6$; (1, 2, 2)
6. (a) $p = -2$ (b) $p = 0$
7. $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$; $(\frac{2}{7}, \frac{3}{7}, -\frac{6}{7})$; $\arccos -\frac{16}{21}$

Exercise 39h – p. 758

1. (a) 30 (b) 0; **a** and **b** are perpendicular (c) −1
2. (a) $7, \frac{1}{3}\sqrt{7}$ (b) $14, \sqrt{(\frac{7}{19})}$ (c) $3, \frac{3}{58}\sqrt{58}$ (d) $1, \frac{1}{5}$
3. (a) $a^2 - \mathbf{a}\cdot\mathbf{b}$ (b) $a^2 + b^2 - 2\mathbf{a}\cdot\mathbf{b}$ (c) $b^2 - a^2$ (d) $2\mathbf{b}\cdot\mathbf{c}$
4. (a) 0 (b) $-b^2$ (c) a^2 (d) $2a^2 - b^2$
5. 4
6. (a) −4 (b) 15 (c) 11 (d) −12 (e) 29
7. $-\sqrt{\frac{7}{34}}$; $\arccos\sqrt{\frac{7}{10}}$

11. (a) $\arccos \frac{19}{21}$ (b) $\arccos \frac{2}{3}$
(c) $\arccos(\frac{8}{15}\sqrt{3})$
12. (a) 78.5° (b) 0, i.e. parallel lines
15. (a) $\frac{4}{3}$ (b) 2 (c) $-\frac{7}{5}$

Exercise 39i – p. 768

1. (a) $x - y - z = 2$
(b) $2x + 3y - 4z = 0$
2. (a) $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 6$
(b) $\mathbf{r}.(\mathbf{i} - 7\mathbf{j} - \mathbf{k}) = 5$
4. (a) $2x - y + z + 1 = 0$;
$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = -1$
(b) $3x - y - z = 2$; $\mathbf{r}.(3\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2$
5. $\mathbf{r}.(7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 3$, $3/\sqrt{62}$
6. $3/\sqrt{5}$ 7. 2 8. $\mathbf{r} = \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
9. $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
10. $\mathbf{r}.(\mathbf{i} - 2\mathbf{j}) = -2$
11. $\mathbf{r}.(\mathbf{j} - \mathbf{k}) = 0$
12. $(\frac{5}{2}, -\frac{1}{2}, -\frac{1}{2})$
13. $(1, -1, 4)$
14. $7\sqrt{3}/3$
15. (a) $\mathbf{r}.(5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 7$
(b) $\mathbf{r}.(-\mathbf{j} + 2\mathbf{k}) = -2$
(c) $\mathbf{r}.(2\mathbf{i} + \mathbf{k}) = 5$
16. $\mathbf{r} = -3\mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
$+ \mu(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
17. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$;
The second line is contained in the plane.
18. (a) intersecting (b) intersecting
(c) parallel
(d) contained in the plane.

Exercise 39j – p. 771

1. $3/\sqrt{11}$
2. $20/\sqrt{442}$
3. $1/\sqrt{87}$
4. $\frac{6}{7}$
5. $\frac{1}{9}\sqrt{3}$
6. $\frac{4}{9}$
7. (a) $a = 2$ (b) $\frac{1}{2}$
9. $\frac{1}{5}\sqrt{15}$

CHAPTER 40

Exercise 40a – p. 779

1. $-\mathrm{i}, \mathrm{i}, \mathrm{i}, -\mathrm{i}, 1, \mathrm{i}$
2. (a) $10 + 4\mathrm{i}$ (b) $7 + 2\mathrm{i}$
(c) $6 - 2\mathrm{i}$ (d) $(a + c) + (b + d)\mathrm{i}$
3. (a) $-4 + 6\mathrm{i}$ (b) $1 - 4\mathrm{i}$
(c) $-2 + 16\mathrm{i}$ (d) $(a - c) + (b - d)\mathrm{i}$
4. (a) $10 - 5\mathrm{i}$ (b) $39 + 23\mathrm{i}$
(c) $11 - 7\mathrm{i}$ (d) 25
(e) $3 - 4\mathrm{i}$ (f) $-2 + 2\mathrm{i}$
(g) $-4 + 3\mathrm{i}$ (h) $x^2 + y^2$
(i) $-3 + \mathrm{i}$ (j) $(a^2 - b^2) + 2ab\mathrm{i}$
5. (a) $1 + \mathrm{i}$ (b) $\frac{9}{25} + \frac{13}{25}\mathrm{i}$
(c) $\frac{4}{17} + \frac{16}{7}\mathrm{i}$ (d) i (e) $-\mathrm{i}$
(f) $\dfrac{x^2 - y^2}{x^2 + y^2} + \dfrac{2xy}{x^2 + y^2}\mathrm{i}$
(g) $1 - 3\mathrm{i}$ (h) $-3 - 2\mathrm{i}$
6. (a) $x = 9, y = -7$
(b) $x = -\frac{3}{2}, y = \frac{7}{2}$
(c) $x = \frac{7}{2}, y = \frac{1}{2}$
(d) $x = 2, y = 0$
(e) $x = 13, y = 0$
(f) $x = 15, y = 8$
(g) $x = 11, y = 3$
(h) $x = 2, y = 1$
or $x = -2, y = -1$
7. (a) $7, -1$ (b) $-2, -2$
(c) 25, 0 (d) $\frac{10}{17}, \frac{11}{17}$
(e) $\frac{9}{5}, -\frac{4}{5}$ (f) $0, \dfrac{-2y}{x^2 + y^2}$
8. (a) $\pm(2 - \mathrm{i})$ (b) $\pm(5 - 2\mathrm{i})$
(c) $\pm(1 + \mathrm{i})$ (d) $\pm(4 + \mathrm{i})$
(e) $\pm(1 + 5\mathrm{i})$

Exercise 40b – p. 782

1. (a) $-\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\sqrt{3}$ (b) $-\frac{7}{4} \pm \frac{1}{4}\sqrt{41}$
(c) $\pm 3\mathrm{i}$ (d) $-\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\sqrt{11}$
(e) $\pm 1, \pm \mathrm{i}$ (f) $-\frac{1}{6} \pm \frac{1}{6}\mathrm{i}\sqrt{35}$
2. (a) $x^2 + 1 = 0$
(b) $x^2 - 4x + 5 = 0$
(c) $x^2 - 2x + 10 = 0$
(d) $x^3 - 4x^2 + 6x - 4 = 0$
3. (a) 4, 5 (b) 6, 25 (c) 0, 1
(d) -24, 169
4. (a) $x^2 + 2x + 2 = 0$
(b) $x^2 + 10x + 26 = 0$
(c) $x^2 - 2x + 10 = 0$
(d) $x^2 - 8x + 17 = 0$

When one root is real, so is the other and there is no special relationship between them.

Exercise 40d – p. 790

Arguments are given in terms of π when exact and in degrees to 3 s.f. otherwise.

1. (a) $\sqrt{13}, -33.7°$ (b) $\sqrt{17}, 166°$
(c) $5, -127°$ (d) $13, 67.4°$
(e) $\sqrt{2}, -\frac{1}{4}\pi$ (f) $\sqrt{2}, \frac{3}{4}\pi$
(g) 6, 0 (h) $4, -\frac{1}{2}\pi$
(i) $2, \frac{1}{2}\pi$ (j) $\sqrt{2}, \frac{1}{4}\pi$
(h) $\sqrt{2}, \frac{3}{4}\pi$ (l) $\sqrt{170}, 32.5°$
(m) $1, \frac{3}{4}\pi$ (n) $2, \frac{2}{3}\pi$
(p) $1, -\frac{5}{6}\pi$
(q) $\sqrt{(a^2 + b^2)}$, $\arctan \dfrac{b}{a}$ (ambiguous until a and b are known)
2. (a) $\sqrt{2}(\cos\frac{1}{4}\pi + \mathrm{i}\sin\frac{1}{4}\pi)$
(b) $2\{\cos(-\frac{1}{6}\pi) + \mathrm{i}\sin(-\frac{1}{6}\pi)\}$
(c) $5\{\cos(-127°) + \mathrm{i}\sin(-127°)\}$
(d) $13\{\cos(-113°) + \mathrm{i}\sin(-113°)\}$
(e) $\sqrt{5}\{\cos(-26.6°) + \mathrm{i}\sin(-26.6°)$
(f) $6\{\cos 0 + \mathrm{i}\sin 0\}$
(g) $3\{\cos\pi + \mathrm{i}\sin\pi\}$
(h) $4\{\cos\frac{1}{2}\pi + \mathrm{i}\sin\frac{1}{2}\pi\}$
(i) $2\sqrt{3}\{\cos(-\frac{5}{6}\pi) + \mathrm{i}\sin(-\frac{5}{6}\pi)\}$
(j) $25\{\cos(16.3°) + \mathrm{i}\sin(16.3°)\}$
3. (a) $\sqrt{3} + \mathrm{i}$ (b) $\frac{3}{2}\sqrt{2} - \frac{3}{2}\mathrm{i}\sqrt{2}$
(c) $-\frac{1}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}$ (d) $3 + 0\mathrm{i}$
(e) $-4 + 0\mathrm{i}$ (f) $-\frac{1}{2}\sqrt{2} - \frac{1}{2}\mathrm{i}\sqrt{2}$
(g) $0 + 2\mathrm{i}$ (h) $0 - 2\mathrm{i}$

Exercise 40e – p. 793

1. (a) $(2\sqrt{3})(\sqrt{2}) = 2\sqrt{6}$;
$-\frac{1}{6}\pi + (-\frac{1}{4}\pi) = -\frac{5}{12}\pi$
(b) $(3 - \sqrt{3}) - (3 + \sqrt{3})\mathrm{i}$
2. (a) $(\sqrt{2})(\sqrt{2}) = 2$; $\frac{1}{4}\pi + (-\frac{1}{4}\pi) = 0$
(b) 2
3. (a) $(\sqrt{7}) \div (\sqrt{7}) = 1$;
$(-139°) - (139°) = -278° \Rightarrow 172°$
(b) $\frac{1}{7} + \frac{4}{7}\mathrm{i}\sqrt{3}$
4. (a) $2 \div (\sqrt{2}) = \sqrt{2}$; $0 - (\frac{1}{4}\pi) = -\frac{1}{4}\pi$
(b) $1 - \mathrm{i}$
5. (a) $\dfrac{1}{z} = \dfrac{2 - \mathrm{i}}{5}$; $z^2 = 3 + 4\mathrm{i}$
(b) $\dfrac{1}{z} = 1 + \mathrm{i}$; $z^2 = -\frac{1}{2}\mathrm{i}$
(c) $\dfrac{1}{z} = \dfrac{3 - 4\mathrm{i}}{25}$; $z^2 = -7 + 24\mathrm{i}$
(d) $\dfrac{1}{z} = \frac{1}{2}\sqrt{3} - \frac{1}{2}\mathrm{i}$; $z^2 = \frac{1}{2}(1 + \mathrm{i}\sqrt{3})$
(e) $\dfrac{1}{z} = \dfrac{5 + 12\mathrm{i}}{169}$; $z^2 = -119 - 120\mathrm{i}$

Mixed Exercise 40 – p. 793

1. (a) $\dfrac{7 - 5\mathrm{i}}{74}$ (b) $11 + 2\mathrm{i}$
(c) $\frac{1}{2}(1 - \mathrm{i})$ (d) $-21 + 20\mathrm{i}$
2. $\frac{3}{10}(11 - 3\mathrm{i})$
3. $\pm\frac{1}{2}\sqrt{2}(5 + \mathrm{i})$
4. (a) $\sqrt{2}, \frac{3}{4}\pi$ (b) 5, 53°
(c) $5\sqrt{2}, -172°$, area = 8.9
5. (a) $2\sqrt{10}$ (b) 10 (c) 1
(d) $\frac{1}{5}\sqrt{10}$
6. $\lambda = 2, \mu = \frac{1}{2}$
7. $a = 4, b = -5$
8. $\dfrac{x(x^2 + y^2 - 1)}{x^2 + y^2}, \dfrac{y(x^2 + y^2 + 1)}{x^2 + y^2}$
9. $\lambda = 4, \mu = -5$
10. $2\sqrt{2}, \frac{3}{4}\pi$; $2\sqrt{2}, -\frac{3}{4}\pi$
11. (a) $-\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 0
12. $2 + \mathrm{i}$, $2 - 3\mathrm{i}$

CONSOLIDATION G

Multiple Choice Exercise G – p. 799

1. B
2. A
3. B
4. E
5. D
6. A
7. D
8. A
9. C
10. E
11. A
12. C
13. D
14. D
15. C
16. E
17. B
18. A
19. C
20. A
21. A
22. B, C
23. B, C
24. B, C
25. A
26. B
27. B, C
28. F
29. F
30. T
31. F
32. F

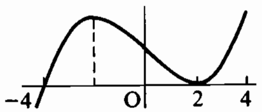
Miscellaneous Exercise G – p. 803

1. 0, 60°, 120°, 180°, 90°
2. (a) $n\pi + \frac{1}{12}\pi$, $n\pi + \frac{5}{12}\pi$
3. (a) 0, 180°, 360°, 54.7°, 125.3°, 234.7°, 305.3°
(b) 51.3°, 128.7°
4. (a) $n\pi$, $2n\pi \pm \frac{1}{6}\pi$
(b) $\frac{1}{2}\pi, \frac{3}{2}\pi$, 0.67 rad, 2.48 rad
5. $x = 6$
6. 0.774

7. $x = 3^{10}, y = 3^2$ or $x = 3^2, y = 3^{10}$

8. (a) 64 or $\frac{1}{64}$ (b) 21

9. $(x-2)^2(x+4)$;



10. 0.433

11. 3.008

12. 1.90

13. $a = 22$; 2.22

14. 1.78

15. $n = 1$; 1.27, 1.24

16. $2\sqrt{2}\,(\cos\frac{1}{4}\pi + \mathrm{i}\sin\frac{1}{4}\pi)$; $\sqrt{104}$; 0.32 rad

17. (a)

a	-2	-4
b	2	1

(b) $\sqrt{2}$; $-\frac{1}{4}\pi$

18. $p = -2, q = -4$

19. ang $z_1 = \frac{2}{3}\pi$, ang $z_2 = \frac{1}{6}\pi$; i, $\frac{1}{2}\pi$

20. (a)

a	5	-5
b	-3	3

; $\pm(5-3\mathrm{i})$

21. $\pm(3-2\mathrm{i})$; $-5+12\mathrm{i}$

22. (a) $\frac{1}{2}\pi$ (b) $-\frac{1}{4}\pi$ (c) $\frac{1}{12}\pi$

23. $x = 3, y = -1$;
(a) $\sqrt{10}$ (b) $-\arctan\frac{1}{3}$

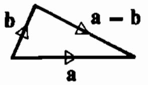
24. (a) $-1, 7$ (b) $-7\mathrm{i}$

25. $\frac{1}{2}(1+3\mathrm{i})$ or $\frac{3}{2}(1+\mathrm{i})$

26. 10, 96.9°; $\frac{5}{2}$, −23.1°

27. 69°

28. (a) 7 (b) 4



29. (b) (7, 2, −6) (c) $\frac{14}{15}$
(d) $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$
or $\mathbf{r}.(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 6$

30. $\frac{2}{3}(4\mathbf{i} - 4\mathbf{j} + \mathbf{k})$
(a) $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ (b) 83.5°
(c) $(6\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})/\sqrt{77}$

31. 77.5°

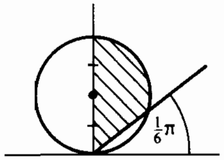
32. $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(4\mathbf{i} - 3\mathbf{j} - \mathbf{k})$; −2

33. (a) 20, $\frac{20}{21}$
(b) $\mathbf{r} = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + t(4\mathbf{i} + \mathbf{j} + \mathbf{k})$;
$-\frac{14}{11}\mathbf{i} + \frac{13}{11}\mathbf{j} + \frac{2}{11}\mathbf{k}$

34. $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})$;
$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k} = 0$

35. (b) $\frac{1}{27}\sqrt{27}(5\mathbf{i} - \mathbf{j} + \mathbf{k})$
(c) 28, 54.2°

36.



37. (a) $\mathbf{r} = a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + ta(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k})$
(b) $\mathbf{r}.(13\mathbf{i} + 17\mathbf{j} + 16\mathbf{k}) = 0$
(c) $\frac{1}{55}\sqrt{55}\sqrt{33}$; $\mathbf{r}.(9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = 15a$
(d) $3x - 2y + 4z = 5a$
(e) $\dfrac{x+4a}{3} = \dfrac{y-4a}{-2} = \dfrac{z+a}{4}$

38. (b) −1 (c) $\sqrt{68}$

INDEX

Core Maths for A-level has been written to cover the whole of the Common Core syllabus for A-level Mathematics, together with the additional material required by most Examining Boards.

- Because students enter A-level courses from a variety of backgrounds, the early chapters provide transitional material of an introductory nature.
- All chapters contain some new work however, to interest those students who come to the course with a higher level of skills and knowledge.
- Exercises are included at frequent intervals, giving ample provision of straightforward questions.
- Consolidation Sections contain a summary of the preceding work and multiple-choice questions as well as questions at the level of sophistication found in A-level examination papers. These questions are intended for practice when the student has, through experience, developed confidence and skill.

ISBN 0-7487-0067-6

STANLEY THORNES

STANLEY THORNES

Ellenborough House
Wellington Street
CHELTENHAM
Glos. GL50 1YD

9 780748 700677